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Representation theory of Khovanov–Lauda–Rouquier algebras

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Abstract

This thesis concerns representation theory of the symmetric groups and related algebras.

In recent years, the study of the "quiver Hecke algebras", constructed independently by Khovanov and Lauda and by Rouquier, has become extremely popular. In this thesis, our motivation for studying these graded algebras largely stems from a result of Brundan and Kleshchev – they proved that (over a field) the KLR algebras have cyclotomic quotients which are isomorphic to the Ariki–Koike algebras, which generalise the Hecke algebras of type A, and thus the group algebras of the symmetric groups. This has allowed the study of the graded representation theory of these algebras. In particular, the Specht modules for the Ariki–Koike algebras can be graded; in this thesis we investigate graded Specht modules in the KLR setting.

First, we conduct a lengthy investigation of the (graded) homomorphism spaces between Specht modules. We generalise the row and column removal results of Lyle and Mathas, producing graded analogues which apply to KLR algebras of arbitrary level. These results are obtained by studying a class of homomorphisms we call *dominated*. Our study provides us with a new result regarding the indecomposability of Specht modules for the Ariki–Koike algebras.

Next, we use homomorphisms to produce some decomposability results pertaining to the Hecke algebra of type A in quantum characteristic two.

In the remainder of the thesis, we use homogeneous homomorphisms to study some graded decomposition numbers for the Hecke algebra of type A. We investigate graded decomposition numbers for Specht modules corresponding to two-part partitions. Our investigation also leads to the discovery of some exact sequences of homomorphisms between Specht modules.

Contents

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Introduction

In group theory, the symmetric groups are among the most classically studied; their importance stems from Cayley's Theorem – every group may be embedded in some symmetric group. Their presence permeates mathematics – from algebra to combinatorics, and even areas of physics, the symmetric groups' ubiquity can be felt. A common perspective when studying groups is the study of their representations, or actions on vector spaces.

Despite the strength of Cayley's Theorem, when studying the representation theory of finite groups, the symmetric groups are especially suitable candidates. Their inherently vast amount of internal symmetry gives rise to their combinatorial nature; studying them via combinatorics eliminates the need for much of the heavy-duty machinery often seen in group theory.

Over the complex numbers, the representation theory of the symmetric group dates back over a century, with the influences of Young [44] and Specht [42] still visible today – the combinatorics of tableaux and indeed the construction of *Specht modules* were their contributions. Decades later, in the 1970s, James developed much of our modern standpoint on the subject. In particular, James constructed all irreducible modules over arbitrary fields, as quotients of Specht modules. An excellent introduction to the subject can be found in [24].

Later, in the series of papers [12, 13, 14], Dipper and James laid the foundations for and built up the theory of the Iwahori–Hecke algebras of symmetric groups, or Hecke algebras of type A. Their principal motivation for studying these algebras was that their modular representation theory provides a bridge between that of the symmetric groups and that of the general linear groups $\mathrm{GL}_n(q)$ over fields of *non-defining* characteristic – that is, over any field whose characteristic does not divide q. In the case of defining characteristic, such a connection has been known since the work of Schur over a century ago, via his construction of the algebras which now bear his name.

Almost a decade later, Ariki and Koike [2] generalised these Hecke algebras beyond type A; here they defined Hecke algebras for the complex reflection group $G(l,1,n) = \mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n$. This theory of *Ariki–Koike algebras* was further developed in [3], among others, and a survey of the subject can be found in [34]. At around the same time, the idea was independently generalised even further to the *cyclotomic Hecke algebras* of Broué and Malle [6]; here, they constructed Hecke algebras for a larger set of complex reflection groups, including some of the exceptional types, motivated by the modular representation theory of algebraic groups. Ariki also generalised his construction to all complex reflection groups $G(l,p,n)$ in [1].

In 2009, Khovanov and Lauda [27], and independently, Rouquier [39], defined a family of $\mathbb{Z}$-graded algebras which categorify the negative part of quantum groups associated to Kac–Moody algebras. These algebras are known as *quiver Hecke algebras*, or *KLR algebras*; in this thesis we shall favour the latter name. Khovanov and Lauda's construction of these algebras was purely diagrammatic, whereas Rouquier presented them algebraically, and in more generality. Remarkably, Brundan and Kleshchev [8] were able to show that the Ariki–Koike algebras were isomorphic to certain *cyclotomic* quotients of these KLR algebras (of affine type A) – this result yields a grading on the Ariki–Koike algebras; in particular, in positive characteristic the group algebras of the symmetric groups are non-trivially $\mathbb{Z}$-graded! This, in part, has given rise to the KLR algebras receiving an abundance of attention since their inception.

In this thesis, Brundan and Kleshchev's isomorphism is seen as our main motivation for studying the KLR algebras. They allow us to study the *graded* representation theory of the Hecke algebras of type A and, more generally, of the Ariki–Koike algebras. We place an emphasis on studying the (graded) representation theory of these algebras via (graded) homomorphisms between their Specht modules. We shall begin, in Chapter 1, with a detailed background of all necessary material. We will introduce our main players – the symmetric groups, Hecke algebras of type A, Ariki–Koike algebras and KLR algebras – and outline the combinatorics and representation theory relevant to their study. We shall also give a brief overview of graded algebras.

In Chapter 2, we look at the KLR algebras of affine type A in full generality. Here, we study the graded homomorphism spaces between Specht modules. Fayers and Lyle [18] and Lyle and Mathas [30] proved row and column removal theorems for these homomorphism spaces, for the symmetric groups and Hecke algebras respectively, in the ungraded setting. In this chapter, we provide graded versions of these theorems, while at the same time generalising them to higher levels so that they apply to all (degenerate) Ariki–Koike algebras. More precisely, we prove analogues of these theorems for the KLR algebras of affine type A.

In fact, our results apply not to all homomorphisms between two given Specht modules but only to those of a certain type, which we call *dominated* homomorphisms. However, in many cases (for example, for the symmetric group in odd characteristic) every homomorphism between two Specht modules is dominated, so our results apply generally; in particular, via the Brundan–Kleshchev isomorphism, we recover the original row and column removal theorems of Fayers and Lyle and of Lyle and Mathas. Along the way, we produce a new result that Specht modules for the Ariki–Koike algebras are always indecomposable, under some minor conditions. This chapter is based on joint work with Fayers, and appears in [19]. The main result of the chapter is Theorem 2.30 (generalised graded column removal).

In Chapter 3, largely taken from [43], we make a contribution to the problem of determining which Specht modules are decomposable. Here, we concentrate on the level 1 (i.e. the Hecke algebra of type A) situation. Over any field whose characteristic is not 2, it is known that Specht modules for the symmetric group are indecomposable. Likewise, when the *quantum characteristic* of a Hecke algebra of type A is not 2, an identical statement holds true. We determine which Specht modules indexed by hook partitions are decomposable in quantum characteristic 2. This is achieved from the KLR algebra perspective, though we do not invoke any *graded* machinery here; instead, the bulk of the work involves constructing an endomorphism for a Specht module and using it to decompose the Specht module into generalised eigenspaces. The result arrived at is Theorem 3.41, which states that when $\mathrm{char}(\mathbb{F}) \neq 2$, the Specht module $\mathrm{S}_{(a,1^b)}$ is indecomposable if and only if $a+b$ is even or $b = 2$ or 3 with $\mathrm{char}(\mathbb{F}) \mid \lceil \frac{a}{2} \rceil$.

Next, in Chapter 4, we study the graded decomposition numbers corresponding to two-part partitions, for Hecke algebras of type A. Once again, we make extensive use of homomorphisms in our endeavour. This chapter contains work which is in progress; here we present a complete solution to the problem in quantum characteristic 2, along with an interesting result regarding exact sequences of homomorphisms between Specht modules.

Finally, Chapter 5 is an attempt at generalising the results of Chapter 2. We concentrate on attempting to prove that in level 1 when $e = 2$, all homomorphisms are dominated, subject to an extra condition on the partition indexing the domain. Our approach begins with us proving a graded branching rule for restriction to a subalgebra, generalising existing results of this flavour. Using this, we are able to prove our desired result with the aid of three conjectures which put conditions on the indexing tableaux occurring in the image of a generator under a homomorphism. We end with a conjecture which extends this beyond level 1.

We conclude the thesis with an index of notation for the reader's reference.

Chapter 1

Background

In this chapter we recall some background and set up some notation.

1.1 The symmetric group

Let $\mathfrak{S}_n$ denote the symmetric group of degree n. Let $s_1, \ldots, s_{n-1}$ denote the standard Coxeter generators of $\mathfrak{S}_n$, i.e. s_i is the transposition $(i, i+1)$. Given $w \in \mathfrak{S}_n$, a *reduced expression* for w is an expression $w = s_{i_1} \ldots s_{i_l}$ with l as small as possible; we call $l = l(w)$ the *length* of w.

We will need to use two natural partial orders on $\mathfrak{S}_n$. If $w, x \in \mathfrak{S}_n$, then we say that x is smaller than w in the *left order* (and write $x \preccurlyeq_{\mathrm{L}} w$) if $l(w) = l(wx^{-1}) + l(x)$; this is equivalent to the statement that there is a reduced expression for w which has a reduced expression for x as a suffix.

More important will be the *Bruhat order* on $\mathfrak{S}_n$: if $w, x \in \mathfrak{S}_n$, then we say that w is smaller than x in the Bruhat order (and write $w \preccurlyeq x$) if there is a reduced expression for x which has a (possibly non-reduced) expression for w as a subsequence. In fact [23, Theorem 5.10], if $w \preccurlyeq x$, then every reduced expression has a reduced expression for w as a subsequence.

The following proposition gives an alternative characterisation of the Bruhat order.

Proposition 1.1 [23, §5.9]. *Suppose $w, x \in \mathfrak{S}_n$. Then $w \preccurlyeq x$ if and only if there are $w =$*

$w_0, w_1, \dots, w_r = x$ such that for each $1 \leqslant i \leqslant r$ we have $w_i = (u_i, v_i)w_{i-1}$, where $1 \leqslant u_i < v_i \leqslant n$ and $w_{i-1}^{-1}(u_i) < w_{i-1}^{-1}(v_i)$.

Later we shall need the following lemma; in fact, this is a special case of Deodhar's 'property Z' [11, Theorem 1.1].

Lemma 1.2. *Suppose $w, x \in \mathfrak{S}_n$ with $x \prec w$. If $l(s_i w) < l(w)$ while $l(s_i x) > l(x)$, then $s_i x \leqslant w$.*

Proof. Since $l(s_i w) < l(w)$, w has a reduced expression s beginning with s_i. We can find a reduced expression for x as a subexpression of s, and this subexpression cannot include the first term s_i, since $l(s_i x) > l(x)$. So we can add the initial s_i to the subexpression to get a reduced expression for $s_i x$ as a subexpression of s. □

Occasionally, it will be useful to talk about *fully commutative* elements of the symmetric group:

Definition 1.3. We call an element $w \in \mathfrak{S}_n$ *fully commutative* if we can go from any reduced expression for w to any other via application of only the commuting braid relations $s_i s_j = s_j s_i$ for $|i - j| > 1$.

We end this section by defining some very natural and useful homomorphisms. Suppose $1 \leqslant m \leqslant n$ and $0 \leqslant k \leqslant n - m$, and define the homomorphism $\mathrm{shift}_k : \mathfrak{S}_m \to \mathfrak{S}_n$ by $s_i \mapsto s_{i+k}$ for every i. Note that if $k = 0$, this is the natural embedding.

1.2 The Iwahori–Hecke algebra

We will fix a field $\mathbb{F}$ throughout this thesis.

Definition 1.4. For any $q \in \mathbb{F}$ we define the *Iwahori–Hecke algebra* $\mathcal{H} = \mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ of the symmetric group $\mathfrak{S}_n$ (also referred to as the Hecke algebra of type A) to be the unital associative $\mathbb{F}$-algebra with presentation

$$\left\langle T_1, \ldots, T_{n-1} \;\middle|\; \begin{array}{c} (T_i - q)(T_i + 1) = 0 \text{ for } 1 \leqslant i \leqslant n-1 \\ T_i T_j = T_j T_i \text{ for } 1 \leqslant i < j-1 \leqslant n-2 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ for } 1 \leqslant i \leqslant n-2 \end{array} \right\rangle.$$

Remark. Note that setting $q = 1$ recovers the Coxeter presentation for the symmetric group. The degenerate case $q = 0$ behaves very differently, and so we will assume that $q \neq 0$ throughout.

Definition 1.5. Define $e \in \{2, 3, 4, \ldots\}$ to be the smallest integer such that $1 + q + q^2 + \cdots + q^{e-1} = 0$. If no such integer exists, we define $e = \infty$. We call e the *quantum characteristic* of $\mathcal{H}$.

An excellent introduction to these algebras and their representation theory can be found in [33].

1.3 The Ariki–Koike algebras

The Iwahori–Hecke algebras, or *Hecke algebras of type A* are deformations of the symmetric group. Soon after their study by Dipper and James, this theme was extended to studying Hecke algebras of type B. In 1994, Ariki and Koike [2] further generalised this work; they defined Hecke algebras for the complex reflection groups $\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n$, or type $G(l, 1, n)$ as given in the Shephard–Todd classification of complex reflection groups [41].

Definition 1.6. Given parameters $q \in \mathbb{F}$ and $Q = (Q_1, \ldots, Q_l) \in \mathbb{F}^l$, we define the *Ariki–Koike algebra* $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$ to be the unital associative $\mathbb{F}$-algebra with presentation

$$\left\langle T_0, T_1, \ldots, T_{n-1} \;\middle|\; \begin{array}{c} (T_0 - Q_1)(T_0 - Q_2)\ldots(T_0 - Q_l) = 0 \\ (T_i - q)(T_i + 1) = 0 \text{ for } 1 \leqslant i \leqslant n-1 \\ T_0T_1T_0T_1 = T_1T_0T_1T_0 \\ T_iT_j = T_jT_i \text{ for } 0 \leqslant i < j-1 \leqslant n-2 \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \text{ for } 1 \leqslant i \leqslant n-2 \end{array} \right\rangle.$$

Remark. Setting $l = 1$, we recover the Hecke algebra of type A. Similarly, when $l = 2$, the Hecke algebra of type B may be recovered from this algebra, which generalises these previous constructions. As in type A, the Ariki–Koike algebra has an associated *quantum characteristic e* defined identically. We will again assume throughout that $q \neq 0$, and similarly that $Q_i \neq 0$ for all i.

Mathas has written a survey [34] of the representation theory of Ariki–Koike algebras (and the associated *cyclotomic q-Schur algebras*). A result of particular interest is [15, Theorem 1.1] – any Ariki–Koike algebra is Morita equivalent to a tensor product of smaller Ariki–Koike algebras, each with the property that $Q_i = q^{a_i}$ for some integers a_i. Thus, we may assume from now on that we work with Ariki–Koike algebras with parameters Q_i each an integral power of q.

Proposition 1.7. *Let $Q = (q^{a_1}, \ldots, q^{a_l})$ for some integers a_i. If $Q' = (q^{b_1}, \ldots, q^{b_l})$ for integers b_i such that $\{q^{a_1}, \ldots, q^{a_l}\} = \{q^{b_1}, \ldots, q^{b_l}\}$ as multisets, then $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n) \cong \mathcal{H}_{\mathbb{F},q,Q'}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$.*

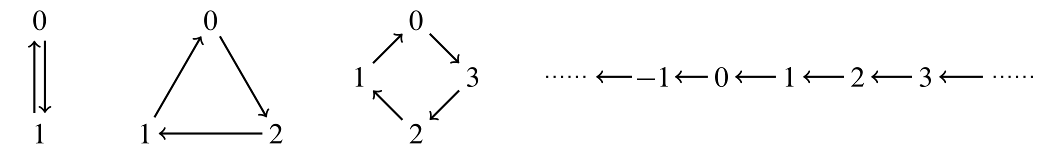
Remark. The above isomorphism is an obvious consequence of the presentation of $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$. Though permuting the parameters leaves an isomorphic algebra, the combinatorics of multipartitions is greatly changed by doing so, and thus the representation theory is somehow distorted (for example, the set of multipartitions indexing the simple modules may change).

Note that, strictly speaking, if $e = \mathrm{char}(\mathbb{F})$ we are interested in *degenerate* Ariki–Koike algebras, which we do not define here. An analogous Morita equivalence result for these degenerations has been proved by Brundan and Kleshchev. Recently, Mathas has given a unifying definition of the Ariki–Koike algebras and their degenerations in [35, Definition 1.1.1].

1.4 Lie-theoretic notation

Throughout this thesis e is a fixed element of the set $\{2,3,4,\dots\}\cup\{\infty\}$. We denoted by e the quantum characteristic of the Hecke algebras of type A, and more generally, the Ariki–Koike algebras; this apparent clash of notation will be resolved later. If $e=\infty$ then we set $I:=\mathbb{Z}$, while if $e<\infty$ then we set $I:=\mathbb{Z}/e\mathbb{Z}$; we may identify I with the set $\{0,\dots,e-1\}$ when convenient. The Cartan matrix $(a_{ij})_{i,j\in I}$ is defined by $a_{ij}=2\delta_{i,j}-\delta_{i,(j+1)}-\delta_{i,(j-1)}$.

Let Γ be the quiver with vertex set I and an arrow from i to $i-1$ for each i. (Note that this convention is the same as that in [29], and opposite to that in [8, 10].) The quiver Γ is pictured below for some values of e.



$e=2$ $e=3$ $e=4$ $e=\infty$

In the relations we give below, we use arrows with reference to Γ; thus we may write $i\rightarrow j$ to mean that $e\neq 2$ and $j=i-1$, or $i\rightleftarrows j$ to mean that $e=2$ and $j=i-1$.

We adopt standard notation from Kac's book [26] for the Kac–Moody algebra associated to the Cartan matrix $(a_{ij})_{i,j\in I}$; in particular, we have fundamental dominant weights Λ_i and simple roots α_i for $i\in I$, and an invariant symmetric bilinear form $(\ |\ )$ satisfying $(\Lambda_i\,|\,\alpha_j)=\delta_{i,j}$ and $(\alpha_i\,|\,\alpha_j)=a_{ij}$ for $i,j\in I$. We let $Q^+:=\bigoplus_{i\in I}\mathbb{Z}_{\geqslant 0}\alpha_i$ be the *positive root lattice*. For $\alpha=\sum_{i\in I}c_i\alpha_i\in Q^+$, we define the *height* of α to be $\sum_{i\in I}c_i$. Given $\alpha,\beta\in Q^+$ with $\alpha=\sum_{i\in I}c_i\alpha_i$ and $\beta=\sum_{i\in I}d_i\alpha_i$, we write $\alpha\geqslant\beta$ if $c_i\geqslant d_i$ for each i.

Let I^l denote the set of all l-tuples of elements of I. We call an element of I^l an *e-multicharge of level l*. The symmetric group $\mathfrak{S}_l$ acts on I^l on the left by place permutations. Given an e-multicharge $\kappa = (\kappa_1, \dots, \kappa_l)$, we define a corresponding dominant weight $\Lambda_\kappa := \Lambda_{\kappa_1} + \dots + \Lambda_{\kappa_l}$. For $\alpha \in Q^+$, we then define the *defect* of α (with respect to κ) to be

$$\operatorname{def}(\alpha) = (\Lambda_\kappa \mid \alpha) - \tfrac{1}{2}(\alpha \mid \alpha).$$

1.5 Multicompositions and multipartitions

A *composition* is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $\lambda_i = 0$ for sufficiently large i. We write $|\lambda|$ for the sum $\lambda_1 + \lambda_2 + \cdots$. When writing compositions, we may omit trailing zeroes and group equal parts together with a superscript. We write $\varnothing$ for the composition $(0, 0, \dots)$. A *partition* is a composition λ for which $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$. We write $\lambda \vdash n$ to mean λ is a partition of n.

Now suppose $l \in \mathbb{N}$. An *l-multicomposition* is an l-tuple $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$ of compositions, which we refer to as the *components* of λ. We write $|\lambda| = |\lambda^{(1)}| + \dots + |\lambda^{(l)}|$, and say that λ is an l-multicomposition of $|\lambda|$. If the components of λ are all partitions, then we say that λ is an *l-multipartition*. We write $\mathscr{P}^l_n$ for the set of l-multipartitions of n. We abuse notation by using $\varnothing$ also for the multipartition $(\varnothing, \dots, \varnothing)$.

If λ and μ are l-multicompositions of n, then we say that λ *dominates* μ, and write $\lambda \trianglerighteq \mu$, if

$$|\lambda^{(1)}| + \dots + |\lambda^{(m-1)}| + \lambda^{(m)}_1 + \dots + \lambda^{(m)}_r \geqslant |\mu^{(1)}| + \dots + |\mu^{(m-1)}| + \mu^{(m)}_1 + \dots + \mu^{(m)}_r$$

for all $1 \leqslant m \leqslant l$ and $r \geqslant 0$.

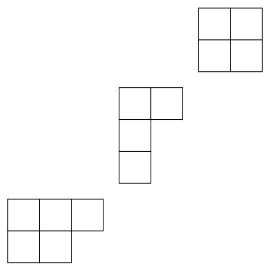
If λ is an l-multicomposition, the *Young diagram* $[\lambda]$ is defined to be the set

$$\left\{ (r, c, m) \in \mathbb{N} \times \mathbb{N} \times \{1, \dots, l\} \;\middle|\; c \leqslant \lambda^{(m)}_r \right\}.$$

We refer to the elements of $[\lambda]$ as the *nodes* of λ. We may also refer to (r, c, m) as the

(r, c)-node of $\lambda^{(m)}$. If $\lambda \in \mathscr{P}^l_n$, a node of λ is *removable* if it can be removed from $[\lambda]$ to leave the Young diagram of a smaller l-multipartition, while a node not in $[\lambda]$ is *addable* if it can be added to $[\lambda]$ to form the Young diagram of an l-multipartition.

We adopt an unusual (but in our view, extremely helpful – see Chapter 2) convention for drawing Young diagrams. We draw the nodes of each component as boxes in the plane, using the English convention, where the first coordinate increases down the page and the second coordinate increases from left to right. Then we arrange the diagrams for the components *in a diagonal line from top right to bottom left*. For example, if $\lambda = \big((2^2), (2, 1^2), (3, 2)\big) \in \mathscr{P}^3_{13}$, then $[\lambda]$ is drawn as follows.



We shall use directions such as left and right with reference to this convention; for example, we shall say that a node (r, c, m) lies to the left of (r', c', m') if either $m > m'$ or ($m = m'$ and $c < c'$). Similarly, we say that (r, c, m) is above, or higher than, (r', c', m') if either $m < m'$ or ($m = m'$ and $r < r'$).

If λ is a partition, the *conjugate partition* λ' is defined by

$$\lambda'_i = \big|\{\, j \geqslant 1 \mid \lambda_j \geqslant i \,\}\big|\,.$$

If λ is an l-multipartition, then the conjugate multipartition λ' is given by

$$\lambda' = (\lambda^{(l)'}, \dots, \lambda^{(1)'}).$$

Observe that with our convention, the Young diagram $[\lambda']$ may be obtained from $[\lambda]$ by reflecting in a diagonal line running from top left to bottom right.

Finally, for some parts of this thesis we will be particularly interested in the level 1 case, and therefore partitions (rather than multipartitions). The following definition is extremely useful in this case:

Definition 1.8. Let $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$. We say that λ is *e-regular* and write $\lambda \vdash_e n$ if it does not have e equal non-zero parts; i.e. we do not have $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+e-1} \neq 0$ for any i. Conversely, if λ does have e equal non-zero parts we call λ *e-singular*.

1.6 Tableaux

If $\lambda \in \mathscr{P}_n^l$, a *λ-tableau* is a bijection $\mathtt{T} : [\lambda] \to \{1, \dots, n\}$. We depict a λ-tableau $\mathtt{T}$ by drawing the Young diagram $[\lambda]$ and filling each box with its image under $\mathtt{T}$. $\mathtt{T}$ is *row-strict* if its entries increase from left to right along each row of the diagram, and *column-strict* if its entries increase down each column. $\mathtt{T}$ is *standard* if it is both row- and column-strict. We write $\mathrm{Std}(\lambda)$ for the set of standard λ-tableaux.

If $\mathtt{T}$ is a λ-tableau, then we define a λ'-tableau $\mathtt{T}'$ by

$$\mathtt{T}'(r, c, m) = \mathtt{T}(c, r, l + 1 - m)$$

for all $(r, c, m) \in [\lambda']$.

We import and modify some notation from [10] and [29]: given a tableau $\mathtt{T}$ and $1 \leqslant i, j \leqslant n$, we write $i \to_{\mathtt{T}} j$ to mean that i and j lie in the same row of the same component, with j to the right of i. We write $i \nearrow_{\mathtt{T}} j$ to mean that i and j lie in the same component of $\mathtt{T}$, with j strictly higher and strictly to the right, and we write $i \Nearrow_{\mathtt{T}} j$ to mean that either $i \nearrow_{\mathtt{T}} j$ or j lies in an earlier component than i. The notations $i \downarrow_{\mathtt{T}} j$, $i \swarrow_{\mathtt{T}} j$ and $i \Swarrow_{\mathtt{T}} j$ are defined similarly.

There are two standard λ-tableaux of particular importance. The tableau $\mathtt{T}_\lambda$ is the standard tableau obtained by writing $1, \dots, n$ in order down successive columns from left to right, while $\mathtt{T}^\lambda$ is the tableau obtained by writing $1, \dots, n$ in order along successive rows from top to bottom. Note that we then have $\mathtt{T}^\lambda = (\mathtt{T}_{\lambda'})'$.

Example. With $\lambda = \big((2^2), (2, 1^2), (3, 2)\big)$ we have

$$\mathtt{T}_\lambda = \begin{array}{ccc} & & \begin{array}{|c|c|}\hline 10 & 12 \\ \hline 11 & 13 \\ \hline \end{array} \\ & \begin{array}{|c|c|}\hline 6 & 9 \\ \hline 7 & \multicolumn{1}{c}{} \\ \cline{1-1} 8 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} & \\ \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} & & \end{array}, \qquad \mathtt{T}^\lambda = \begin{array}{ccc} & & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \\ & \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 7 & \multicolumn{1}{c}{} \\ \cline{1-1} 8 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} & \\ \begin{array}{|c|c|c|}\hline 9 & 10 & 11 \\ \hline 12 & 13 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} & & \end{array}.$$

The symmetric group $\mathfrak{S}_n$ acts naturally on the left on the set of λ-tableaux. Given a λ-tableau $\mathtt{T}$, we define the permutations $w_\mathtt{T}$ and $w^\mathtt{T}$ in $\mathfrak{S}_n$ by

$$w_\mathtt{T}\mathtt{T}_\lambda = \mathtt{T} = w^\mathtt{T}\mathtt{T}^\lambda.$$

We define the *column reading word* of a tableau $\mathtt{T}$ to be the word obtained by reading the entries of $\mathtt{T}$ down successive columns from left to right. Occasionally, the following result comparing fully commutative elements with the tableaux they correspond to will be useful.

Lemma 1.9 [5, Theorem 2.1]. *Let $\lambda \vdash n$. A permutation $w \in \mathfrak{S}_n$ is fully commutative if and only if the column reading word of $w\mathtt{T}_\lambda$ has no decreasing subsequence of length 3.*

Remark. Note that the above definition is independent of the choice of λ, since the column reading word of $w\mathtt{T}_\lambda$ is. In fact, in [5, Theorem 2.1], the result is not given in terms of the reading word of a tableau – instead the condition is that there exist $i < j < k$ with $w(i) > w(j) > w(k)$.

Later we shall also need the following lemma; recall that $\leqslant_\mathrm{L}$ denotes the left order on $\mathfrak{S}_n$.

Lemma 1.10. *Suppose $\lambda \in \mathscr{P}_n^l$ and $\mathtt{S}, \mathtt{T}$ are λ-tableaux with $w_\mathtt{S} \leqslant_\mathrm{L} w_\mathtt{T}$. If $\mathtt{T}$ is standard, then $\mathtt{S}$ is standard.*

Proof. Using induction on $l(w_\mathtt{T}) - l(w_\mathtt{S})$, we may assume $l(w_\mathtt{T}) = l(w_\mathtt{S}) + 1$, which means in particular that $\mathtt{T} = s_i\mathtt{S}$ for some i. Since $\mathtt{T}$ is standard, the only way $\mathtt{S}$ could fail to be standard is if $i+1$ occupies the node immediately below or immediately to the right of i in $\mathtt{T}$. But either possibility means that i occurs before $i+1$ in the column reading word of $\mathtt{T}$. In other words, $w_{\mathtt{T}}^{-1}(i) < w_{\mathtt{T}}^{-1}(i+1)$, but this means that $l(w_{\mathtt{S}}) > l(w_{\mathtt{T}})$, a contradiction. □

Now we introduce a dominance order on tableaux. If $\mathtt{S},\mathtt{T}$ are λ-tableaux, then we write $\mathtt{S} \trianglerighteq \mathtt{T}$ if and only if $w_{\mathtt{S}} \geqslant w_{\mathtt{T}}$ (recall that $\geqslant$ denotes the Bruhat order on $\mathfrak{S}_n$). There should be no ambiguity in using the symbol $\trianglerighteq$ for both the dominance order on multipartitions and the dominance order on tableaux.

There is an alternative description of the dominance order on tableaux which will be very useful. If $\mathtt{T}$ is a λ-tableau and $0 \leqslant m \leqslant n$, we define $\mathtt{T}_{\downarrow m}$ to be the set of nodes of $[\lambda]$ whose entries are less than or equal to m. If $\mathtt{T}$ is row-strict, then $\mathtt{T}_{\downarrow m}$ is the Young diagram of an l-multicomposition of m, which we call $\operatorname{Shape}(\mathtt{T}_{\downarrow m})$. If $\mathtt{T}$ is standard, then $\operatorname{Shape}(\mathtt{T}_{\downarrow m})$ is an l-multipartition of m.

Now we have the following proposition. This is proved in the case $l=1$ in [33, Theorem 3.8] (where it is attributed to Ehresmann and James); in fact, the proof in [33] carries over to the case of arbitrary l without any modification.

Proposition 1.11. *Suppose* $\lambda \in \mathscr{P}_n^l$ *and* $\mathtt{S},\mathtt{T}$ *are row-strict* λ*-tableaux. Then* $\mathtt{S} \trianglelefteq \mathtt{T}$ *if and only if* $\operatorname{Shape}(\mathtt{S}_{\downarrow m}) \trianglelefteq \operatorname{Shape}(\mathtt{T}_{\downarrow m})$ *for* $m=1,\dots,n$.

In Chapter 2, we shall briefly consider a natural analogue of this notion for column-strict tableaux. Suppose $\lambda \in \mathscr{P}_n^l$ and $\mathtt{T}$ is a column-strict λ-tableau; define the diagram $\mathtt{T}_{\downarrow m}$ as above, and define $\mathtt{T}'_{\downarrow m}$ to be the 'conjugate diagram' to $\mathtt{T}_{\downarrow m}$, that is

$$\mathtt{T}'_{\downarrow m} = \left\{(c,r,l+1-k) \;\middle|\; (r,c,k) \in \mathtt{T}_{\downarrow m}\right\}.$$

Then $\mathtt{T}'_{\downarrow m}$ is the Young diagram of an l-multicomposition of m, which we denote $\operatorname{Shape}(\mathtt{T}_{\downarrow m})'$. Now we have the following statement, which can be deduced from Proposition 1.11 by conjugating tableaux.

Proposition 1.12. *Suppose $\lambda \in \mathscr{P}^l_n$ and $\mathtt{S}, \mathtt{T}$ are column-strict λ-tableaux. Then $\mathtt{S} \trianglelefteq \mathtt{T}$ if and only if* $\mathrm{Shape}(\mathtt{S}_{\downarrow m})' \trianglerighteq \mathrm{Shape}(\mathtt{T}_{\downarrow m})'$ *for $m = 1, \ldots, n$.*

1.7 Residues and degrees

In this section we connect the Lie-theoretic set-up above with multipartitions and tableaux. We fix an e-multicharge $\kappa = (\kappa_1, \ldots, \kappa_l)$. We define the *residue* $\mathrm{res}\, A = \mathrm{res}^\kappa A$ of a node $A = (r, c, m) \in \mathbb{N} \times \mathbb{N} \times \{1, \ldots, l\}$ by

$$\mathrm{res}\, A = \kappa_m + (c - r) \pmod{e}.$$

We say that A is an i-*node* if it has residue i. Given $\lambda \in \mathscr{P}^l_n$, we define the *content* of λ to be the element

$$\mathrm{cont}(\lambda) = \sum_{A \in [\lambda]} \alpha_{\mathrm{res}\, A} \in Q^+.$$

We then define the *defect* $\mathrm{def}(\lambda)$ of λ to be $\mathrm{def}(\mathrm{cont}(\lambda))$.

If $\mathtt{T}$ is a λ-tableau, we define its *residue sequence* to be the sequence $i(\mathtt{T}) = (i_1, \ldots, i_n)$, where i_r is the residue of the node $\mathtt{T}^{-1}(r)$, for each r. The residue sequences of the tableaux $\mathtt{T}_\lambda$ and $\mathtt{T}^\lambda$ will be of particular importance, and we set $i_\lambda := i(\mathtt{T}_\lambda)$ and $i^\lambda := i(\mathtt{T}^\lambda)$.

Example. Take $\lambda = \big((2^2), (2, 1^2), (3, 2)\big)$ as in the last example, and suppose $e = 4$ and $\kappa = (1, 2, 0)$. Then the residues of the nodes of λ are given by the following diagram.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 0 & 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1} 0 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 3 & 0 \\ \cline{1-2} \end{array}$$

So we have

$$i_\lambda = (0,3,1,0,2,2,1,0,3,1,0,2,1), \qquad i^\lambda = (1,2,0,1,2,3,1,0,0,1,2,3,0).$$

Now we recall from [10, §3.5] the degree and codegree of a standard tableau. Suppose $\lambda \in \mathscr{P}_n^l$ and A is an i-node of λ. Set

$$d_A(\lambda) := \left| \left\{ \begin{matrix} \text{addable } i\text{-nodes of } \lambda \\ \text{strictly below } A \end{matrix} \right\} \right| - \left| \left\{ \begin{matrix} \text{removable } i\text{-nodes of} \\ \lambda \text{ strictly below } A \end{matrix} \right\} \right|,$$

and

$$d^A(\lambda) := \left| \left\{ \begin{matrix} \text{addable } i\text{-nodes of } \lambda \\ \text{strictly above } A \end{matrix} \right\} \right| - \left| \left\{ \begin{matrix} \text{removable } i\text{-nodes of} \\ \lambda \text{ strictly above } A \end{matrix} \right\} \right|.$$

For $\mathtt{T} \in \mathrm{Std}(\lambda)$ we define the *degree* of $\mathtt{T}$ recursively, setting $\deg(\mathtt{T}) := 0$ when $\mathtt{T}$ is the unique $\varnothing$-tableau. If $\mathtt{T} \in \mathrm{Std}(\lambda)$ with $|\lambda| > 0$, let $A = \mathtt{T}^{-1}(n)$, let $\mathtt{T}_{<n}$ be the tableau obtained by removing this node and set

$$\deg(\mathtt{T}) := d_A(\lambda) + \deg(\mathtt{T}_{<n}).$$

Similarly, define the *codegree* of $\mathtt{T}$ by setting $\mathrm{codeg}(\mathtt{T}) := 0$ if $\mathtt{T}$ is the unique $\varnothing$-tableau, and

$$\mathrm{codeg}(\mathtt{T}) := d^A(\lambda) + \mathrm{codeg}(T_{<n})$$

for $\mathtt{T} \in \mathrm{Std}(\lambda)$ with $|\lambda| > 0$. We note that the definitions of degree and codegree depend on the e-multicharge κ, and therefore we write $\deg^\kappa$ and codeg^κ when we wish to emphasise κ.

Example. Suppose $e = 3$, $\kappa = (1,1)$ and $\mathtt{T}$ is the $((2),(2,1))$-tableau

$$\begin{array}{ll} & \boxed{3}\boxed{4} \\ \begin{array}{l}\boxed{1}\boxed{5} \\ \boxed{2}\end{array} & \end{array}$$

which has residue sequence $i(\mathtt{T}) = (1,0,1,2,2)$. Letting $A = \mathtt{T}^{-1}(5) = (1,2,2)$, we find that $d_A(\lambda) = 1$ and $d^A(\lambda) = -1$. Recursively one finds that for the tableau

$$\mathtt{T}_{<5} = \quad \begin{array}{ccc} & & \boxed{3}\boxed{4} \\ & \begin{array}{c}\boxed{1}\\ \boxed{2}\end{array} & \end{array}$$

we have $\deg(\mathtt{T}_{<5}) = 2$ and $\operatorname{codeg}(\mathtt{T}_{<5}) = 1$, so that $\deg(\mathtt{T}) = 3$ and $\operatorname{codeg}(\mathtt{T}) = 0$.

The degree, codegree of a standard λ-tableau are related to the defect of λ by the following result.

Lemma 1.13 [10, Lemma 3.12]. *Suppose* $\lambda \in \mathscr{P}_n^l$ *and* $\mathtt{T} \in \operatorname{Std}(\lambda)$. *Then*

$$\deg(\mathtt{T}) + \operatorname{codeg}(\mathtt{T}) = \operatorname{def}(\lambda).$$

1.8 Graded algebras

In this thesis we shall be concerned with algebras which are $\mathbb{Z}$-graded. In general, one can define gradings by any group G, but we limit our definitions to the situation we are interested in. Recall that we have fixed a field $\mathbb{F}$ throughout.

Definition 1.14. Let A be an $\mathbb{F}$-algebra. A *($\mathbb{Z}$-)grading* on A is a decomposition $A = \bigoplus_{i\in\mathbb{Z}} A_i$ as vector spaces such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}$. A *($\mathbb{Z}$-)graded algebra* is an algebra with a chosen grading.

Example.

1. Any algebra A is a graded algebra with the trivial grading – i.e. $A = A_0$.
2. The archetypal example of a graded algebra is $A = \mathbb{F}[x]$. Here, we define $A_i = \langle x^i \rangle_{\mathbb{F}}$ for all $i \geqslant 0$ and $A_i = 0$ if $i < 0$. In particular, A is positively graded.
3. More generally, we can take A to be the ring of Laurent polynomials $\mathbb{F}[x, x^{-1}]$. Here we set $A_i = \langle x^i \rangle_{\mathbb{F}}$ for all i to obtain a graded algebra structure.
4. Let $A = M_{n\times n}(\mathbb{F})$ – the ring of n by n matrices over $\mathbb{F}$. Set $A_i = \langle E_{kl} \mid k - l = i \rangle_{\mathbb{F}}$, where E_{kl} is the matrix unit with a 1 in position (k,l) and zeroes everywhere else.

Definition 1.15. If $a \in A_i$, we say that a is *homogeneous of degree* i and write $\deg(a) = i$.

Definition 1.16. Let A be a graded $\mathbb{F}$-algebra and M be an A-module. A *grading* on M is a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ as vector spaces such that $A_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$. A *graded module* is a module with a chosen grading. We write $\underline{M}$ for the module obtained from M by forgetting the grading.

Example. 1. For any graded algebra A, the (left) regular module ${}_A A$ is a graded A-module.

2. $A = \mathbb{F}[x]$ and $M = \mathbb{F}^2$, with x acting as

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Set $M_1 = \langle \binom{1}{0} \rangle_{\mathbb{F}}$, $M_2 = \langle \binom{0}{1} \rangle_{\mathbb{F}}$ and $M_j = 0$ for all $j \neq 1, 2$.

3. $A = M_{n \times n}(\mathbb{F})$ and $M = \mathbb{F}^n$. Set $M_i = \langle e_i \rangle_{\mathbb{F}}$ if $1 \leqslant i \leqslant n$ and $M_i = 0$ otherwise.

Definition 1.17. If M is a graded module and $k \in \mathbb{Z}$, define $M\langle k \rangle$ to be the same module with $(M\langle k \rangle)_i = M_{i-k}$. We call this a *degree shift* by k.

Definition 1.18. Let v be an indeterminate. We define the *graded dimension* of a graded module M to be

$$\operatorname{grdim} M = \sum_{i \in \mathbb{Z}} \dim M_i v^i.$$

Note that $\operatorname{grdim} M\langle k \rangle = v^k \operatorname{grdim} M$.

Definition 1.19. Let A be a graded algebra. We say that a graded A-module is *irreducible*, or *simple*, if it has no non-trivial proper graded submodules.

Next we quote a useful result which tells us that, in terms of the representation theory, we do not lose information by considering grading, but in fact gain some.

Theorem 1.20. [38, Theorems 4.4.4(v) & 9.6.8] and [4, Lemma 2.5.3]. *Let A be a finite dimensional graded algebra. Then:*

1. *If M is an irreducible graded A-module, then $\underline{M}$ is an irreducible A-module.*

2. *If M is an irreducible A-module, then M can be graded; the grading is unique, up to degree shift and automorphism of M.*

Definition 1.21. Suppose A is a graded algebra and let M and N be graded A-modules. A map $\varphi : M \to N$ is a *homogeneous homomorphism of degree r* if φ is a homomorphism of A-modules and $\varphi(M_i) \subseteq N_{i+r}$ for all i.

Example. If φ is the identity map on ungraded modules $\underline{M} \to \underline{M\langle i\rangle}$, then φ lifts naturally to a homogeneous homomorphism of degree i.

Proposition 1.22. *If A is a graded algebra and M is a finitely generated (graded) A-module, then* $\mathrm{Hom}(M, N)$ *can be graded. That is,* $\mathrm{Hom}(M, N)$ *has a basis of homogeneous homomorphisms.*

Proof. Suppose M is generated by homogeneous elements $x_1, x_2, \ldots, x_r$ and $\varphi \in \mathrm{Hom}(M, N)$. Then φ is completely determined by what it maps the generators to. Say

$$\varphi(x_i) = \sum_{j \in \mathbb{Z}} n_{i,j} \quad \text{where } n_{i,j} \in N_j \text{ for each } j \text{ and only finitely many } n_{i,j} \text{ are non-zero.}$$

Now, if we define $\varphi_j : M \to N$ to be the map such that $\varphi_j(x_i) = n_{i,\deg(x_i)+j}$, then φ_j is a homogeneous linear map of degree j. As only finitely many degrees arise in the image of φ, φ is a sum of finitely many homogeneous maps φ_j. To see that each φ_j is a homomorphism, we simply consider degrees; suppose $m \in M$ is homogeneous. Then $m\varphi_j(x_i)$ is homogeneous of degree $\deg(x_i) + j + \deg(m)$. But since φ is a homomorphism, we know that $m\varphi(x_i) = \varphi(mx_i) = \sum mn_{i,j}$ and thus that the constituent of $\varphi(mx_i)$ of degree $\deg(x_i) + j + \deg(m)$ is $mn_{i,\deg(x_i)+j}$. Thus $\varphi_j(mx_i) = mn_{i,\deg(x_i)+j}$ and the proof is complete. □

1.9 KLR algebras

We now give the definition of the algebras which will be our main object of study.

Suppose $\alpha \in Q^+$ has height n, and set

$$I^\alpha = \{ i \in I^n \mid \alpha_{i_1} + \cdots + \alpha_{i_n} = \alpha \}.$$

Now define $\mathscr{H}_\alpha$ to be the unital associative $\mathbb{F}$-algebra with generating set

$$\{ e(i) \mid i \in I^\alpha \} \cup \{ y_1, \ldots, y_n \} \cup \{ \psi_1, \ldots, \psi_{n-1} \}$$

and relations

$$\begin{aligned}
e(i)e(j) &= \delta_{i,j} e(i); \\
\sum_{i \in I^\alpha} e(i) &= 1; \\
y_r e(i) &= e(i) y_r; \\
\psi_r e(i) &= e(s_r i) \psi_r; \\
y_r y_s &= y_s y_r; \\
\psi_r y_s &= y_s \psi_r && \text{if } s \neq r, r+1; \\
\psi_r \psi_s &= \psi_s \psi_r && \text{if } |r-s| > 1; \\
y_r \psi_r e(i) &= (\psi_r y_{r+1} - \delta_{i_r, i_{r+1}}) e(i); \\
y_{r+1} \psi_r e(i) &= (\psi_r y_r + \delta_{i_r, i_{r+1}}) e(i); \\
\psi_r^2 e(i) &= \begin{cases}
0 & \text{if } i_r = i_{r+1}, \\
e(i) & \text{if } i_{r+1} \neq i_r, i_r \pm 1, \\
(y_{r+1} - y_r) e(i) & \text{if } i_r \to i_{r+1}, \\
(y_r - y_{r+1}) e(i) & \text{if } i_r \leftarrow i_{r+1}, \\
(y_{r+1} - y_r)(y_r - y_{r+1}) e(i) & \text{if } i_r \rightleftarrows i_{r+1};
\end{cases}
\end{aligned}$$

$$\psi_r\psi_{r+1}\psi_r e(i) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1}+1)e(i) & \text{if } i_{r+2}=i_r \to i_{r+1}, \\ (\psi_{r+1}\psi_r\psi_{r+1}-1)e(i) & \text{if } i_{r+2}=i_r \leftarrow i_{r+1}, \\ (\psi_{r+1}\psi_r\psi_{r+1}+y_r-2y_{r+1}+y_{r+2})e(i) & \text{if } i_{r+2}=i_r \rightleftarrows i_{r+1}, \\ (\psi_{r+1}\psi_r\psi_{r+1})e(i) & \text{otherwise;} \end{cases}$$

for all admissible r, s, i, j.

The *affine Khovanov–Lauda–Rouquier algebra* or *quiver Hecke algebra* $\mathscr{H}_n$ is defined to be the direct sum $\bigoplus_\alpha \mathscr{H}_\alpha$, where the sum is taken over all $\alpha \in Q^+$ of height n.

Remarks.

1. We use the same notation for the generators ψ_r and y_s for different α; when using these generators, we shall always make it clear which algebra $\mathscr{H}_\alpha$ these generators are taken from.

2. When $e < \infty$, we can modify the above presentation of $\mathscr{H}_\alpha$ to give a presentation for $\mathscr{H}_n$: we take the generating set $\{e(i) \mid i \in I^n\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}$, and replace the relation $\sum_{i \in I^\alpha} e(i) = 1$ with $\sum_{i \in I^n} e(i) = 1$. The generator ψ_r in this presentation is just the sum of the corresponding generators ψ_r of the individual algebras $\mathscr{H}_\alpha$ in the direct sum $\bigoplus_\alpha \mathscr{H}_\alpha$, and similarly for y_s. When $e = \infty$ we cannot do this, since the set I^n is infinite (in fact, $\mathscr{H}_n$ is non-unital in this case).

The following result can easily be checked from the definition of $\mathscr{H}_\alpha$.

Lemma 1.23 [8, Corollary 1]. *There is a $\mathbb{Z}$-grading on the algebra $\mathscr{H}_\alpha$ such that for all admissible r and i,*

$$\deg(e(i)) = 0, \quad \deg(y_r) = 2, \quad \deg(\psi_r e(i)) = -a_{i_r i_{r+1}}.$$

Shift maps

Recall from Section 1.1 that $\mathrm{shift}_k : \mathfrak{S}_m \to \mathfrak{S}_n$ denotes the homomorphism defined by $s_i \mapsto s_{i+k}$. We now define the corresponding maps for the algebras $\mathscr{H}_\alpha$.

Definition 1.24. Suppose $1 \leqslant m \leqslant n$ and $0 \leqslant k \leqslant n-m$, and that $\alpha, \beta \in Q^+$ with α of height n and β of height m. Given $i \in I^\beta$, define $J_i := \{\, j \in I^\alpha \mid j_{s+k} = i_s \text{ for } 1 \leqslant s \leqslant m\}$, and let $e(i)^{+k} = \sum_{j \in J_i} e(j)$. Now define the homomorphism $\mathrm{shift}_k : \mathscr{H}_\beta \to \mathscr{H}_\alpha$ by

$$e(i) \mapsto e(i)^{+k}, \qquad \psi_r e(i) \mapsto \psi_{r+k} e(i)^{+k}, \qquad y_r e(i) \mapsto y_{r+k} e(i)^{+k}.$$

It is easy to check from the definition of $\mathscr{H}_\alpha$ that shift_k is a degree-preserving (non-unital) homomorphism of algebras. Moreover, the PBW-type basis theorem for $\mathscr{H}_\alpha$ in [27, Theorem 2.5] and [39, Theorem 3.7] shows that if $\beta \leqslant \alpha$ then shift_k is injective (obviously shift_k is the zero map if $\beta \nleqslant \alpha$).

Cyclotomic algebras and the Brundan–Kleshchev isomorphism theorem

Given $\alpha \in Q^+$ and an e-multicharge $\kappa = (\kappa_1, \dots, \kappa_l) \in I^l$, we define $\mathscr{H}_\alpha^\kappa$ to be the quotient of $\mathscr{H}_\alpha$ by the *cyclotomic relations*

$$y_1^{(\Lambda_\kappa \mid \alpha_{i_1})} e(i) = 0 \quad \text{for } i \in I^\alpha.$$

The *cyclotomic KLR algebra* $\mathscr{H}_n^\kappa$ is then defined to be the sum $\bigoplus_\alpha \mathscr{H}_\alpha^\kappa$. Here we sum over all $\alpha \in Q^+$ of height n, though in fact only finitely many of the summands will be non-zero, so (even when $e = \infty$) $\mathscr{H}_n^\kappa$ is a unital algebra. Note that the algebra $\mathscr{H}_n^\kappa$ depends only on $\{\kappa_1, \dots, \kappa_l\}$ and not on κ.

Example. Of particular interest to us in some parts of this thesis will be the case when $l = 1$. Here, the cyclotomic relations simplify to

$$y_1 = 0,$$
$$e(i) = 0 \quad \text{for } i_1 \neq 0.$$

Note that the embedding shift_0 passes naturally into the cyclotomic quotients.

Next, we state a stunning result of Brundan and Kleshchev.

Theorem 1.25 [8, Main Theorem]. *If $e = \infty$ or e is not divisible by* $\mathrm{char}(\mathbb{F})$*, and $\kappa_i \equiv a_i$* $(\mathrm{mod}\ e)$*, then $\mathscr{H}_n^\kappa$ is isomorphic to the Ariki–Koike algebra $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$ with q a primitive eth root of unity and parameters $Q = (q^{a_i}, \dots, q^{a_l})$. Similarly, if $e =$* $\mathrm{char}(\mathbb{F})$*, then $\mathscr{H}_n^\kappa$ is isomorphic to a* degenerate Ariki–Koike algebra*; in particular, when $l = 1$, $\mathscr{H}_n^\kappa$ is isomorphic to the group algebra $\mathbb{F}\mathfrak{S}_n$.*

As a consequence, the Ariki–Koike algebras, and in particular the Hecke algebras of type A and (in positive characteristic) $\mathbb{F}\mathfrak{S}_n$ are non-trivially $\mathbb{Z}$-graded. This theorem motivates our choice of notation $\mathscr{H}_n$ for the KLR algebra.

Corollary 1.26. *Suppose $q \neq q' \in \mathbb{F}$ are primitive eth roots of unity. Then $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n) \cong \mathcal{H}_{\mathbb{F},q',Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$ as $\mathbb{F}$-algebras.*

1.10 Specht modules

We now recall the universal graded row and column Specht modules introduced by Kleshchev, Mathas and Ram; we closely follow [29, §§5,7], and refer the reader there for further details.

Fix an e-multicharge κ. Suppose $\lambda \in \mathscr{P}_n^l$, and let $\alpha = \mathrm{cont}(\lambda)$. Say that a node $A = (r, c, m) \in [\lambda]$ is a *column Garnir node* if $(r, c+1, m) \in [\lambda]$. The *column Garnir belt* $\mathbf{B}_A$ is defined to be the set of nodes

$$\mathbf{B}_A = \{(s, c, m) \in [\lambda] \mid s \geqslant r\} \cup \{(s, c+1, m) \in [\lambda] \mid s \leqslant r\}.$$

Suppose $\mathtt{T}_\lambda(r, c, m) = a$ and $\mathtt{T}_\lambda(r, c+1, m) = b$. Then we define the *column Garnir tableau* $\mathtt{G}_A$ to be the λ-tableau which agrees with $\mathtt{T}_\lambda$ outside of $\mathbf{B}_A$ and has entries $a, a+1, \dots, b$ in $\mathbf{B}_A$ in order from top right down to bottom left.

A *column (A-)brick* is a set of e nodes $\{(i, j, m), (i+1, j, m), \dots, (i+e-1, j, m)\} \subseteq \mathbf{B}_A$ such that $\mathrm{res}(i, j, m) = \mathrm{res}\, A$. Thus $\mathbf{B}_A$ is a disjoint union of the bricks it contains along with less than e nodes at the bottom of column c and less than e nodes at the top of column $c+1$, none of which are contained in a brick.

Example. Let $e = 2$ and $\lambda = ((3, 2^2, 1^4), (2, 1))$. Look at the Garnir node $A = (3, 1, 1)$. Then

$$\mathtt{T}_\lambda = \begin{array}{l} \begin{array}{|c|c|c|} \hline 4 & 11 & 14 \\ \hline 5 & 12 \\ \cline{1-2} 6 & 13 \\ \cline{1-2} 7 \\ \cline{1-1} 8 \\ \cline{1-1} 9 \\ \cline{1-1} 10 \\ \cline{1-1} \end{array} \\ \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \end{array} \quad \text{and} \quad \mathtt{G}_A = \begin{array}{l} \begin{array}{|c|c|c|} \hline 4 & 6 & 14 \\ \hline 5 & 7 \\ \cline{1-2} 9 & 8 \\ \cline{1-2} 10 \\ \cline{1-1} 11 \\ \cline{1-1} 12 \\ \cline{1-1} 13 \\ \cline{1-1} \end{array} \\ \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \end{array} .$$

For each $w \in \mathfrak{S}_n$ we fix a *preferred reduced expression* $w = s_{r_1} \dots s_{r_a}$, and define $\psi_w := \psi_{r_1} \dots \psi_{r_a}$. Note that the elements ψ_w may depend on the choice of preferred reduced expressions, since the ψ_r do not satisfy the braid relations. However, if w is fully commutative, ψ_w is uniquely determined.

Let k be the number of bricks in $\mathbf{B}_A$. Label the bricks $B_A^1, B_A^2, \dots, B_A^k$ in $\mathbf{B}_A$ from top right to bottom left.

If $k > 0$, let d be the smallest entry of B_A^1 in $\mathtt{G}_A$. For each $1 \leqslant r < k$, define *brick transpositions*

$$w_A^r := \prod_{a=d+re-e}^{d+re-1} (a, a+e)$$

which transpose the bricks B_A^r and B_A^{r+1}, and the related elements

$$\sigma_A^r := (-1)^e \psi_{w_A^r} \in \mathscr{H}_\alpha \quad \text{and} \quad \tau_A^r := (\sigma_A^r + 1) \in \mathscr{H}_\alpha.$$

Define Gar_A to be the set of all column-strict λ-tableaux obtained from $\mathtt{G}_A$ by brick permutations (i.e. products of elements w_A^r). We recall some basic facts from [29]:

- Every $\mathtt{T} \in \mathrm{Gar}_A \setminus \{\mathtt{G}_A\}$ is standard.

- There exists a unique minimal tableau in Gar_A, which we denote $\mathtt{T}_A$.

- For each $\mathtt{S} \in \mathrm{Gar}_A$, we can write $w_{\mathtt{S}} = u_{\mathtt{S}} w_{\mathtt{T}_A}$, where $l(w_{\mathtt{S}}) = l(u_{\mathtt{S}}) + l(w_{\mathtt{T}_A})$, and $w_{\mathtt{S}}, u_{\mathtt{S}}$ and $w_{\mathtt{T}_A}$ are all fully commutative (by Lemma 1.9, for example). We therefore have elements $\psi_{\mathtt{S}}, \psi_{u_{\mathtt{S}}}$ and $\psi_{\mathtt{T}_A}$ of $\mathscr{H}_\alpha$ with $\psi_{\mathtt{S}} = \psi_{u_{\mathtt{S}}} \psi_{\mathtt{T}_A}$ all independent of choice of reduced expression.

- If $u_{\mathtt{S}} = w_A^{r_1} w_A^{r_2} \dots w_A^{r_a}$ then $\tau_A^{u_{\mathtt{S}}} = \tau_A^{r_1} \tau_A^{r_2} \dots \tau_A^{r_a}$ is also independent of the choice of reduced expression, as $s_{r_1} \dots s_{r_a}$ is fully commutative. If $\mathtt{S} = \mathtt{T}_A$ then by convention we set $\tau_A^{u_{\mathtt{S}}} = 1$.

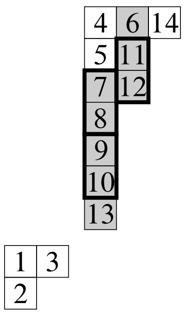
Definition 1.27. Let $A \in [\lambda]$ be a column Garnir node. The *column Garnir element* is

$$\mathtt{g}_A := \sum_{\mathtt{S} \in \mathrm{Gar}_A} \tau_A^{u_{\mathtt{S}}} \psi_{\mathtt{T}_A} \in \mathscr{H}_\alpha.$$

In fact, as defined in [29], the column Garnir element $\mathtt{g}_A$ also involves an idempotent $e(i)$ which depends on λ and makes $\mathtt{g}_A$ homogeneous, but this term can be omitted without affecting the Garnir relation given below.

Example. Continuing the previous example, we have

$\mathtt{T}_A =$



and $\mathrm{Gar}_A = \{\mathtt{T}_A, w_A^2 \mathtt{T}_A, w_A^1 w_A^2 \mathtt{T}_A\}$. Now

$$\begin{aligned}
\mathtt{g}_A &= (1 + \tau_A^2 + \tau_A^1 \tau_A^2) \psi_{\mathtt{T}_A} \\
&= (1 + (\sigma_A^2 + 1) + (\sigma_A^1 + 1)(\sigma_A^2 + 1)) \psi_{\mathtt{T}_A} \\
&= (3 + 2\sigma_A^2 + \sigma_A^1 + \sigma_A^1 \sigma_A^2) \psi_{\mathtt{T}_A} \\
&= (3 + 2\psi_{10}\psi_{11}\psi_9\psi_{10} + \psi_8\psi_9\psi_7\psi_8 + \psi_8\psi_9\psi_7\psi_8\psi_{10}\psi_{11}\psi_9\psi_{10}) \cdot \\
&\quad \psi_6\psi_7\psi_8\psi_9\psi_{12}\psi_{11}\psi_{10}.
\end{aligned}$$

Now define the *column Specht module* $\mathrm{S}_{\lambda|\kappa}$ to be the graded $\mathscr{H}_\alpha$-module generated by the vector z_λ of degree $\operatorname{codeg}(\mathtt{T}_\lambda)$ subject to the following relations:

1. $e(i_\lambda)z_\lambda = z_\lambda$;
2. $y_r z_\lambda = 0$ for all $r = 1, \ldots, n$;
3. $\psi_r z_\lambda = 0$ for all $r = 1, \ldots, n-1$ such that $r \downarrow_{\mathtt{T}_\lambda} r+1$;
4. $\mathsf{g}_A z_\lambda = 0$ for all column Garnir nodes $A \in \lambda$.

We may relax notation and just write S_λ, if the e-multicharge κ is understood. In Chapter 2 we shall mostly consider S_λ as an $\mathscr{H}_n$-module, by setting $\mathscr{H}_\beta \mathrm{S}_\lambda = 0$ for $\beta \neq \alpha$. Thus we have $\mathscr{H}_n$-modules $\mathrm{S}_{\lambda|\kappa}$ for all e-multicharges κ and all $\lambda \in \mathscr{P}^l_n$.

Remark. In the previous example, our Garnir element involved a superfluous term – the term $\sigma^1_A = \psi_8\psi_9\psi_7\psi_8$ acts as zero on the Specht module's generator and can thus be omitted. Similarly, terms arising in the Garnir relations are not, in general, reduced expressions. Work to clarify reduced expressions for Garnir relations can be found in [17].

The main purpose of Chapter 2 will be to study the space of $\mathscr{H}_n$-homomorphisms $\mathrm{S}_\lambda \to \mathrm{S}_\mu$, for $\lambda, \mu \in \mathscr{P}^l_n$. The following result is obvious from the definitions.

Lemma 1.28. *Suppose $\lambda, \mu \in \mathscr{P}^l_n$, and let $\alpha = \operatorname{cont}(\lambda)$. If $\operatorname{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) \neq 0$, then $\operatorname{cont}(\mu) = \alpha$ (and in particular $\operatorname{def}(\lambda) = \operatorname{def}(\mu)$), and $\operatorname{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) = \operatorname{Hom}_{\mathscr{H}_\alpha}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$.*

We shall also need to consider row Specht modules; for these, the definitions are largely obtained by 'conjugating' the definitions for column Specht modules. Fix κ, λ and α as above. Say that a node $A = (r, c, m) \in [\lambda]$ is a *row Garnir node* if $(r+1, c, m) \in [\lambda]$, and define the *row Garnir belt*

$$\mathbf{B}^A = \{(r, d, m) \in [\lambda] \mid d \geqslant c\} \cup \{(r+1, d, m) \in [\lambda] \mid d \leqslant c\}.$$

This belt is used to define a *row Garnir element* g^A. We refer the reader to [29, Definition 5.8] for the definition of this – it is morally the same as the column Garnir element, with conjugation applied throughout. In Chapter 2, where we consider these row Specht modules, we will note the key facts we need to know about g^A.

Now we can define the *row Specht module* S^λ, which is the graded $\mathscr{H}_\alpha$-module generated by the vector z^λ of degree $\deg(\mathtt{T}^\lambda)$ subject to the relations

1. $e(i^\lambda)z^\lambda = z^\lambda$;
2. $y_r z^\lambda = 0$ for all $r = 1, \dots, n$;
3. $\psi_r z^\lambda = 0$ for all $r = 1, \dots, n-1$ such that $r \rightarrow_{\mathtt{T}^\lambda} r+1$;
4. $\mathsf{g}^A z^\lambda = 0$ for all row Garnir nodes $A \in \lambda$.

We define basis elements for the row and column Specht modules as follows. Recall that for each tableau $\mathtt{T} \in \mathrm{Std}(\lambda)$ we have fixed a preferred reduced expression $s_{r_1} \dots s_{r_a}$ for the permutation $w_\mathtt{T}$, and define $\psi_\mathtt{T} := \psi_{r_1} \dots \psi_{r_a}$ and $v_\mathtt{T} := \psi_\mathtt{T} z_\lambda$. Similarly, we fix a preferred reduced expression $s_{t_1} \dots s_{t_b}$ for $w^\mathtt{T}$, and set $\psi^\mathtt{T} := \psi_{t_1} \dots \psi_{t_b}$ and $v^\mathtt{T} := \psi^\mathtt{T} z^\lambda$.

Note that the elements $v_\mathtt{T}$ and $v^\mathtt{T}$ may depend on the choice of preferred reduced expressions, since the ψ_r do not satisfy the braid relations. However, the following results are independent of the choices made.

Lemma 1.29 [29, Propositions 5.14 & 7.14]. *Suppose* $\lambda \in \mathscr{P}_n^l$ *and* $\mathtt{T} \in \mathrm{Std}(\lambda)$. *Then* $\deg(v^\mathtt{T}) = \deg(\mathtt{T})$ *and* $\deg(v_\mathtt{T}) = \mathrm{codeg}(\mathtt{T})$.

Lemma 1.30 [29, Corollaries 6.24 & 7.20]. *Suppose* $\lambda \in \mathscr{P}_n^l$. *Then* $\{v^\mathtt{T} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ *is an* $\mathbb{F}$*-basis for* S^λ*, and* $\{v_\mathtt{T} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ *is an* $\mathbb{F}$*-basis for* S_λ.

Lemma 1.31 [10, Lemma 4.4]. *Suppose* $\lambda \in \mathscr{P}_n^l$. *Then for any* $\mathtt{T} \in \mathrm{Std}(\lambda)$, $e(i)v_\mathtt{T} = \delta_{i,i_\mathtt{T}} v_\mathtt{T}$.

In spite of the dependence of these bases on the choices of preferred reduced expressions, we refer to the bases $\{v^\mathtt{T} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ and $\{v_\mathtt{T} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ as the *standard bases* for S^λ and S_λ respectively.

For the remainder of this section we summarise some basic results about the action of $\mathscr{H}_\alpha$ on S_λ. Many of these results are cited from [10], where they are stated for row

Specht modules. In this thesis we concentrate as far as possible on column Specht modules, so we translate all the results to this setting. Throughout we fix $\lambda \in \mathscr{P}^l_n$, and let $\psi_1, \dots, \psi_{n-1}$ refer to the generators of $\mathscr{H}_\alpha$, where $\alpha = \mathrm{cont}(\lambda)$. Recall that if $\mathtt{S}, \mathtt{T}$ are standard λ-tableaux, then we write $\mathtt{S} \trianglerighteq \mathtt{T}$ to mean that $w_\mathtt{S} \geqslant w_\mathtt{T}$.

Lemma 1.32 [10, Theorem 4.10(i)]. *Suppose* $\mathtt{T} \in \mathrm{Std}(\lambda)$, *and* $s_{j_1} \dots s_{j_r}$ *is any reduced expression for* $w_\mathtt{T}$. *Then* $\psi_{j_1} \dots \psi_{j_r} z_\lambda - v_\mathtt{T}$ *is a linear combination of basis elements* $v_\mathtt{U}$ *for* $\mathtt{U} \triangleleft \mathtt{T}$.

Lemma 1.33 [10, Lemma 4.9]. *Suppose* $\mathtt{T} \in \mathrm{Std}(\lambda)$ *and that* $j-1 \rightarrow_\mathtt{T} j$ *or* $j-1 \downarrow_\mathtt{T} j$. *Then* $\psi_{j-1} v_\mathtt{T}$ *is a linear combination of basis elements* $v_\mathtt{U}$ *for* $\mathtt{U} \triangleleft \mathtt{T}$.

Lemma 1.34 [10, Lemma 4.8]. *Suppose* $\mathtt{T} \in \mathrm{Std}(\lambda)$ *and* $1 \leqslant i \leqslant n$. *Then* $y_i v_\mathtt{T}$ *is a linear combination of basis elements* $v_\mathtt{U}$ *for* $\mathtt{U} \triangleleft \mathtt{T}$.

We'll use Lemmas 1.32 and 1.34 to prove the following similar result, which is suggested but not proved in the proof of [10, Theorem 4.10].

Lemma 1.35. *Suppose* $\mathtt{T} \in \mathrm{Std}(\lambda)$ *and* $j-1 \not\swarrow_\mathtt{T} j$. *Then* $\psi_{j-1} v_\mathtt{T}$ *is a linear combination of basis elements* $v_\mathtt{U}$ *for* $\mathtt{U} \trianglelefteq \mathtt{T}$.

We begin with the following simple observation.

Lemma 1.36. *Suppose* $\mathtt{T} \in \mathrm{Std}(\lambda)$. *Then* $j-1 \not\swarrow_\mathtt{T} j$ *if and only if* $w_\mathtt{T}$ *has a reduced expression beginning with* s_{j-1}.

Proof. Both conditions are equivalent to the condition that $w_\mathtt{T}^{-1}(j-1) > w_\mathtt{T}^{-1}(j)$. □

Proof of Lemma 1.35. By Lemma 1.36, $w_\mathtt{T}$ has a reduced expression of the form $s_{j-1} s_{k_1} \dots s_{k_r}$. Using Lemma 1.32 we have

$$v_\mathtt{T} = \psi_{j-1}\psi_{k_1} \dots \psi_{k_r} z_\lambda + \sum_{\substack{\mathtt{U} \in \mathrm{Std}(\lambda) \\ \mathtt{U} \triangleleft \mathtt{T}}} a_\mathtt{U} v_\mathtt{U}$$

for some $a_{\mathtt{U}} \in \mathbb{F}$. So

$$\psi_{j-1}v_{\mathtt{T}} = \psi_{j-1}^2\psi_{k_1}\dots\psi_{k_r}z_\lambda + \sum_{\substack{\mathtt{U}\in\mathrm{Std}(\lambda),\\ \mathtt{U}\lhd\mathtt{T}}} a_{\mathtt{U}}\psi_{j-1}v_{\mathtt{U}}. \qquad (*)$$

Using the KLR relations (and moving the appropriate idempotent $e(i)$ through), the first term on the right-hand side becomes $g\psi_{k_1}\dots\psi_{k_r}z_\lambda$, where g is a polynomial in $y_1,\dots,y_n$. Now $s_{k_1}\dots s_{k_r}$ is a reduced expression for the standard tableau $\mathtt{S} = s_{j-1}\mathtt{T}$, so by Lemma 1.32 we have

$$\psi_{k_1}\dots\psi_{k_r}z_\lambda = v_{\mathtt{S}} + \sum_{\substack{\mathtt{V}\in\mathrm{Std}(\lambda),\\ \mathtt{V}\lhd\mathtt{S}}} b_{\mathtt{V}}v_{\mathtt{V}}$$

for some $b_{\mathtt{V}} \in \mathbb{F}$. So (since $\mathtt{S} \lhd \mathtt{T}$) the first term on the right-hand side of $(*)$ is a linear combination of terms of the form $gv_{\mathtt{V}}$ for $\mathtt{V} \in \mathrm{Std}(\lambda)$ with $\mathtt{V} \lhd \mathtt{T}$. By Lemma 1.34 this reduces to a linear combination of basis elements $v_{\mathtt{V}}$ for $\mathtt{V} \lhd \mathtt{T}$.

Now consider each of the remaining terms $\psi_{j-1}v_{\mathtt{U}}$ in $(*)$. If $j-1 \swarrow\!\!\swarrow_{\mathtt{U}} j$, then by induction on the Bruhat order $\psi_{j-1}v_{\mathtt{U}}$ is a linear combination of basis elements $v_{\mathtt{V}}$ for $\mathtt{V} \trianglelefteq \mathtt{U} \lhd \mathtt{T}$, so we can ignore any such $\mathtt{U}$. If $j-1 \rightarrow_{\mathtt{U}} j$ or $j-1 \downarrow_{\mathtt{U}} j$, then we apply Lemma 1.33 to get the same conclusion. If $j-1 \nearrow\!\!\nearrow_{\mathtt{U}} j$, let $\mathtt{R}$ be the tableau obtained by swapping $j-1$ and j in $\mathtt{U}$; then a reduced expression for $w_{\mathtt{R}}$ may be obtained by adding s_{j-1} at the start of a reduced expression for $w_{\mathtt{U}}$, and we have $\mathtt{R} \trianglelefteq \mathtt{T}$ by Lemma 1.2. So by Lemma 1.32 again,

$$\psi_{j-1}v_{\mathtt{U}} = v_{\mathtt{R}} + \sum_{\mathtt{W}\lhd\mathtt{R}} c_{\mathtt{W}}v_{\mathtt{W}}$$

for some $c_{\mathtt{W}} \in \mathbb{F}$, and we are done. □

Lemma 1.37. *Suppose* $\lambda \in \mathscr{P}_n^l$, *and* $\mathtt{T} \in \mathrm{Std}(\lambda)$. *Suppose* $j_1,\dots,j_r \in \{1,\dots,n-1\}$, *and that when* $\psi_{j_1}\dots\psi_{j_r}z_\lambda$ *is expressed as a linear combination of standard basis elements,* $v_{\mathtt{T}}$ *appears with non-zero coefficient. Then the expression* $s_{j_1}\dots s_{j_r}$ *has a reduced expression for* $w_{\mathtt{T}}$ *as a subexpression.*

Proof. We proceed by induction on r, with the case $r = 0$ trivial. Let $j = j_1$. Then by assumption $v_{\mathtt{T}}$ appears with non-zero coefficient in $\psi_j v_{\mathtt{S}}$, where $\mathtt{S} \in \mathrm{Std}(\lambda)$ and $v_{\mathtt{S}}$ appears with non-zero coefficient in $\psi_{j_2} \dots \psi_{j_r} z_\lambda$. By induction the expression $s_{j_2} \dots s_{j_r}$ has a subexpression which is a reduced expression for $w_{\mathtt{S}}$, so if $w_{\mathtt{T}} \leqslant w_{\mathtt{S}}$ (i.e. if $\mathtt{T} \trianglelefteq \mathtt{S}$) then we are done. By Lemma 1.33 and Lemma 1.35, this happens if $j \to_{\mathtt{S}} j+1$, $j \downarrow_{\mathtt{S}} j+1$ or $j \swarrow\!\!\swarrow_{\mathtt{S}} j+1$. So we can assume that $j \nearrow\!\!\nearrow_{\mathtt{S}} j+1$. But in this case $w_{\mathtt{T}} = s_j w_{\mathtt{S}}$, with $l(w_{\mathtt{T}}) = l(w_{\mathtt{S}}) + 1$, so $w_{\mathtt{T}}$ has a reduced expression obtained by adding s_j at the start of a reduced expression for $w_{\mathtt{S}}$. So again the result follows by induction. □

1.11 Specht modules for $\mathscr{H}_n^\kappa$ and homomorphisms

Throughout Chapter 2 we consider the Specht module S_λ as a module for the *affine* algebra $\mathscr{H}_\alpha$ (where $\alpha = \mathrm{cont}(\lambda)$) and by extension for the algebra $\mathscr{H}_n$. In fact, it is not hard to show that S_λ is annihilated by the element $y_1^{(\Lambda_\kappa | \alpha_{i_1})} e(i)$ for every i, so that S_λ is a module for the cyclotomic algebra $\mathscr{H}_n^\kappa$ introduced in Section 1.9; we show this with the following lemmas and proposition.

Lemma 1.38. *Suppose* $\mathtt{T}$ *is a* λ*-tableau. Then* $v_{\mathtt{T}}$ *is a linear combination of basis elements* $v_{\mathtt{U}}$ *labelled by tableaux* $\mathtt{U} \triangleleft \mathtt{T}$.

Proof. This proof proceeds almost identically to that of [37, Corollary 5.10], and we omit it here. □

Lemma 1.39. *Suppose* $\mathtt{T} \in \mathrm{Std}(\lambda)$. *Then* $y_1 v_{\mathtt{T}}$ *is a linear combination of basis elements* $v_{\mathtt{U}}$ *labelled by tableaux* $\mathtt{U}$ *in which the number 1 lies strictly to the left of where it lies in* $\mathtt{T}$.

Proof. Suppose that $w_{\mathtt{T}}(k) = 1$ – i.e. for some m the entry 1 appears in the node $(1, 1, m)$ of $\mathtt{T}$, and k appears in the node $(1, 1, m)$ of $\mathtt{T}_\lambda$. Then $w_{\mathtt{T}}$ has a reduced expression $w s_1 s_2 \dots s_{k-1}$, where $w = s_{i_1} \dots s_{i_r}$ for some $i_1, \dots, i_r > 1$. By Lemma 1.32,

$$v_{\mathtt{T}} = \psi_w \psi_1 \psi_2 \dots \psi_{k-1} z_\lambda + \sum_{\mathtt{S} \triangleleft \mathtt{T}} a_{\mathtt{S}} v_{\mathtt{S}} \quad \text{for some } a_{\mathtt{S}} \in \mathbb{F}.$$

Note that the condition $\mathtt{S} \lhd \mathtt{T}$ is equivalent to $w_{\mathtt{S}} \prec w s_1 s_2 \dots s_{k-1}$. It follows that

$$\begin{aligned} y_1 v_{\mathtt{T}} &= \psi_{i_1} \dots \psi_{i_r} y_1 \psi_1 \psi_2 \dots \psi_{k-1} z_\lambda + \sum_{\mathtt{S} \lhd \mathtt{T}} a_{\mathtt{S}} y_1 v_{\mathtt{S}} \\ &= \sum_{i=0}^{k-1} b_i \psi_{i_1} \dots \psi_{i_r} \psi_1 \psi_2 \dots \psi_{i-1} \psi_{i+1} \dots \psi_{k-1} z_\lambda + \sum_{\substack{1 \leqslant i \leqslant k-1 \\ w_{\mathtt{U}} \prec w s_1 \dots s_{i-1} s_{i+1} \dots s_{k-1}}} c_{\mathtt{U},i} v_{\mathtt{U}} \quad \text{for some } c_{\mathtt{U},i} \in \mathbb{F} \end{aligned}$$

for $b_i = 0$ or 1. Now we note that for each i,

$$w_{\mathtt{U}} \prec w s_1 \dots s_{i-1} s_{i+1} \dots s_{k-1} = w s_{i+1} \dots s_{k-1} s_1 \dots s_{i-1}$$

and thus

$$\psi_{i_1} \dots \psi_{i_r} \psi_1 \psi_2 \dots \psi_{i-1} \psi_{i+1} \dots \psi_{k-1} z_\lambda = v_{\mathtt{V}} + \sum_{\mathtt{W} \lhd \mathtt{V}} d_{\mathtt{W}} v_{\mathtt{W}} \quad \text{for some } d_{\mathtt{W}} \in \mathbb{F}$$

where $w_{\mathtt{V}}(i) = 1$, whenever $\psi_{i_1} \dots \psi_{i_r} \psi_1 \psi_2 \dots \psi_{i-1} \psi_{i+1} \dots \psi_{k-1} z_\lambda \neq 0$. For if $\mathtt{V}^{-1}(i) \neq (1, c, m')$ for some c, m', it follows from the presentation of S_λ that $\psi_{i-1} z_\lambda = 0$. Similarly, if $\mathtt{V}^{-1}(i) = (1, c, m')$ for some $c > 1$ there is a Garnir relation $\psi_x \psi_{x+1} \dots \psi_{i-1} z_\lambda = 0$ for some $x \geqslant 1$. So certainly $\mathtt{V}$ is a tableau with $\mathtt{V}^{-1}(i) = (1, 1, m')$ for some $m' < m$, since $i < k$. If $\mathtt{V}$ is non-standard, then the most dominant term will in fact be indexed by an even less dominant tableau, by Lemma 1.38. Since all other basis elements occurring in $y_1 v_{\mathtt{T}}$ are $v_{\mathtt{S}}$ for $\mathtt{S} \lhd \mathtt{V}$, we have that they must also satisfy the desired property, since $\mathrm{Shape}(\mathtt{V}_{\downarrow 1}) \trianglerighteq \mathrm{Shape}(\mathtt{S}_{\downarrow 1})$. □

Proposition 1.40. $y_1^{(\Lambda_\kappa | \alpha_{i_1})} e(i) \mathrm{S}_\lambda = 0$ *for all* i.

Proof. We need to show that $y_1^{x(i)} e(i) \mathrm{S}_\lambda = 0$ for any i, where $x(i) = (\Lambda | \alpha_{i_1})$ is the number of times i_1 appears in the e-multicharge κ – note that this means there are at most $x(i)$ different places where the number 1 can appear in a standard λ-tableau with residue sequence i.

So suppose for a contradiction that there is a standard tableau $\mathtt{T}$ such that $y_1^{x(i)} e(i) v_{\mathtt{T}} \neq 0$. Applying Lemma 1.39 $x(i)$ times, we find a sequence of standard

λ-tableaux $\mathtt{T}, \mathtt{U}_1, \mathtt{U}_2, \ldots, \mathtt{U}_{x(i)}$, which all have residue sequence i and which have the number 1 in successively more leftward positions. Hence there are at least $x(i)+1$ different places where the number 1 can occur in a standard λ-tableau of residue sequence i – a contradiction! □

In Chapter 2 we shall almost entirely be studying the space of $\mathscr{H}_n$-homomorphisms between two Specht modules S_λ and S_μ defined for the same e-multicharge κ, and clearly in this situation $\mathscr{H}_n$-homomorphisms between these two modules are the same as $\mathscr{H}_n^\kappa$-homomorphisms. In view of the Brundan–Kleshchev isomorphism theorem Theorem 1.25, the results of this chapter can therefore be viewed as statements about homomorphisms between Specht modules for (degenerate) Ariki–Koike algebras, and so they generalise the results of Fayers and Lyle for homomorphisms between Specht modules for the symmetric group [18, Theorem 2.1], and of Lyle and Mathas for Hecke algebras of type A [30, Theorem 1.1].

In Chapter 3 we will be interested in the case $l = 1$; in fact we will be studying decomposability of Specht modules for the Hecke algebra of type A. The following basic results for Specht modules in this setting will be useful for our purposes.

Theorem 1.41. *S_λ is decomposable if and only if $S_{\lambda'}$ is.*

Proof. The result follows from [13, Theorem 3.5]. □

Definition 1.42. For each $\lambda \vdash n$ we define a module $\mathrm{D}_\lambda := \mathrm{hd}(\mathrm{S}_\lambda)$.

Theorem 1.43 [12, Theorem 7.6]. *If $e = \infty$, $\{\mathrm{S}_\lambda \mid \lambda \vdash n\}$ is a complete set of pairwise non-isomorphic simple modules for $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$. If $e \neq \infty$, $\{\mathrm{D}_\lambda \mid \lambda \vdash_e n\}$ is a complete set of pairwise non-isomorphic simple modules for $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$.*

Remark. For the Ariki–Koike algebra (of level $l > 1$) the corresponding result is much more complicated. Even in the $e = \infty$ case, the Specht modules are not in general simple.

Theorem 1.44. *If $e \neq 2$, or if λ is 2-regular, then S_λ is indecomposable.*

Proof. The result follows from [14, Corollary 8.7] using a similar argument to that used by James to prove the analogous result for the symmetric group in [24, Theorem 13.13]. □

In view of this last result, we would like to determine which Specht modules S_λ are decomposable when λ is 2-singular and $e = 2$. In Chapter 3 we will focus on (and fully solve) the special case where $\lambda = (a, 1^b)$ for $b \geqslant 2$.

Note that throughout this thesis, GAP [20] has been used for calculations and examples; in particular, we thank Matt Fayers for his GAP packages which have allowed the computations of homomorphisms between Specht modules to take place – this has been a great source for examples and conjectures!

1.12 Decomposition numbers when $l = 1$

In Chapter 4 we will be interested in the graded decomposition numbers for $\mathcal{H} = \mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$. Here we shall recall some basic definitions and results pertaining to decomposition numbers. Note that this framework may be extended to higher levels – i.e. to arbitrary KLR algebras. However, as we do not discuss decomposition numbers (or even simple modules) when $l > 1$, we introduce them only in the simpler context of cyclotomic KLR algebras for $l = 1$. We will let $p = \mathrm{char}(\mathbb{F})$ throughout.

We begin by discussing the classical (ungraded) decomposition numbers.

Definition 1.45. Let $\lambda \vdash n$ and $\mu \vdash_e n$. The *decomposition number* $d_{\lambda\mu} = [\mathrm{S}_\lambda : \mathrm{D}_\mu]$ is defined to be the multiplicity of the simple module D_μ as a composition factor of S_λ. The *decomposition matrix* $D^{e,p} = (d_{\lambda\mu})$ has rows indexed by partitions and columns indexed by e-regular partitions.

Theorem 1.46 [12, Theorem 7.6]. *Let $\lambda \vdash n$ and $\mu \vdash_e n$. Then*

- $d_{\mu\mu} = 1$;
- $d_{\lambda\mu} = 0$ *if* $\lambda \rhd \mu$.

Theorem 1.47 [33, Theorem 6.35]. *There exists a square unitriangular matrix $A_p = (a_{\lambda\mu})_{\lambda,\mu\vdash_e n}$ with entries $a_{\lambda\mu} \in \mathbb{Z}$, whose rows and columns are indexed by e-regular partitions, such that $D^{e,p} = D^{e,0}A_p$. Furthermore, the entries $a_{\lambda\mu}$ of A_p are in fact non-negative integers. A_p is called the* adjustment matrix.

Next, we shall look at these decomposition matrices with the extra structure afforded by the grading on $\mathcal{H}$.

A graded version of the famous Jordan–Hölder Theorem exists and may be proved completely analogously to the classical version. This in turn means we have a well-defined notion of graded composition factors. Furthermore, Theorem 1.20 tells us exactly what these composition factors may be (i.e. ungraded simple modules with their unique gradings, up to degree shifts). Note that when defined as the head of the Specht module S_μ, the simple module D_μ has a canonical grading, not just up to shift. In our graded setting, let α_i be the number of times $\mathrm{D}_\mu\langle i\rangle$ appears as a composition factor of S_λ. Note that $\sum_{i\in\mathbb{Z}}\alpha_i = [\mathrm{S}_\lambda : \mathrm{D}_\mu]$. This motivates the following definition:

Definition 1.48. We define the *graded decomposition number*

$$d_{\lambda\mu}(v) = [\mathrm{S}_\lambda : \mathrm{D}_\mu]_v := \sum_{i\in\mathbb{Z}} \alpha_i v^i.$$

The *graded decomposition matrix $D^{e,p}(v) = (d_{\lambda\mu}(v))$* has rows indexed by partitions and columns indexed by e-regular partitions.

Remark. Setting $v = 1$ in the above definition recovers the decomposition number $[\mathrm{S}_\lambda : \mathrm{D}_\mu]$.

Theorem 1.49 [9, Theorem 5.17]. *There exists a square unitriangular matrix $A_p(v) = (a_{\lambda\mu}(v))_{\lambda,\mu\vdash_e n}$ with entries $a_{\lambda\mu}(v) \in \mathbb{Z}[v, v^{-1}]$ symmetric in v, v^{-1}, whose rows and columns are indexed by e-regular partitions, such that $D^{e,p}(v) = D^{e,0}(v)A_p(v)$. Furthermore, the entries $a_{\lambda\mu}(v)$ of $A_p(v)$ in fact have non-negative coefficients. $A_p(v)$ is called the* graded adjustment matrix.

Chapter 2

Graded column removal

In this chapter we consider the space of homomorphisms between two given Specht modules for the (affine) KLR algebra. However, our results concerning row and column removal will only apply to homomorphisms of a certain kind, which we call *dominated* homomorphisms. But as we shall see in Theorem 2.7, in many cases all homomorphisms between Specht modules are dominated.

In spite of the comments in Section 1.11, we restrict attention entirely to the affine algebra $\mathscr{H}_n$ in this chapter. This is because we occasionally (in particular, in Theorem 2.17) need to compare Specht modules defined for different e-multicharges.

Recall the presentation given in Section 1.10 for Specht modules. For our purposes in this chapter, it will suffice to give $\mathtt{g}_A$ explicitly in a special case which we will use in the proof of Proposition 2.11, and record some useful properties of $\mathtt{g}_A$ which apply in general.

For our special case, we suppose that A is a Garnir node of λ of the form $(1, c, m)$. If a is the entry in node A of $\mathtt{T}_\lambda$ and b is the entry in node $(1, c+1, m)$, then $\mathtt{g}_A = \psi_a \psi_{a+1} \dots \psi_{b-1}$, regardless of the value of e.

Now suppose $A = (r, c, m)$ is an arbitrary Garnir node of λ. Then in $\mathtt{T}_\lambda$ the nodes of $\mathbf{B}_A$ are occupied by the integers $a, a+1, \dots, b$ for some $a < b$. We will only rely on the following properties:

- g_A is a linear combination of products of the form $\psi_{i_1} \dots \psi_{i_d}$ where $a \leqslant i_1, \dots, i_d < b$;
- g_A depends only on e, r, a and the length of the column containing A.

Similarly, we make note of the following facts about *row Garnir element* g^A:

- in $\mathtt{T}^\lambda$ the nodes of $\mathbf{B}^A$ are occupied by the integers $a, a+1, \dots, b$ for some $a < b$;
- g^A is a linear combination of products of the form $\psi_{i_1} \dots \psi_{i_d}$ where $a \leqslant i_1, \dots, i_d < b$;
- g^A depends only on e, c, a and the length of the row containing A.

2.1 λ-dominated tableaux

Suppose $\lambda, \mu \in \mathscr{P}^l_n$ and $\mathtt{T} \in \operatorname{Std}(\mu)$. Given $0 \leqslant j \leqslant n$, we say that $\mathtt{T}$ is *λ-column-dominated on* $1, \dots, j$ if each $i \in \{1, \dots, j\}$ appears at least as far to the left in $\mathtt{T}$ as it does in $\mathtt{T}_\lambda$. We say simply that $\mathtt{T}$ is *λ-column-dominated* if it is λ-column-dominated on $1, \dots, n$. We remind the reader of our unusual convention for drawing Young diagrams, in which a node (r, c, m) lies to the left of (r', c', m') if either $m > m'$ or ($m = m'$ and $c \leqslant c'$).

We write $\operatorname{Std}_\lambda(\mu)$ for the set of λ-column-dominated standard μ-tableaux. It is easy to see that $\operatorname{Std}_\lambda(\mu)$ is non-empty if and only if $\lambda \trianglerighteq \mu$, and that $\operatorname{Std}_\mu(\mu) = \{\mathtt{T}_\mu\}$.

We say $\mathtt{T}$ is *weakly λ-column-dominated on* $1, \dots, j$ if each $i \in \{1, \dots, j\}$ appears in a component at least as far to the left in $\mathtt{T}$ as it does in $\mathtt{T}_\lambda$. We say that $\mathtt{T}$ is weakly λ-column-dominated if it is weakly λ-column-dominated on $1, \dots, n$.

We also introduce row-dominance. Say that $\mathtt{T} \in \operatorname{Std}(\mu)$ is *λ-row-dominated* if each $i \in \{1, \dots, n\}$ appears at least as high in $\mathtt{T}$ as it does in $\mathtt{T}^\lambda$. We write $\operatorname{Std}^\lambda(\mu)$ for the set of λ-row-dominated standard μ-tableaux, which is non-empty if and only if $\lambda \trianglelefteq \mu$.

Example. Let $\lambda = ((3,2),(2,1))$. Then $\mathtt{T}_\lambda = \begin{array}{ll} & \begin{array}{|c|c|c|}\hline 4&6&8\\\hline 5&7\\\cline{1-2}\end{array}\\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} & \end{array}$. The tableau $\begin{array}{ll} & \begin{array}{|c|c|}\hline 4&5\\\hline 7\\\cline{1-1} 8\\\cline{1-1}\end{array}\\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 6\\\cline{1-1}\end{array} & \end{array}$ is λ-column-dominated on $1,\ldots,4$ but not λ-column-dominated, as the entry 5 appears further to the right (second column of the first component) than it does in $\mathtt{T}_\lambda$ (where it appears in the *first* column of the first component). Transposing the entries 5 and 6 yields a λ-column-dominated tableau.

Now, $\mathtt{T}^\lambda = \begin{array}{ll} & \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\\ \begin{array}{|c|c|}\hline 6&7\\\hline 8\\\cline{1-1}\end{array} & \end{array}$ and so the tableau $\begin{array}{ll} & \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 6\\\cline{1-1}\end{array}\\ \begin{array}{|c|c|c|}\hline 5&7&8\\\hline\end{array} & \end{array}$ is λ-row-dominated on $1,\ldots,4$ but not λ-row-dominated, as the entry 5 appears lower (in the first row of the second component) than it does in $\mathtt{T}^\lambda$ (where it appears in the second row of the *first* component). Transposing the entries 5 and 6 yields a λ-row-dominated tableau.

Since we shall primarily be considering column Specht modules, we shall often simply say 'λ-dominated' meaning 'λ-column-dominated'.

We give a helpful alternative characterisation of the λ-dominated and λ-row-dominated properties.

Lemma 2.1. *Suppose* $\lambda, \mu \in \mathscr{P}^l_n$*, and* $\mathtt{S} \in \mathrm{Std}(\mu)$.

1. $\mathtt{S}$ *is λ-column-dominated on* $1,\ldots,j$ *if and only if* $\mathrm{Shape}((\mathtt{T}_\lambda)_{\downarrow m}) \trianglerighteq \mathrm{Shape}(\mathtt{S}_{\downarrow m})$ *for all* $m = 1,\ldots,j$.

2. $\mathtt{S}$ *is λ-row-dominated on* $1,\ldots,j$ *if and only if* $\mathrm{Shape}((\mathtt{T}^\lambda)_{\downarrow m}) \trianglelefteq \mathrm{Shape}(\mathtt{S}_{\downarrow m})$ *for all* $m = 1,\ldots,j$.

Proof. We prove only (2); the proof of (1) is analogous. Suppose first that $\mathtt{S}$ is not λ-row-dominated on $1,\ldots,j$. Choose an entry $m \leqslant j$ which appears strictly lower in $\mathtt{S}$ than in $\mathtt{T}^\lambda$, and let $\tau = \mathrm{Shape}((\mathtt{T}^\lambda)_{\downarrow m})$ and $\sigma = \mathrm{Shape}(\mathtt{S}_{\downarrow m})$. Suppose that m appears in position (r,c,k) in $\mathtt{T}^\lambda$. The construction of $\mathtt{T}^\lambda$ means that the entries $1,\ldots,m-1$ all appear at least as high as m in $\mathtt{T}^\lambda$, and so

$$|\tau^{(1)}| + \cdots + |\tau^{(k-1)}| + \tau_1^{(k)} + \cdots + \tau_r^{(k)} = m.$$

On the other hand, m appears below row r of component k in $\mathtt{S}$, so

$$|\sigma^{(1)}| + \cdots + |\sigma^{(k-1)}| + \sigma_1^{(k)} + \cdots + \sigma_r^{(k)} < m.$$

Hence $\tau \ntrianglelefteq \sigma$.

Conversely, suppose $\mathrm{Shape}(\mathtt{T}^\lambda_{\downarrow m}) \ntrianglelefteq \mathrm{Shape}(\mathtt{S}_{\downarrow m})$ for some $m \leqslant j$; choose such an m, and let $\tau = \mathrm{Shape}(\mathtt{T}^\lambda_{\downarrow m})$ and $\sigma = \mathrm{Shape}(\mathtt{S}_{\downarrow m})$. Since $\tau \ntrianglelefteq \sigma$, there are r, k such that

$$|\tau^{(1)}| + \cdots + |\tau^{(k-1)}| + \tau_1^{(k)} + \cdots + \tau_r^{(k)} > |\sigma^{(1)}| + \cdots + |\sigma^{(k-1)}| + \sigma_1^{(k)} + \cdots + \sigma_r^{(k)}.$$

If we let $d = |\tau^{(1)}| + \cdots + |\tau^{(k-1)}| + \tau_1^{(k)} + \cdots + \tau_r^{(k)}$, then $d \leqslant m$ and the integers $1, \ldots, d$ all appear in row r of component k or higher in $\mathtt{T}^\lambda$. Since $|\sigma^{(1)}| + \cdots + |\sigma^{(k-1)}| + \sigma_1^{(k)} + \cdots + \sigma_r^{(k)} < d$, at least one of the integers $1, \ldots, d$ appears in $\mathtt{S}$ below row r of component k. So there is some $i \leqslant j$ which appears lower in $\mathtt{S}$ than in $\mathtt{T}^\lambda$, so $\mathtt{S}$ is not λ-row-dominated on $1, \ldots, j$. □

Corollary 2.2. *Suppose $\lambda, \mu \in \mathscr{P}^l_n$, and $\mathtt{S}, \mathtt{T} \in \mathrm{Std}(\mu)$.*

1. *If $\mathtt{S}$ is λ-dominated on $1, \ldots, j$ and $\mathtt{S} \trianglerighteq \mathtt{T}$, then $\mathtt{T}$ is λ-dominated on $1, \ldots, j$. In particular, if $\mathtt{S} \in \mathrm{Std}_\lambda(\mu)$ and $\mathtt{S} \trianglerighteq \mathtt{T}$, then $\mathtt{T} \in \mathrm{Std}_\lambda(\mu)$.*

2. *If $\mathtt{S}$ is λ-row-dominated on $1, \ldots, j$ and $\mathtt{S} \trianglelefteq \mathtt{T}$, then $\mathtt{T}$ is λ-row-dominated on $1, \ldots, j$. In particular, if $\mathtt{S} \in \mathrm{Std}^\lambda(\mu)$ and $\mathtt{S} \trianglelefteq \mathtt{T}$, then $\mathtt{T} \in \mathrm{Std}^\lambda(\mu)$.*

Lemma 2.3. *Suppose $\lambda, \mu \in \mathscr{P}^l_n$, and $\mathtt{T}, \mathtt{U} \in \mathrm{Std}(\mu)$ with $\mathtt{U} \trianglelefteq \mathtt{T}$. If $\mathtt{T}$ is weakly λ-dominated on $1, \ldots, j$, then so is $\mathtt{U}$.*

Proof. The proof follows almost identically to that of Corollary 2.2(1), with the exception of m_i needing to denote the sum of sizes of components which are at least as far to the left as the component containing i in $\mathtt{T}_\lambda$. □

2.2 Dominated homomorphisms

Given $\lambda,\mu \in \mathscr{P}^l_n$, we want to consider the space of $\mathscr{H}_n$-homomorphisms $\varphi : \mathrm{S}_\lambda \to \mathrm{S}_\mu$ with the property that $\varphi(z_\lambda)$ lies in the $\mathbb{F}$-span of $\{v_{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}_\lambda(\mu)\}$. But we need to show that this notion is well-defined.

Proposition 2.4. *Suppose $\lambda,\mu \in \mathscr{P}^l_n$. Then the subspace $\langle v_{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}_\lambda(\mu)\rangle_{\mathbb{F}}$ of S_μ is independent of the choice of standard basis elements $v_{\mathtt{S}}$.*

Proof. Let V denote the space $\langle v_{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}_\lambda(\mu)\rangle_{\mathbb{F}}$, and take $\mathtt{T} \in \mathrm{Std}_\lambda(\mu)$. Let $s_{j_1}\dots s_{j_r}$ be a new reduced expression for $w_{\mathtt{T}}$, and let $v'_{\mathtt{T}} = \psi_{j_1}\dots\psi_{j_r} z_\mu$ (where $\psi_1,\dots,\psi_{n-1}$ are taken to lie in $\mathscr{H}_{\mathrm{cont}(\lambda)}$). Let V' be the space obtained from V by replacing $v_{\mathtt{T}}$ with $v'_{\mathtt{T}}$ in the spanning set $\{v_{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}_\lambda(\mu)\}$; it suffices to show that $V = V'$. By Lemma 1.32,

$$v'_{\mathtt{T}} = v_{\mathtt{T}} + \sum_{\mathtt{U} \lhd \mathtt{T}} a_{\mathtt{U}} v_{\mathtt{U}} \quad \text{for some } a_{\mathtt{U}} \in \mathbb{F}.$$

By Corollary 2.2(1), each $v_{\mathtt{U}}$ with $\mathtt{U} \trianglelefteq \mathtt{T}$ lies in V, and so $v'_{\mathtt{T}} \in V$. Hence $V' \subseteq V$; but since the elements $v_{\mathtt{S}}$ are linearly independent, $\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} V' = |\mathrm{Std}_\lambda(\mu)|$. So $V' = V$. □

In view of Proposition 2.4 and an analogue for row-dominated tableaux, the following definition makes sense.

Definition 2.5. Suppose $\lambda,\mu \in \mathscr{P}^l_n$. If $\varphi \in \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda,\mathrm{S}_\mu)$, we say that φ is *(column-) dominated* if $\varphi(z_\lambda) \in \langle v_{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}_\lambda(\mu)\rangle_{\mathbb{F}}$. We write $\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda,\mathrm{S}_\mu)$ for the space of dominated homomorphisms from S_λ to S_μ.

Similarly, if $\chi \in \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\lambda,\mathrm{S}^\mu)$, we say that χ is *row-dominated* if $\chi(z^\lambda) \in \langle v^{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}^\lambda(\mu)\rangle_{\mathbb{F}}$, and we write $\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}^\lambda,\mathrm{S}^\mu)$ for the space of row-dominated homomorphisms from S^λ to S^μ.

Proposition 2.6. *$\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ and $\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$ are graded subspaces of $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ and $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$ respectively. That is, $\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ and $\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$ are spanned by homogeneous homomorphisms.*

Proof. The proof proceeds almost identically to the proof of the fact that $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ is graded; see Proposition 1.22. The important additional observation is that $\langle v_{\mathsf{S}} \mid \mathsf{S} \in \mathrm{Std}_\lambda(\mu)\rangle_{\mathbb{F}}$ is a graded subspace of S_μ by Proposition 2.4. □

The rest of this section is devoted to showing that in certain cases every Specht homomorphism is dominated. Specifically, we shall prove the following.

Theorem 2.7. *Suppose $e \neq 2$ and that $\kappa_1, \dots, \kappa_l$ are distinct. Then $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) = \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$.*

Remark. The hypotheses that $e \neq 2$ and that $\kappa_1, \dots, \kappa_l$ are distinct are equivalent to the condition that $\mathscr{H}_n^\kappa$ has exactly $2l$ isomorphism classes of one-dimensional modules. These hypotheses also appear in Rouquier's work [40, Theorems 6.6, 6.8, 6.13] on 1-faithful quasi-hereditary covers of cyclotomic Hecke algebras. The following small examples show that these hypotheses are essential in Theorem 2.7; in fact, they show that Specht modules labelled by different multipartitions can be isomorphic without these assumptions.

1. Take $e = 2$, $\kappa = (0)$, $\lambda = ((1^2))$ and $\mu = ((2))$. Then there is a non-zero homomorphism $\mathrm{S}_\lambda \to \mathrm{S}_\mu$ defined by $z_\lambda \mapsto z_\mu$, though the tableau $\mathtt{T}_\mu = \boxed{1}\boxed{2}$ is not λ-dominated. So $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) \neq \{0\} = \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$.

2. For any e, take $\kappa = (0, 0)$, $\lambda = (\varnothing, (1))$ and $\mu = ((1), \varnothing)$. Then $z_\lambda \mapsto z_\mu$ again defines a non-zero homomorphism $\mathrm{S}_\lambda \to \mathrm{S}_\mu$, though $\mathtt{T}_\mu$ is not λ-dominated.

The proof of Theorem 2.7 requires several preliminary results. We fix $\lambda, \mu \in \mathscr{P}_n^l$ and an e-multicharge κ of level l throughout. If $\mathrm{cont}(\lambda) \neq \mathrm{cont}(\mu)$, then by Lemma 1.28 $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) = 0$, so that Theorem 2.7 is trivially true. So we assume that $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$. In the results below, $\psi_1, \dots, \psi_{n-1}$ are elements of $\mathscr{H}_{\mathrm{cont}(\lambda)}$.

Lemma 2.8. *Suppose $j \in \{2,\dots,n\}$ with $j-1 \downarrow_{\mathtt{T}_\lambda} j$, and $\mathtt{T} \in \mathrm{Std}(\mu)$ is λ-dominated on $1,\dots,j$. Then $\psi_{j-1}v_{\mathtt{T}}$ is a linear combination of basis elements $v_{\mathtt{U}}$ for standard tableaux $\mathtt{U}$ which are λ-dominated on $1,\dots,j$.*

Proof. If $j-1 \rightarrow_{\mathtt{T}} j$ or $j-1 \downarrow_{\mathtt{T}} j$ or $j-1 \swarrow\!\!\swarrow_{\mathtt{T}} j$, then the result follows from Corollary 2.2(1) together with either Lemma 1.33 or Lemma 1.35. The remaining possibility is that $j-1 \nearrow\!\!\nearrow_{\mathtt{T}} j$. But now if we let $\mathtt{S}$ be the standard tableau $s_{j-1}\mathtt{T}$, then by Lemma 1.32 $\psi_{j-1}v_{\mathtt{T}} = v_{\mathtt{S}} + \sum_{\mathtt{U}\lhd\mathtt{S}} b_{\mathtt{U}}v_{\mathtt{U}}$ for some $b_{\mathtt{U}} \in \mathbb{F}$. Clearly, since $\mathtt{T}$ is λ-dominated on $1,\dots,j$ and $j-1, j$ lie in the same column of $\mathtt{T}_\lambda$, $\mathtt{S}$ is also λ-dominated on $1,\dots,j$. Corollary 2.2(1) completes the proof. □

Proposition 2.9. *Suppose $e \neq 2$, and that $\varphi : \mathrm{S}_\lambda \to \mathrm{S}_\mu$ is a homomorphism, and write*

$$\varphi(z_\lambda) = \sum_{\mathtt{T}\in\mathrm{Std}(\mu)} a_{\mathtt{T}}v_{\mathtt{T}} \quad \textit{for some } a_{\mathtt{T}} \in \mathbb{F}.$$

Suppose $j \in \{2,\dots,n\}$ with $j-1 \downarrow_{\mathtt{T}_\lambda} j$, and that each $\mathtt{T}$ for which $a_{\mathtt{T}} \neq 0$ is λ-dominated on $1,\dots,j-1$. Then each $\mathtt{T}$ for which $a_{\mathtt{T}} \neq 0$ is λ-dominated on $1,\dots,j$.

Proof. The fact that $j-1 \downarrow_{\mathtt{T}_\lambda} j$ means that $\psi_{j-1}z_\lambda = 0$, so we must have $\sum_{\mathtt{T}\in\mathrm{Std}(\mu)} a_{\mathtt{T}}\psi_{j-1}v_{\mathtt{T}} = 0$. Assuming the proposition is false, there is at least one $\mathtt{T}$ which is not λ-dominated on $1,\dots,j$ such that $a_{\mathtt{T}} \neq 0$; choose such a $\mathtt{T}$ which is $\trianglerighteq$-maximal. Since $\mathtt{T}$ is λ-dominated on $1,\dots,j-1$, the entry j lies in a column strictly to the right of $j-1$ in $\mathtt{T}$. We claim that we cannot have $j-1 \rightarrow_{\mathtt{T}} j$. If this is the case, then the residue sequence $i(\mathtt{T})$ satisfies $i(\mathtt{T})_j = i(\mathtt{T})_{j-1}+1$. However, since f is a homomorphism and $v_{\mathtt{T}}$ appears with non-zero coefficient in $\varphi(z_\lambda)$, we must have $i(\mathtt{T}) = i_\lambda$, and the fact that $j-1 \downarrow_{\mathtt{T}_\lambda} j$ means that $(i_\lambda)_j = (i_\lambda)_{j-1}-1$. Since $e \neq 2$, this is a contradiction.

Hence $j-1 \nearrow\!\!\nearrow_{\mathtt{T}} j$, so the tableau $\mathtt{S} := s_{j-1}\mathtt{T}$ is standard, and if we write $\psi_{j-1}v_{\mathtt{T}}$ as a linear combination of standard basis elements, then $v_{\mathtt{S}}$ occurs with coefficient 1. We claim that $v_{\mathtt{S}}$ does not occur in any other $\psi_{j-1}v_{\mathtt{T}'}$ when $a_{\mathtt{T}'} \neq 0$: if $\mathtt{T}'$ is not λ-dominated on $1,\dots,j$, then (defining $\mathtt{S}'$ analogously to $\mathtt{S}$) we have $\psi_{j-1}v_{\mathtt{T}'} = v_{\mathtt{S}'} + \sum_{\mathtt{U}\lhd\mathtt{S}'} c_{\mathtt{U}}v_{\mathtt{U}}$ for some $c_{\mathtt{U}} \in \mathbb{F}$; but the fact that $\mathtt{T} \ntrianglelefteq \mathtt{T}'$ (by our choice of $\mathtt{T}$ being $\trianglerighteq$-maximal) means that $\mathtt{S} \ntrianglelefteq \mathtt{S}'$, so $v_{\mathtt{S}}$ cannot occur. On the other hand, if $\mathtt{T}'$ is λ-dominated on $1, \dots, j$, then the result follows from Lemma 2.8, since $\mathtt{S}$ is not λ-dominated on $1, \dots, j$.

So $v_{\mathtt{S}}$ occurs with non-zero coefficient in $\sum_{\mathtt{T} \in \mathrm{Std}(\mu)} a_{\mathtt{T}} \psi_{j-1} v_{\mathtt{T}}$, a contradiction. □

We now turn our attention to the case where j is in the top row of its component in $\mathtt{T}_\lambda$.

Lemma 2.10. *Suppose $1 \leqslant a \leqslant j \leqslant n$, $j-1 \nearrow_{\mathtt{T}_\lambda} j$ and that the entries a and j appear in the same component of $\mathtt{T}_\lambda$. If $\mathtt{T} \in \mathrm{Std}(\mu)$ is weakly λ-dominated on $1, \dots, j$ then $\psi_a \psi_{a+1} \dots \psi_{j-1} v_{\mathtt{T}}$ is a linear combination of basis elements $v_{\mathtt{U}}$ for standard tableaux $\mathtt{U}$ which are weakly λ-dominated on $1, \dots, j$.*

Proof. We argue by induction on $l(s_a s_{a+1} \dots s_{j-1}) = j - a$. If $j - a = 0$, the result is trivial. So suppose $a < j$, and assume by induction that $\psi_{a+1} \dots \psi_{j-1} v_{\mathtt{T}}$ is a linear combination of basis elements $v_{\mathtt{U}}$ which are weakly λ-dominated on $1, \dots, j$. We want to show that for each $v_{\mathtt{U}}$, $\psi_a v_{\mathtt{U}}$ is a linear combination of basis elements $v_{\mathtt{U}'}$ for standard tableaux $\mathtt{U}'$ which are weakly λ-dominated on $1, \dots, j$.

If $a \rightarrow_{\mathtt{U}} a+1$ or $a \downarrow_{\mathtt{U}} a+1$ or $a \not\nearrow_{\mathtt{U}} a+1$, then the result follows from Lemma 2.3 together with either Lemma 1.33 or Lemma 1.35. The remaining possibility is that $a \nearrow_{\mathtt{U}} a+1$. Let $\mathtt{S}$ be the standard tableau $s_a \mathtt{U}$. Then by Lemma 1.32, $\psi_a v_{\mathtt{U}} = v_{\mathtt{S}} + \sum_{\mathtt{U}' \triangleleft \mathtt{S}} a_{\mathtt{U}'} v_{\mathtt{U}'}$ for some $a_{\mathtt{U}'} \in \mathbb{F}$.

Recalling that $\mathtt{U}$ is weakly λ-dominated on $1, \dots, j$ and that $a, a+1$ are in the same component of $\mathtt{T}_\lambda$, $\mathtt{S}$ is weakly λ-dominated on $1, \dots, j$ and Lemma 2.3 completes the proof. □

Proposition 2.11. *Suppose $\varphi : \mathrm{S}_\lambda \to \mathrm{S}_\mu$ is a homomorphism with*

$$\varphi(z_\lambda) = \sum_{\mathtt{T} \in \mathrm{Std}(\mu)} a_{\mathtt{T}} v_{\mathtt{T}} \quad \textit{for some } a_{\mathtt{T}} \in \mathbb{F}.$$

Suppose $j \in \{2, \dots, n\}$ with either $j-1 \nearrow_{\mathtt{T}_\lambda} j$ or $j-1 \rightarrow_{\mathtt{T}_\lambda} j$, and that each $\mathtt{T}$ for which

$a_{\mathtt{T}} \neq 0$ is λ-dominated on $1, \dots, j-1$. Then each $\mathtt{T}$ for which $a_{\mathtt{T}} \neq 0$ is λ-dominated on $1, \dots, j$.

Proof. The proof follows the same lines as Proposition 2.9. The condition that $j-1 \nearrow_{\mathtt{T}_\lambda} j$ or $j-1 \rightarrow_{\mathtt{T}_\lambda} j$ means that S_λ satisfies a Garnir relation $\psi_a\psi_{a+1}\dots\psi_{j-1}z_\lambda = 0$, where a is the entry immediately to the left of j in $\mathtt{T}_\lambda$; since f is a homomorphism, we therefore have $\sum_{\mathtt{T}\in\mathrm{Std}(\mu)} a_{\mathtt{T}}\psi_a \dots \psi_{j-1}v_{\mathtt{T}} = 0$. Assuming the result is false, there is at least one $\mathtt{T}$ which is not λ-dominated on $1, \dots, j$ such that $a_{\mathtt{T}} \neq 0$; choose such a $\mathtt{T}$ which is $\trianglerighteq$-maximal. Since $\mathtt{T}$ is λ-dominated on $1, \dots, j-1$, but not $1, \dots, j$, we have $j-1 \not\nearrow_{\mathtt{T}} j$. In fact $j-1$ and j are in different components of $\mathtt{T}$: if not, what is the entry immediately to the left of j in $\mathtt{T}$? It must be some $k < j$, since $\mathtt{T}$ is standard, but by assumption k is strictly left of j in $\mathtt{T}_\lambda$ and hasn't moved to the right in $\mathtt{T}$.

Let $\mathtt{S}$ denote the standard tableau $s_a s_{a+1} \dots s_{j-1}\mathtt{T}$. Then $l(w_{\mathtt{S}}) = l(w_{\mathtt{T}}) + j - a$, so that when we write $\psi_a\psi_{a+1}\dots\psi_{j-1}v_{\mathtt{T}}$ as a linear combination of standard basis elements, $v_{\mathtt{S}}$ occurs with coefficient 1. We claim that $v_{\mathtt{S}}$ does not occur with non-zero coefficient in $\psi_a\psi_{a+1}\dots\psi_{j-1}v_{\mathtt{T}'}$ for any other $\mathtt{T}'$ with $a_{\mathtt{T}'} \neq 0$: if $\mathtt{T}'$ is not λ-dominated on $1, \dots, j$, then (defining $\mathtt{S}'$ analogously to $\mathtt{S}$) we have $\psi_a\psi_{a+1}\dots\psi_{j-1}v_{\mathtt{T}'} = v_{\mathtt{S}'} + \sum_{\mathtt{U}\triangleleft\mathtt{S}'} b_{\mathtt{U}}v_{\mathtt{U}}$ for some $b_{\mathtt{U}} \in \mathbb{F}$; but the fact that $\mathtt{T} \not\trianglelefteq \mathtt{T}'$ (by our choice of $\mathtt{T}$) means that $\mathtt{S} \not\trianglelefteq \mathtt{S}'$, so $v_{\mathtt{S}}$ cannot occur. On the other hand, if $\mathtt{T}'$ is λ-dominated on $1, \dots, j$, then the result follows from Lemma 2.10, since $\mathtt{S}$ is not weakly λ-dominated on $1, \dots, j$ as $j-1$ and j are in different components of $\mathtt{T}$.

So $v_{\mathtt{S}}$ occurs with non-zero coefficient in $\sum_{\mathtt{T}\in\mathrm{Std}(\mu)} a_{\mathtt{T}}\psi_a\psi_{a+1}\dots\psi_{j-1}v_{\mathtt{T}}$, a contradiction. □

The last thing we need for the proof of Theorem 2.7 is the following.

Lemma 2.12. *Suppose $\kappa_1, \dots, \kappa_l$ are distinct, and that $\mathtt{T} \in \mathrm{Std}(\mu)$ satisfies $i(\mathtt{T}) = i_\lambda$. If $\mathtt{T}$ is λ-dominated on $1, \dots, j-1$ and j appears in the $(1,1)$-position of its component in $\mathtt{T}_\lambda$, then $\mathtt{T}$ is λ-dominated on $1, \dots, j$.*

Proof. Suppose not; then j appears in $\mathtt{T}$ strictly to the right of where it appears in $\mathtt{T}_\lambda$. This means that j must appear in the $(1,1)$-node of some component of $\mathtt{T}$, since otherwise there would be a smaller entry immediately above or to the left of j, contradicting the assumption that $\mathtt{T}$ is λ-dominated on $1, \dots, j-1$.

So there are $1 \leqslant r < s \leqslant l$ such that $\mathtt{T}_\lambda(1,1,s) = j = \mathtt{T}(1,1,r)$. Hence $\kappa_s = (i_\lambda)_j = i(\mathtt{T})_j = \kappa_r$, contrary to the assumption. □

Proof of Theorem 2.7. Suppose $\varphi : \mathrm{S}_\lambda \to \mathrm{S}_\mu$ is a homomorphism, and write

$$\varphi(z_\lambda) = \sum_{\mathtt{T}\in\mathrm{Std}(\mu)} a_\mathtt{T} v_\mathtt{T} \quad \text{for some } a_\mathtt{T} \in \mathbb{F}.$$

We must show that every $\mathtt{T}$ for which $a_\mathtt{T} \neq 0$ is λ-dominated. In fact, we show by induction on j that every such $\mathtt{T}$ is λ-dominated on $1, \dots, j$, with the case $j = 0$ being vacuous. So suppose $j \geqslant 1$, and assume by induction that $\mathtt{T}$ is λ-dominated on $1, \dots, j-1$. Note that since φ is a homomorphism, we have $i(\mathtt{T}) = i_\lambda$.

If $j = 1$ or j lies in an earlier component of $\mathtt{T}_\lambda$ than $j-1$, then j lies in the $(1,1)$-node of its component in $\mathtt{T}_\lambda$. So by Lemma 2.12 $\mathtt{T}$ is λ-dominated on $1, \dots, j$. The remaining possibilities are that $j > 1$ and that one of

$$j-1 \downarrow_{\mathtt{T}_\lambda} j, \qquad j-1 \rightarrow_{\mathtt{T}_\lambda} j, \qquad j-1 \nearrow_{\mathtt{T}_\lambda} j$$

occurs; these cases are dealt with in Propositions 2.9 and 2.11. □

We immediately see the following interesting result.

Corollary 2.13. *Suppose $e \neq 2$ and that $\kappa_1, \dots, \kappa_l$ are distinct. If $\lambda, \mu \in \mathscr{P}_n^l$ with* $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) \neq \{0\}$*, then $\lambda \trianglerighteq \mu$. Furthermore (since* $\mathrm{Std}_\lambda(\lambda) = \{\mathtt{T}_\lambda\}$*)* $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\lambda)$ *is one-dimensional. In particular, S_λ is indecomposable.*

Remark. Note that if $e = 2$ then S_λ may be decomposable. For example, when $l = 1$ and $\mathrm{char}(\mathbb{F}) \neq 3$, the Specht module $\mathrm{S}_{((5,1^2))}$ is decomposable; this was shown in [24, Example 23.10(iii)] in the case $\mathrm{char}(\mathbb{F}) = 2$, and we show it in Theorem 3.37 in odd characteristic.

Similarly, when $\kappa_i = \kappa_j$ for some $i \neq j$, we can have decomposable Specht modules: take $\kappa = (0,0)$, $e = 3$ and $\lambda = ((3),(3))$. Then $\{\mathrm{id}, \varphi\}$ form a basis for $\mathrm{End}_{\mathscr{H}_6}(\mathrm{S}_\lambda)$, where φ is given by $\varphi(z_\lambda) = \psi_3\psi_2\psi_1\psi_4\psi_3\psi_2\psi_5\psi_4\psi_3 z_\lambda$. It can be checked that $\varphi^2(z_\lambda) = -2\varphi(z_\lambda)$, and thus the endomorphisms $\mathrm{id} + 1/2\varphi$ and $-1/2\varphi$ are idempotents whenever $\mathrm{char}(\mathbb{F}) \neq 2$.

In particular, $\mathrm{S}_{((3),(3))}$ is decomposable if and only if $\mathrm{char}(\mathbb{F}) \neq 2$.

In exactly the same way, we can prove the corresponding result for row Specht modules.

Theorem 2.14. *Suppose $e \neq 2$ and that $\kappa_1, \dots, \kappa_l$ are distinct, and $\lambda, \mu \in \mathscr{P}^l_n$. Then* $\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu) = \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$. *Hence* $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu) \neq \{0\}$ *only if $\lambda \trianglelefteq \mu$,* $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\lambda, \mathrm{S}^\lambda)$ *is one-dimensional, and* S^λ *is indecomposable.*

2.3 Duality for dominated homomorphisms

In this section we consider the relationship between row and column Specht modules, as well as between Specht modules labelled by conjugate multipartitions. These relationships are encapsulated in [29, Theorems 7.25 and 8.5], from which it follows that a (generalised) column-removal theorem for homomorphisms between Specht modules is equivalent to the corresponding row-removal theorem. The main result of this section, which requires considerable additional work, is that the same is true for dominated homomorphisms.

Following [29, §3.2], let $\tau : \mathscr{H}_\alpha \to \mathscr{H}_\alpha$ denote the anti-automorphism which fixes all the generators $e(i), y_r, \psi_s$, and define $\tau : \mathscr{H}_n \to \mathscr{H}_n$ by combining these maps for all α. If $M = \bigoplus_{d\in\mathbb{Z}} M_d$ is a graded $\mathscr{H}_n$-module, let $M^\circledast$ denote the graded module with $M^\circledast_d = \mathrm{Hom}_\mathbb{F}(M_{-d}, \mathbb{F})$ for each d, with $\mathscr{H}_n$-action given by $(hf)m = f(\tau(h)m)$ for $m \in M$, $f \in M^\circledast$ and $h \in \mathscr{H}_n$. Recall from Section 1.8 that for $k \in \mathbb{Z}$, $M\langle k\rangle$ denotes the same module with the grading shifted by k, i.e. $M\langle k\rangle_d = M_{d-k}$. Finally, recall the defect $\mathrm{def}(\lambda)$ of a multipartition from Section 1.7.

Theorem 2.15 [29, Theorem 7.25]. *Suppose* $\lambda \in \mathscr{P}^l_n$. *Then*

$$\mathrm{S}^\lambda \cong (\mathrm{S}_\lambda)^\circledast\langle\mathrm{def}(\lambda)\rangle \qquad \textit{and} \qquad \mathrm{S}_\lambda \cong (\mathrm{S}^\lambda)^\circledast\langle\mathrm{def}(\lambda)\rangle.$$

Now suppose $\lambda, \mu \in \mathscr{P}^l_n$. Applying Theorem 2.15 to both λ and μ gives an isomorphism of graded vector spaces

$$\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\mu, \mathrm{S}^\lambda) \cong \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\circledast_\mu\langle\mathrm{def}(\mu)\rangle, \mathrm{S}^\circledast_\lambda\langle\mathrm{def}(\lambda)\rangle);$$

since by Lemma 1.28 $\mathrm{def}(\lambda) = \mathrm{def}(\mu)$ for any λ and μ with $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\mu, \mathrm{S}^\lambda) \neq \{0\}$, this yields an isomorphism of graded vector spaces

$$\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\mu, \mathrm{S}^\lambda) \cong \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\circledast_\mu, \mathrm{S}^\circledast_\lambda).$$

The anti-automorphism τ is homogeneous of degree zero, so $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\circledast_\mu, \mathrm{S}^\circledast_\lambda)$ is canonically isomorphic as a graded vector space to $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$, and hence we have an isomorphism of graded vector spaces

$$\Theta : \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\mu, \mathrm{S}^\lambda).$$

Our aim is to prove the following.

Proposition 2.16. *Suppose* $\lambda, \mu \in \mathscr{P}^l_n$, *and let* $\Theta : \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) \to \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^\mu, \mathrm{S}^\lambda)$ *be the bijection above. Then* $\Theta(\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)) = \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}^\mu, \mathrm{S}^\lambda)$.

We shall prove Proposition 2.16 below. First we examine the consequences for row and column removal. In order to be able to compare row and column removal, we combine Proposition 2.16 with a result which relates to an analogue of the sign representation of the symmetric group. Following [29, §3.3], let $\mathrm{sgn} : \mathscr{H}_\alpha \to \mathscr{H}_\alpha$ denote the automorphism which maps $e(i) \mapsto e(-i)$, $y_r \mapsto -y_r$ and $\psi_s \mapsto -\psi_s$ for all i, r, s, and define $\mathrm{sgn} : \mathscr{H}_n \to \mathscr{H}_n$ by combining these maps for all α. Given a graded $\mathscr{H}_n$-module M, let M^{sgn} denote the same graded vector space with the action of $\mathscr{H}_n$ twisted by sgn.

Recall that if λ is a multipartition, then λ' denotes the conjugate multipartition to λ, and that if $\mathtt{S} \in \mathrm{Std}(\lambda)$, then $\mathtt{S}' \in \mathrm{Std}(\lambda')$ denotes the conjugate tableau to $\mathtt{S}$. Also define the *conjugate e-multicharge* $\kappa' := (-\kappa_l, \dots, -\kappa_1)$. Now the following is immediate from the construction of row and column Specht modules.

Theorem 2.17 [29, Theorem 8.5]. *Suppose* $\lambda \in \mathscr{P}_n^l$. *Then there is an isomorphism* $(\mathrm{S}^{\lambda|\kappa})^{\mathrm{sgn}} \cong \mathrm{S}_{\lambda'|\kappa'}$ *of* $\mathscr{H}_n$*-modules, given by* $v^{\mathtt{S}} \mapsto v_{\mathtt{S}'}$.

Remark. Theorem 2.17 is one place where it is essential that we consider Specht modules as modules for $\mathscr{H}_n$, rather than its cyclotomic quotients, since the two modules involved are defined relative to different e-multicharges.

Now suppose $\lambda, \mu \in \mathscr{P}_n^l$. Since sgn is a homogeneous automorphism of $\mathscr{H}_n$, we have an equality of graded vector spaces

$$\mathrm{Hom}_{\mathscr{H}_n}((\mathrm{S}^{\mu|\kappa})^{\mathrm{sgn}}, (\mathrm{S}^{\lambda|\kappa})^{\mathrm{sgn}}) = \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^{\mu|\kappa}, \mathrm{S}^{\lambda|\kappa}), \qquad (*)$$

Combining this with Theorem 2.17, we have an isomorphism of graded vector spaces

$$\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_{\mu'|\kappa'}, \mathrm{S}_{\lambda'|\kappa'}) \cong \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}^{\mu|\kappa}, \mathrm{S}^{\lambda|\kappa}). \qquad (\dagger)$$

Applying Theorem 2.15 yields an isomorphism of graded vector spaces

$$\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_{\mu'|\kappa'}, \mathrm{S}_{\lambda'|\kappa'}) \cong \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa}). \qquad (\ddagger)$$

We want to show that the same holds for dominated homomorphisms; this is immediate when $e > 2$ and $\kappa_1, \dots, \kappa_l$ are distinct, by Theorem 2.7. In general, we observe that $(*)$ remains true with Hom replaced by DHom, and the explicit form of the isomorphism in Theorem 2.17 shows that $(\dagger)$ does too, since $\mathtt{S} \in \mathrm{Std}_{\mu'}(\lambda')$ if and only if $\mathtt{S}' \in \mathrm{Std}^{\mu}(\lambda)$. Finally, Proposition 2.16 shows that $(\ddagger)$ remains true for DHom too. So we have the following theorem.

Theorem 2.18. *Suppose $\lambda, \mu \in \mathscr{P}^l_n$. Then there is an isomorphism of graded vector spaces*

$$\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa}) \cong \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\mu'|\kappa'}, \mathrm{S}_{\lambda'|\kappa'}).$$

It remains to prove Proposition 2.16; for the remainder of this section, all Specht modules are defined for the e-multicharge κ.

We begin by recalling how the isomorphism $\mathrm{S}^\lambda \cong \mathrm{S}^{\circledast}_\lambda\langle \mathrm{def}(\lambda)\rangle$ in Theorem 2.15 is constructed. Given the standard basis $\{v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ for S_λ, let $\{f^{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ be the dual basis for $\mathrm{S}^{\circledast}_\lambda$; although the elements $f^{\mathtt{T}}$ in general depend on the choice of the elements $v_{\mathtt{T}}$ (i.e. on the choice of preferred reduced expressions), it is easy to see that the element $f^{\mathtt{T}^\lambda}$ does not: by Lemma 1.32 we know that if $s_{j_1} \dots s_{j_r}$ is a reduced expression for $w_{\mathtt{T}}$ then $\psi_{j_1} \dots \psi_{j_r} z_\lambda$ and $v_{\mathtt{T}}$ only differ by a linear combination of $v_{\mathtt{U}}$ for $\mathtt{U} \lhd \mathtt{T}$. Since $\mathtt{T}^\lambda$ is the $\trianglelefteq$-maximal standard λ-tableau, $v_{\mathtt{T}^\lambda}$ will never appear as such an error term when choosing different reduced expressions for the elements $v_{\mathtt{T}}$, and thus $f^{\mathtt{T}^\lambda}$ is independent of such a choice.

The isomorphism $\theta^\lambda : \mathrm{S}^\lambda \to \mathrm{S}^{\circledast}_\lambda\langle \mathrm{def}(\lambda)\rangle$ in Theorem 2.15 is defined (see [29, Theorem 7.25]) by $\theta(z^\lambda) = f^{\mathtt{T}^\lambda}$.

Lemma 2.19. *Suppose $\lambda \in \mathscr{P}^l_n$, and let $\theta^\lambda : \mathrm{S}^\lambda \to \mathrm{S}^{\circledast}_\lambda\langle \mathrm{def}(\lambda)\rangle$ be the isomorphism constructed above.*

1. *For any $\mathtt{S} \in \mathrm{Std}(\lambda)$ we have $\theta^\lambda(v^{\mathtt{S}}) \in \langle f^{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda),\ \mathtt{T} \trianglerighteq \mathtt{S}\rangle_{\mathbb{F}}$.*

2. *θ^λ maps the space $\langle v^{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}^\mu(\lambda)\rangle_{\mathbb{F}}$ bijectively to the space $\langle f^{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}^\mu(\lambda)\rangle_{\mathbb{F}}$.*

Proof.

1. For each $\mathtt{T} \in \mathrm{Std}(\lambda)$, write $\tau(\psi^{\mathtt{S}})v_{\mathtt{T}} = \sum_{\mathtt{U} \in \mathrm{Std}(\lambda)} a_{\mathtt{TU}} v_{\mathtt{U}}$. Then one can check that the definitions give $\theta^\lambda(v^{\mathtt{S}}) = \sum_{\mathtt{T} \in \mathrm{Std}(\lambda)} a_{\mathtt{TT}^\lambda} f^{\mathtt{T}}$. So it suffices to show that $a_{\mathtt{TT}^\lambda} = 0$ when $\mathtt{T} \not\trianglerighteq \mathtt{S}$. Clearly, to prove this, it is sufficient to show this in the case where $\mathbb{F} = \mathbb{C}$, and so (as in the proof of [29, Theorem 7.25]) we can invoke the proof of [21, Proposition 6.19]; here θ^λ is given in the form $x \mapsto \{x, -\}$, for a bilinear form

$\{\ ,\ \} : S^\lambda \times S_\lambda\langle \mathrm{def}(\lambda)\rangle \to \mathbb{C}$ satisfying $\{v^{\mathtt{S}}, v_{\mathtt{T}}\} = 0$ unless $\mathtt{T} \trianglerighteq \mathtt{S}$, which is exactly what we want.

2. From (1) and Corollary 2.2(2) we have $\theta^\lambda(v^{\mathtt{S}}) \in \langle f^{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}^\mu(\lambda)\rangle_{\mathbb{F}}$ whenever $\mathtt{S} \in \mathrm{Std}^\mu(\lambda)$, so $\theta^\lambda\left(\langle v^{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}^\mu(\lambda)\rangle\right) \subseteq \langle f^{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}^\mu(\lambda)\rangle$. But θ^λ is an isomorphism of vector spaces and

$$\dim_{\mathbb{F}}\langle v^{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}^\mu(\lambda)\rangle_{\mathbb{F}} = \left|\mathrm{Std}^\mu(\lambda)\right| = \dim_{\mathbb{F}}\langle f^{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}^\mu(\lambda)\rangle_{\mathbb{F}},$$

so in fact $\theta^\lambda\left(\langle v^{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}^\mu(\lambda)\rangle_{\mathbb{F}}\right) = \langle f^{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}^\mu(\lambda)\rangle_{\mathbb{F}}$. □

Lemma 2.20. *Suppose $\lambda, \mu \in \mathscr{P}^l_n$. Suppose $\mathtt{S} \in \mathrm{Std}(\lambda)$ and $\mathtt{U}$ is a λ-tableau such that $w_{\mathtt{S}} \geqslant w_{\mathtt{U}}$ and that for every $1 \leqslant i \leqslant n$ the number i appears in $\mathtt{U}$ weakly to the right of where it appears in $\mathtt{T}^\mu$. Then $\mathtt{S} \in \mathrm{Std}^\mu(\lambda)$.*

Proof. Using Lemma 2.1(2) we just need to show that $\mathrm{Shape}(\mathtt{S}_{\downarrow m}) \trianglerighteq \mathrm{Shape}(\mathtt{T}^\mu_{\downarrow m})$ for all m. Let $\mathtt{U}_{\mathrm{c}}$ be the column-strict tableau which is column-equivalent to $\mathtt{U}$. Then by Proposition 1.1 $w_{\mathtt{U}} \geqslant w_{\mathtt{U}_{\mathrm{c}}}$. By Proposition 1.12, we have that $\mathrm{Shape}((\mathtt{U}_{\mathrm{c}})_{\downarrow m})' \trianglerighteq \mathrm{Shape}(\mathtt{S}_{\downarrow m})'$ for all m. Furthermore, the condition that every entry in $\mathtt{U}_{\mathrm{c}}$ lies weakly to the right of where it lies in $\mathtt{T}^\mu$ is equivalent to every entry in $(\mathtt{U}_{\mathrm{c}})'$ lying weakly below where it lies in $(\mathtt{T}^\mu)'$, so we necessarily have that $\mathrm{Shape}((\mathtt{U}_{\mathrm{c}})_{\downarrow m})' \trianglelefteq \mathrm{Shape}(\mathtt{T}^\mu_{\downarrow m})'$ for all m. Reapplying Proposition 1.12, we have $w_{\mathtt{U}_{\mathrm{c}}} \geqslant w_{\mathtt{T}^\mu}$. □

Lemma 2.21. *Suppose $\lambda, \mu \in \mathscr{P}^l_n$, $\mathtt{S} \in \mathrm{Std}(\lambda) \setminus \mathrm{Std}^\mu(\lambda)$ and $\mathtt{T} \in \mathrm{Std}_\lambda(\mu)$. Then when $\psi_{\mathtt{S}} v_{\mathtt{T}}$ is expressed in terms of the standard basis $\{ v_{\mathtt{U}} \mid \mathtt{U} \in \mathrm{Std}(\mu)\}$, the coefficient of $v_{\mathtt{T}^\mu}$ is zero.*

Proof. Suppose to the contrary that $v_{\mathtt{T}^\mu}$ does appear with non-zero coefficient in $\psi_{\mathtt{S}} v_{\mathtt{T}} = \psi_{\mathtt{S}} \psi_{\mathtt{T}} z_\mu$. Let $s_{i_1} \dots s_{i_a}$ and $s_{j_1} \dots s_{j_b}$ be the preferred reduced expressions for $w_{\mathtt{S}}$ and $w_{\mathtt{T}}$ respectively. Then by Lemma 1.37 there is a reduced expression for $w_{\mathtt{T}^\mu}$ occurring as a subexpression of $s_{i_1} \dots s_{i_a} s_{j_1} \dots s_{j_b}$. If we separate this reduced expression into two parts, which occur as subexpressions of $s_{i_1} \dots s_{i_a}$ and $s_{j_1} \dots s_{j_b}$ respectively, and let w, x

denote the corresponding elements of $\mathfrak{S}_n$, then we have

$$w \preccurlyeq w_{\mathtt{S}},\ x \preccurlyeq w_{\mathtt{T}},\ wx = w_{\mathtt{T}^\mu},\ \text{ and }\ l(w)+l(x) = l(w_{\mathtt{T}^\mu}).$$

Putting $\mathtt{V} = x\mathtt{T}_\mu$, we have $\mathtt{V} \in \mathrm{Std}(\mu)$ by Lemma 1.10, and in fact $\mathtt{V} \in \mathrm{Std}_\lambda(\mu)$ (using Corollary 2.2(1), because $w_{\mathtt{V}} \preccurlyeq w_{\mathtt{T}}$ and $\mathtt{T} \in \mathrm{Std}_\lambda(\mu)$). If we let $\mathtt{U} = w\mathtt{T}_\lambda$ then, as functions $[\mu] \to [\lambda]$,

$$\mathtt{U}^{-1}\mathtt{T}^\mu = \mathtt{T}_\lambda^{-1}x\mathtt{T}_\mu = \mathtt{T}_\lambda^{-1}\mathtt{V}.$$

The fact that $\mathtt{V}$ is λ-dominated can be expressed as saying that the map $\mathtt{T}_\lambda^{-1}\mathtt{V} : [\mu] \to [\lambda]$ maps any node of μ to a node weakly to the right. So each entry of $\mathtt{U}$ appears weakly to the right of where it appears in $\mathtt{T}^\mu$, i.e. $\mathtt{U}$ satisfies the hypotheses of Lemma 2.20. Hence by Lemma 2.20 $\mathtt{S} \in \mathrm{Std}^\mu(\lambda)$, contrary to the hypothesis. □

Proof of Proposition 2.16. We shall prove that $\Theta(\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)) \subseteq \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}^\mu, \mathrm{S}^\lambda)$; the same argument with λ and μ interchanged and with row and column Specht modules interchanged proves the opposite containment.

Suppose $\varphi \in \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$, and write $\varphi(z_\lambda) = \sum_{\mathtt{T}\in\mathrm{Std}_\lambda(\mu)} a_{\mathtt{T}} v_{\mathtt{T}}$ for some $a_{\mathtt{T}} \in \mathbb{F}$. Let $\varphi^\circledast : \mathrm{S}_\mu^\circledast \to \mathrm{S}_\lambda^\circledast$ denote the dual map. We want to show that the homomorphism $\Theta(\varphi)$ which corresponds to $\varphi^\circledast$ via Theorem 2.15 is row-dominated, i.e. $\Theta(\varphi)(z^\mu) \in \langle v^{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}^\mu(\lambda)\rangle_{\mathbb{F}}$. By the construction of the isomorphism $\mathrm{S}^\mu \to \mathrm{S}_\mu^\circledast$ and by Lemma 2.19, this is the same as saying that $\varphi^\circledast(f^{\mathtt{T}^\mu}) \in \langle f^{\mathtt{S}} \mid \mathtt{S} \in \mathrm{Std}^\mu(\lambda)\rangle_{\mathbb{F}}$; in other words, $\varphi^\circledast(f^{\mathtt{T}^\mu})(v_{\mathtt{S}}) = 0$ when $\mathtt{S} \in \mathrm{Std}(\lambda) \setminus \mathrm{Std}^\mu(\lambda)$.

The dual map $\varphi^\circledast$ is given by $f \mapsto f \circ \varphi$. In particular, $\varphi^\circledast(f^{\mathtt{T}^\mu}) = f^{\mathtt{T}^\mu} \circ \varphi$, which maps $v_{\mathtt{S}}$ to the coefficient of $v_{\mathtt{T}^\mu}$ in $\varphi(v_{\mathtt{S}}) = \sum_{\mathtt{T}\in\mathrm{Std}_\lambda(\mu)} a_{\mathtt{T}}\psi_{\mathtt{S}} v_{\mathtt{T}}$. By Lemma 2.21 this coefficient is zero when $\mathtt{S} \notin \mathrm{Std}^\mu(\lambda)$, and the result follows. □

2.4 Generalised column removal for multipartitions

Now we come to the main results of the chapter, which give row and column removal theorems for dominated homomorphisms between Specht modules.

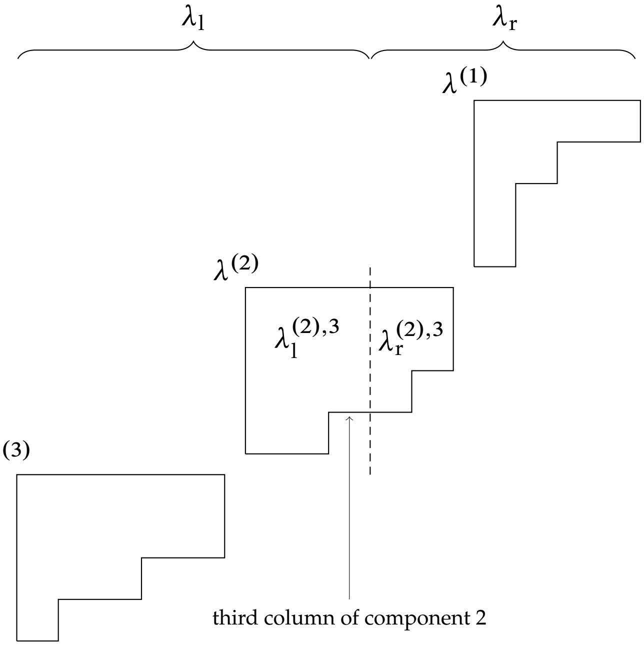
Definition 2.22. Suppose $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(l)}) \in \mathscr{P}_n^l$. For any $1 \leqslant m \leqslant l$ and any $c \geqslant 0$, define $\lambda_{\mathrm{l}}^{(m),c}$ to be the partition consisting of all nodes in the first c columns of $\lambda^{(m)}$, and $\lambda_{\mathrm{r}}^{(m),c}$ the partition consisting of all nodes after the first c columns of $\lambda^{(m)}$. That is,

$$(\lambda_{\mathrm{l}}^{(m),c})_i = \min\left\{\lambda_i^{(m)}, c\right\}, \qquad (\lambda_{\mathrm{r}}^{(m),c})_i = \max\left\{\lambda_i^{(m)} - c, 0\right\} \qquad \text{for all } i \geqslant 1.$$

Now define

$$\lambda_{\mathrm{r}} = \lambda_{\mathrm{r}}(c,m) = (\lambda^{(1)}, \ldots, \lambda^{(m-1)}, \lambda_{\mathrm{r}}^{(m),c}),$$
$$\lambda_{\mathrm{l}} = \lambda_{\mathrm{l}}(c,m) = (\lambda_{\mathrm{l}}^{(m),c}, \lambda^{(m+1)}, \ldots, \lambda^{(l)}).$$

Here is an enlightening pictorial representation of this construction, with $l = 3$, $m = 2$ and $c = 3$.



Now we consider tableaux. Suppose $\lambda_{\mathrm{l}}, \lambda_{\mathrm{r}}$ are as above, and let $n_{\mathrm{l}} = |\lambda_{\mathrm{l}}|$ and $n_{\mathrm{r}} = |\lambda_{\mathrm{r}}|$.

Given a λ_l-tableau $\mathtt{T}_\mathrm{l}$ and a λ_r-tableau $\mathtt{T}_\mathrm{r}$, define $\mathtt{T}_\mathrm{l}\#\mathtt{T}_\mathrm{r}$ to be the λ-tableau obtained by filling in the entries $1, \ldots, n_\mathrm{l}$ as they appear in $\mathtt{T}_\mathrm{l}$, and then filling in the entries $n_\mathrm{l}+1, \ldots, n$ as $1, \ldots, n_\mathrm{r}$, respectively, appear in $\mathtt{T}_\mathrm{r}$. If $\mathtt{T}_\mathrm{l} \in \mathrm{Std}(\lambda_\mathrm{l})$ and $\mathtt{T}_\mathrm{r} \in \mathrm{Std}(\lambda_\mathrm{r})$ then $\mathtt{T}_\mathrm{l}\#\mathtt{T}_\mathrm{r} \in \mathrm{Std}(\lambda)$. Conversely, observe that if $\mathtt{T} \in \mathrm{Std}(\lambda)$ and the integers $1, \ldots, n_\mathrm{l}$ all appear in $\mathtt{T}$ in column c of component m or further to the left, then $\mathtt{T}$ has the form $\mathtt{T}_\mathrm{l}\#\mathtt{T}_\mathrm{r}$ for some $\mathtt{T}_\mathrm{l} \in \mathrm{Std}(\lambda_\mathrm{l})$ and $\mathtt{T}_\mathrm{r} \in \mathrm{Std}(\lambda_\mathrm{r})$. We write $\mathrm{Std}_{\mathrm{lr}}(\lambda)$ for the set of $\mathtt{T} \in \mathrm{Std}(\lambda)$ with this property.

Example. Take $l = 3$ and $\lambda = \big((3), (2^2), (2, 1)\big)$. Taking $m = 2$ and $c = 1$, we get

$$\lambda_\mathrm{l} = \big((1^2), (2, 1)\big)\,, \qquad \lambda_\mathrm{r} = \big((3), (1^2)\big)\,.$$

If we choose

$$\mathtt{T}_\mathrm{l} = \begin{array}{rl} & \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\,, \\ \begin{array}{|c|c|}\hline 2&4\\\hline 5\\\cline{1-1}\end{array} & \end{array} \qquad \mathtt{T}_\mathrm{r} = \begin{array}{rl} & \begin{array}{|c|c|c|}\hline 2&3&5\\\hline\end{array}\,, \\ \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} & \end{array}$$

then we obtain

$$\mathtt{T}_\mathrm{l}\#\mathtt{T}_\mathrm{r} = \begin{array}{rrl} & & \begin{array}{|c|c|c|}\hline 7&8&10\\\hline\end{array}\,. \\ & \begin{array}{|c|c|}\hline 1&6\\\hline 3&9\\\hline\end{array} & \\ \begin{array}{|c|c|}\hline 2&4\\\hline 5\\\cline{1-1}\end{array} & & \end{array}$$

2.5 Simple row and column removal

Theorem 2.23 (Graded Column Removal). *Suppose $\lambda, \mu \in \mathscr{P}^l_n$ and $1 \leqslant m \leqslant l$. Suppose that $\lambda^{(m+1)} = \cdots = \lambda^{(l)} = \mu^{(m+1)} = \cdots = \mu^{(l)} = \varnothing$, and $k := (\lambda^{(m)'})_1 = (\mu^{(m)'})_1$. Let $\lambda_\mathrm{r} = \lambda_\mathrm{r}(1, m)$, $\mu_\mathrm{r} = \mu_\mathrm{r}(1, m)$ and $\kappa_\mathrm{r} = (\kappa_1, \ldots, \kappa_{m-1}, \kappa_m + 1)$. Then*

$$\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa}) \cong \mathrm{DHom}_{\mathscr{H}_{n-k}}(\mathrm{S}_{\lambda_\mathrm{r}|\kappa_\mathrm{r}}, \mathrm{S}_{\mu_\mathrm{r}|\kappa_\mathrm{r}})$$

as graded vector spaces over $\mathbb{F}$.

Remarks.

1. Recalling Theorem 2.7, this result in fact implies that

$$\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa}) \cong \mathrm{Hom}_{\mathscr{H}_{n-k}}(\mathrm{S}_{\lambda_{\mathrm{r}}|\kappa_{\mathrm{r}}}, \mathrm{S}_{\mu_{\mathrm{r}}|\kappa_{\mathrm{r}}})$$

   when $e \neq 2$ and $\kappa_1, \ldots, \kappa_l$ are distinct.

2. In light of the first example after Theorem 2.7, the above result is clearly false if we instead consider all homomorphisms, without any restrictions on e or κ. Indeed, building on this same example, we see that when $e = 2$ and $\kappa = (0)$, $\mathrm{Hom}_{\mathscr{H}_4}(\mathrm{S}_{((2^2))}, \mathrm{S}_{((3,1))}) = \{0\}$ (whereas $\mathrm{Hom}_{\mathscr{H}_2}(\mathrm{S}_{((1^2))}, \mathrm{S}_{((2))}) \neq \{0\}$). To see that $\mathrm{Hom}_{\mathscr{H}_4}(\mathrm{S}_{((2^2))}, \mathrm{S}_{((3,1))}) = \{0\}$, note that the only $(3,1)$-tableaux with residue sequence $i_{((2^2))} = (0,1,1,0)$ are $\mathtt{T}_{((3,1))}$ and $\mathtt{S} = s_2\mathtt{T}_{((3,1))}$, which have degrees $+1$ and -1 respectively. The Garnir element $\psi_3\psi_2$ does not kill $z_{((3,1))}$ (as $\psi_3\psi_2 z_{((3,1))}$ is an element of the standard basis of $\mathrm{S}_{((3,1))}$), and $y_2\psi_2 z_{((3,1))} = -z_{((3,1))} \neq 0$.

Proof. We construct the isomorphism explicitly in the KLR setting. First note that we may assume $\lambda \trianglerighteq \mu$, since otherwise $\mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}}) = \mathrm{Std}_{\lambda}(\mu) = \emptyset$ and the result is immediate. We also observe that $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$ if and only if $\mathrm{cont}(\lambda_{\mathrm{r}}) = \mathrm{cont}(\mu_{\mathrm{r}})$; if these conditions do not hold then the result is trivial since both homomorphism spaces are zero, so we assume $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$, and set $\alpha := \mathrm{cont}(\lambda)$, $\beta := \mathrm{cont}(\lambda_{\mathrm{r}})$.

For this proof we make an assumption about the choice of preferred reduced expressions defining the standard bases for $\mathrm{S}_{\mu_{\mathrm{r}}|\kappa_{\mathrm{r}}}$ and $\mathrm{S}_{\mu|\kappa}$. Given $\mathtt{T} \in \mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})$, we define $\mathtt{T}^+ := \mathtt{T}_{\mu_{\mathrm{l}}}\#\mathtt{T}$, where

$$\mu_{\mathrm{l}} = \mu_{\mathrm{l}}(1,m) = \Big((1^k), \varnothing, \ldots, \varnothing\Big) \in \mathscr{P}_k^{l-m+1}.$$

In other words, $\mathtt{T}^+$ is obtained from $\mathtt{T}$ by increasing each entry by k, adding the column $\begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline k \\ \hline\end{array}$ at the left of component m, and then adding $l-m$ empty components at the end. Now recall the maps (both denoted shift_k) from $\mathfrak{S}_{n-k}$ to $\mathfrak{S}_n$ and from $\mathscr{H}_\beta$ to $\mathscr{H}_\alpha$.

Observe that for $\mathtt{T} \in \mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})$ we have $w_{\mathtt{T}^+} = \mathrm{shift}_k(w_{\mathtt{T}})$. By choosing compatible reduced expressions for $w_{\mathtt{T}^+}$ and $w_{\mathtt{T}}$, we may assume that $\psi_{\mathtt{T}^+} = \mathrm{shift}_k(\psi_{\mathtt{T}})$ as well.

Now let $c = (\lambda_{\mathrm{r}}^{(m)})'_1$. Then the entries $1,\dots,c$ all appear in the first column of component m in $\mathtt{T}_{\lambda_{\mathrm{r}}}$, and hence if $\mathtt{T} \in \mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})$ these entries all appear in the first column of component m of $\mathtt{T}$. In particular, $w_{\mathtt{T}}$ fixes $1,\dots,c$, so $\psi_{\mathtt{T}}$ only involves terms ψ_j for $j > c$; hence $\psi_{\mathtt{T}^+}$ only involves terms ψ_j for $j > k + c$.

Now suppose $\varphi_{\mathrm{r}} \in \mathrm{DHom}_{\mathscr{H}_{n-k}}(\mathrm{S}_{\lambda_{\mathrm{r}}|\kappa_{\mathrm{r}}}, \mathrm{S}_{\mu_{\mathrm{r}}|\kappa_{\mathrm{r}}})$. Then

$$\varphi_{\mathrm{r}}(z_{\lambda_{\mathrm{r}}}) = \sum_{\mathtt{T}\in\mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})} a_{\mathtt{T}} v_{\mathtt{T}} \quad \text{for some } a_{\mathtt{T}} \in \mathbb{F}.$$

We define $\varphi : \mathrm{S}_{\lambda|\kappa} \to \mathrm{S}_{\mu|\kappa}$ by

$$\varphi(z_\lambda) = \sum_{\mathtt{T}\in\mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})} a_{\mathtt{T}} v_{\mathtt{T}^+}.$$

We must verify that this does indeed define a homomorphism, i.e. that $h\varphi(z_\lambda) = 0$ whenever $h \in \mathrm{Ann}(z_\lambda)$. (Here and henceforth we write $\mathrm{Ann}(z_\lambda)$ for the annihilator of z_λ.) Firstly, note that if $\mathtt{T} \in \mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})$ with $a_{\mathtt{T}} \neq 0$, then $\mathtt{T}$ has residue sequence $i_{\lambda_{\mathrm{r}}}$; this implies that $\mathtt{T}^+$ has residue sequence i_λ, so that $e(i_\lambda)\varphi(z_\lambda) = \varphi(z_\lambda)$, as required. For the other relations, observe from the defining relations for the column Specht module that $\mathrm{shift}_k(\mathrm{Ann}(z_{\lambda_{\mathrm{r}}})) \subseteq \mathrm{Ann}(z_\lambda)$ (and similarly for μ_{r} and μ). Now for $k < j \leqslant n$ we have $y_{j-k} \in \mathrm{Ann}(z_{\lambda_{\mathrm{r}}})$, so (since φ_{r} is a homomorphism) $y_{j-k} \sum_{\mathtt{T}} a_{\mathtt{T}} \psi_{\mathtt{T}} \in \mathrm{Ann}(z_{\mu_{\mathrm{r}}})$. Hence

$$\mathrm{Ann}(z_\mu) \ni \mathrm{shift}_k\left(y_{j-k} \sum_{\mathtt{T}} a_{\mathtt{T}} \psi_{\mathtt{T}}\right) = y_j \sum_{\mathtt{T}} a_{\mathtt{T}} \psi_{\mathtt{T}^+},$$

so that $y_j\varphi(z_\lambda) = 0$. A similar statement applies to ψ_j whenever $k < j < n$ with $j \downarrow_{\mathtt{T}_\lambda} j+1$, and to any Garnir element $\mathtt{g}_A$ where A does not lie in the first column of component m.

It remains to check the generators of $\mathrm{Ann}(z_\lambda)$ which do not lie in $\mathrm{shift}_k(\mathrm{Ann}(z_{\lambda_{\mathrm{r}}}))$, i.e. the elements $y_1,\dots,y_k$, $\psi_1,\dots,\psi_{k-1}$ and $\mathtt{g}_A$ for A of the form $(j,1,m)$ with $1 \leqslant j \leqslant c$.

Let h denote any of these elements, and observe that since each $\psi_{\mathtt{T}+}$ is a product of terms ψ_i with $i > k + c$, h commutes with $\psi_{\mathtt{T}+}$ (note that if $h = \mathsf{g}_{(j,1,m)}$, then h only involves terms ψ_i for $i < k + c$). Hence

$$h\varphi(z_\lambda) = h\sum_{\mathtt{T}} a_{\mathtt{T}}\psi_{\mathtt{T}+}z_\mu = \sum_{\mathtt{T}} a_{\mathtt{T}}\psi_{\mathtt{T}+}hz_\mu = 0,$$

since $h \in \mathrm{Ann}(z_\mu)$.

So $\mathrm{Ann}(z_\lambda)\varphi(z_\lambda) = 0$, and φ is a well-defined homomorphism. So we have a map $\Phi : \mathrm{DHom}_{\mathscr{H}_{n-k}}(\mathrm{S}_{\lambda_{\mathrm{r}}|\kappa_{\mathrm{r}}}, \mathrm{S}_{\mu_{\mathrm{r}}|\kappa_{\mathrm{r}}}) \to \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa})$ given by $\varphi_{\mathrm{r}} \mapsto \varphi$, and Φ is obviously linear. To show that Φ is bijective, we construct its inverse. Any $\mathtt{S} \in \mathrm{Std}_\lambda(\mu)$ must have entries $1, \ldots, k$ in order down the first column of its mth component; that is, $\mathtt{S} = \mathtt{T}^+$ for some $\mathtt{T} \in \mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})$. So given $\theta \in \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa})$, we can write

$$\theta(z_\lambda) = \sum_{\mathtt{T}\in\mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})} a_{\mathtt{T}}v_{\mathtt{T}+} \quad \text{for some } a_{\mathtt{T}} \in \mathbb{F}.$$

Applying (a simpler version of) the above argument in reverse, we see that we have a homomorphism $\theta_{\mathrm{r}} : \mathrm{S}_{\lambda_{\mathrm{r}}} \to \mathrm{S}_{\mu_{\mathrm{r}}}$ given by

$$\theta_{\mathrm{r}}(z_{\lambda_{\mathrm{r}}}) = \sum_{\mathtt{T}\in\mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})} a_{\mathtt{T}}v_{\mathtt{T}}.$$

So we get a linear map $\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa}) \to \mathrm{DHom}_{\mathscr{H}_{n-k}}(\mathrm{S}_{\lambda_{\mathrm{r}}|\kappa_{\mathrm{r}}}, \mathrm{S}_{\mu_{\mathrm{r}}|\kappa_{\mathrm{r}}})$ which is a two-sided inverse to Φ, and hence Φ is a bijection.

Finally, to show that we have an isomorphism of graded vector spaces, we show that Φ is homogeneous of degree 0. That is, if $0 \neq \varphi_{\mathrm{r}} \in \mathrm{DHom}_{\mathscr{H}_{n-k}}(\mathrm{S}_{\lambda_{\mathrm{r}}|\kappa_{\mathrm{r}}}, \mathrm{S}_{\mu_{\mathrm{r}}|\kappa_{\mathrm{r}}})$ is homogeneous, then φ is also homogeneous with $\deg(\varphi) = \deg(\varphi_{\mathrm{r}})$. To see this, we write

$$\varphi_{\mathrm{r}}(z_{\lambda_{\mathrm{r}}}) = \sum_{\mathtt{T}\in\mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})} a_{\mathtt{T}}v_{\mathtt{T}} \quad \text{for some } a_{\mathtt{T}} \in \mathbb{F}.$$

Then

$$\varphi(z_\lambda) = \sum_{\mathtt{T}\in\mathrm{Std}_{\lambda_\mathrm{r}}(\mu_\mathrm{r})} a_\mathtt{T} v_{\mathtt{T}^+},$$

and for each T with $a_\mathtt{T} \neq 0$ we have

$$\mathrm{codeg}^\kappa(\mathtt{T}^+) - \mathrm{codeg}^\kappa(\mathtt{T}_\lambda) = \mathrm{codeg}^{\kappa R}(\mathtt{T}) - \mathrm{codeg}^{\kappa R}(\mathtt{T}_{\lambda_\mathrm{r}}) = \deg(\varphi_\mathrm{r}).$$

Hence φ is homogeneous of degree $\deg(\varphi_\mathrm{r})$. □

Now we make corresponding definitions for row removal.

Definition 2.24. Suppose $\lambda \in \mathscr{P}_n^l$. For any $1 \leqslant m \leqslant l$ and any $r \geqslant 0$, define

$$\lambda_\mathrm{t}^{(m),r} = (\lambda_1^{(m)}, \dots, \lambda_r^{(m)}, 0, 0, \dots), \qquad \lambda_\mathrm{b}^{(m),r} = (\lambda_{r+1}^{(m)}, \lambda_{r+2}^{(m)}, \dots).$$

Now let

$$\lambda_\mathrm{t} = \lambda_\mathrm{t}(r,m) = (\lambda^{(1)}, \dots, \lambda^{(m-1)}, \lambda_\mathrm{t}^{(m),r}),$$
$$\lambda_\mathrm{b} = \lambda_\mathrm{b}(r,m) = (\lambda_\mathrm{b}^{(m),r}, \lambda^{(m+1)}, \dots, \lambda^{(l)}),$$

and set $n_\mathrm{t} = |\lambda_\mathrm{t}|$ and $n_\mathrm{b} = |\lambda_\mathrm{b}|$.

Corollary 2.25 (Graded Row Removal). *Suppose $\lambda, \mu \in \mathscr{P}_n^l$ and $1 \leqslant m \leqslant l$. Suppose that $\lambda^{(1)} = \cdots = \lambda^{(m-1)} = \mu^{(1)} = \cdots = \mu^{(m-1)} = \varnothing$, and $k := \lambda_1^{(m)} = \mu_1^{(m)}$. Let $\lambda_\mathrm{b} = \lambda_\mathrm{b}(1,m)$, $\mu_\mathrm{b} = \mu_\mathrm{b}(1,m)$ and $\kappa_\mathrm{b} = (\kappa_m - 1, \kappa_{m+1}, \dots, \kappa_l)$. Then*

$$\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa}) \cong \mathrm{DHom}_{\mathscr{H}_{n-k}}(\mathrm{S}_{\lambda_\mathrm{b}|\kappa_\mathrm{b}}, \mathrm{S}_{\mu_\mathrm{b}|\kappa_\mathrm{b}})$$

as graded vector spaces over $\mathbb{F}$.

Proof.

$\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa}) \cong \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\mu'|\kappa'}, \mathrm{S}_{\lambda'|\kappa'})$ by Theorem 2.18,

$$\cong \mathrm{DHom}_{\mathscr{H}_{n-k}}(\mathrm{S}_{(\mu_{\mathrm{b}})'|(\kappa_{\mathrm{b}})'}, \mathrm{S}_{(\lambda_{\mathrm{b}})'|(\kappa_{\mathrm{b}})'}) \qquad \text{by Theorem 2.23,}$$

$$\cong \mathrm{DHom}_{\mathscr{H}_{n-k}}(\mathrm{S}_{\lambda_{\mathrm{b}}|\kappa_{\mathrm{b}}}, \mathrm{S}_{\mu_{\mathrm{b}}|\kappa_{\mathrm{b}}}) \qquad \text{by Theorem 2.18 again.} \qquad \square$$

Now we prove a 'final column removal' theorem, where we assume that the right-most non-empty columns of λ and μ are in the same place and of the same length.

Theorem 2.26 (Final Column Removal)**.** *Suppose $\lambda, \mu \in \mathscr{P}^l_n$ and $1 \leqslant m \leqslant l$. Suppose $\lambda^{(1)} = \cdots = \lambda^{(m-1)} = \mu^{(1)} = \cdots = \mu^{(m-1)} = \varnothing$, $d := \lambda^{(m)}_1 = \mu^{(m)}_1$ and $k := (\lambda^{(m)})'_d = (\mu^{(m)})'_d$. Let $\lambda_{\mathrm{l}} = \lambda_{\mathrm{l}}(d-1, m)$, $\mu_{\mathrm{l}} = \mu_{\mathrm{l}}(d-1, m)$ and $\kappa_{\mathrm{l}} = (\kappa_m, \ldots, \kappa_l)$. Then*

$$\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa}) \cong \mathrm{DHom}_{\mathscr{H}_{n-k}}(\mathrm{S}_{\lambda_{\mathrm{l}}|\kappa_{\mathrm{l}}}, \mathrm{S}_{\mu_{\mathrm{l}}|\kappa_{\mathrm{l}}})$$

as graded vector spaces over $\mathbb{F}$.

Proof. We first use Corollary 2.25 to remove the first k rows of length d from both $\lambda^{(m)}$ and $\mu^{(m)}$. We obtain

$$\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa}) \cong \mathrm{DHom}_{\mathscr{H}_{n-dk}}(\mathrm{S}_{\lambda_{\mathrm{b}}|\kappa_{\mathrm{b}}}, \mathrm{S}_{\mu_{\mathrm{b}}|\kappa_{\mathrm{b}}})$$

where $\lambda_{\mathrm{b}} = \lambda_{\mathrm{b}}(k, m)$, $\mu_{\mathrm{b}} = \mu_{\mathrm{b}}(k, m)$ and $\kappa_{\mathrm{b}} = (\kappa_m - k, \kappa_2, \ldots, \kappa_l)$. We then use Corollary 2.25 again to add k rows of length $d-1$ to the top of both $\lambda^{(m)}_{\mathrm{b}}$ and $\mu^{(m)}_{\mathrm{b}}$. We obtain

$$\mathrm{DHom}_{\mathscr{H}_{n-dk}}(\mathrm{S}_{\lambda_{\mathrm{b}}|\kappa_{\mathrm{b}}}, \mathrm{S}_{\mu_{\mathrm{b}}|\kappa_{\mathrm{b}}}) \cong \mathrm{DHom}_{\mathscr{H}_{n-k}}(\mathrm{S}_{\lambda_{\mathrm{l}}|\kappa_{\mathrm{l}}}, \mathrm{S}_{\mu_{\mathrm{l}}|\kappa_{\mathrm{l}}}),$$

which gives the result. $\square$

It will be helpful below to be able to give a direct construction for final column removal, as done in the proof of Theorem 2.23 for first column removal. We assume the hypotheses and notation of Theorem 2.26, and for ease of notation we assume that S_λ and S_μ are defined using the e-multicharge κ, while $\mathrm{S}_{\lambda_{\mathrm{l}}}$ and $\mathrm{S}_{\mu_{\mathrm{l}}}$ are defined using κ_{l}. We can also assume that $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu) =: \alpha$, and hence $\mathrm{cont}(\lambda_{\mathrm{l}}) = \mathrm{cont}(\mu_{\mathrm{l}}) =: \beta$.

We identify $\mathfrak{S}_{n-k}$ with its image under the map $\mathrm{shift}_0 : \mathfrak{S}_{n-k} \to \mathfrak{S}_n$, and similarly for $\mathscr{H}_\beta$ and $\mathscr{H}_\alpha$. As in the proof of Theorem 2.23 we make an assumption on preferred reduced expressions: given a standard μ_1-tableau $\mathtt{T}$, we define $\mathtt{T}^+$ to be the standard μ-tableau obtained by adding a column with entries $n-k+1,\dots,n$ at the right of component m; then we have $w_{\mathtt{T}^+} = w_{\mathtt{T}}$, and we assume that our preferred reduced expressions have been chosen in such a way that $\psi_{\mathtt{T}^+} = \psi_{\mathtt{T}}$.

Lemma 2.27. *With the above notation, we have* $\mathrm{Ann}(z_{\lambda_1}) = \mathrm{Ann}(z_\lambda) \cap \mathscr{H}_\beta$.

Proof. It follows directly from the presentation for column Specht modules that $\mathrm{Ann}(z_{\lambda_1}) \subseteq \mathrm{Ann}(z_\lambda) \cap \mathscr{H}_\beta$, so we must show the opposite containment. Consider the $\mathscr{H}_\beta$-submodule $\mathscr{H}_\beta z_\lambda$ of S_λ generated by z_λ. For any $\mathtt{T} \in \mathrm{Std}(\lambda_1)$ we have $v_{\mathtt{T}^+} = \psi_{\mathtt{T}^+} z_\lambda = \psi_{\mathtt{T}} z_\lambda \in \mathscr{H}_\beta z_\lambda$, and the $v_{\mathtt{T}^+}$ are linearly independent, so $\dim_{\mathbb{F}} \mathscr{H}_\beta z_\lambda \geqslant |\mathrm{Std}(\lambda_1)| = \dim_{\mathbb{F}} \mathrm{S}_{\lambda_1}$. So we have

$$\dim_{\mathbb{F}} \mathscr{H}_\beta z_\lambda \geqslant \dim_{\mathbb{F}} \mathscr{H}_\beta z_{\lambda_1},$$

i.e.

$$\dim_{\mathbb{F}} \frac{\mathscr{H}_\beta}{\mathrm{Ann}(z_\lambda) \cap \mathscr{H}_\beta} \geqslant \dim_{\mathbb{F}} \frac{\mathscr{H}_\beta}{\mathrm{Ann}(z_{\lambda_1})},$$

and so $\mathrm{Ann}(z_{\lambda_1}) \supseteq \mathrm{Ann}(z_\lambda) \cap \mathscr{H}_\beta$. □

Now we consider dominated homomorphisms. Observe that since λ and μ have the same last column, $\mathrm{Std}_\lambda(\mu) = \{\mathtt{T}^+ \mid \mathtt{T} \in \mathrm{Std}_{\lambda_1}(\mu_1)\}$. So if $\varphi \in \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$, then we can write

$$\varphi(z_\lambda) = \sum_{\mathtt{T} \in \mathrm{Std}_{\lambda_1}(\mu_1)} a_{\mathtt{T}} v_{\mathtt{T}^+} \quad \text{with } a_{\mathtt{T}} \in \mathbb{F}.$$

Then we can define a homomorphism

$$\varphi^- : \mathrm{S}_{\lambda_1|\kappa_1} \longrightarrow \mathrm{S}_{\mu_1|\kappa_1}$$

$$z_{\lambda_{\mathrm{l}}} \longmapsto \sum_{\mathrm{T}\in\mathrm{Std}_{\lambda_{\mathrm{l}}}(\mu_{\mathrm{l}})} a_{\mathrm{T}} v_{\mathrm{T}}.$$

To see that this definition yields a well-defined homomorphism, we must show that $h\sum_{\mathrm{T}} a_{\mathrm{T}} v_{\mathrm{T}} = 0$ whenever $h \in \mathrm{Ann}(z_{\lambda_{\mathrm{l}}})$. By Lemma 2.27 we have $h \in \mathrm{Ann}(z_\lambda)$, and hence (since φ is a homomorphism) $h\sum_{\mathrm{T}} a_{\mathrm{T}} v_{\mathrm{T}^+} = 0$; in other words, $h\sum_{\mathrm{T}} a_{\mathrm{T}} \psi_{\mathrm{T}} \in \mathrm{Ann}(z_\mu)$. We also have $h\sum_{\mathrm{T}} a_{\mathrm{T}}\psi_{\mathrm{T}} \in \mathscr{H}_\beta$, so by Lemma 2.27 again (with λ replaced by μ) $h\sum_{\mathrm{T}} a_{\mathrm{T}}\psi_{\mathrm{T}} \in \mathrm{Ann}(z_{\mu_{\mathrm{l}}})$, as required.

So we have a map $\varphi \mapsto \varphi^- : \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa}) \to \mathrm{DHom}_{\mathscr{H}_{n-k}}(\mathrm{S}_{\lambda_{\mathrm{l}}|\kappa_{\mathrm{l}}}, \mathrm{S}_{\mu_{\mathrm{l}}|\kappa_{\mathrm{l}}})$. This is obviously an injective map of degree 0, and hence (by Theorem 2.26) a graded isomorphism.

2.6 Generalised column removal

Armed with first column removal and final column removal, we can now consider generalised column removal. In what follows, we fix $c \geqslant 0$ and $1 \leqslant m \leqslant l$, and for any $\nu \in \mathscr{P}^l_n$ we write $\nu_{\mathrm{l}} = \nu_{\mathrm{l}}(c,m)$ and $\nu_{\mathrm{r}} = \nu_{\mathrm{r}}(c,m)$. We suppose $\lambda, \mu \in \mathscr{P}^l_n$, and assume that $|\lambda_{\mathrm{l}}| = |\mu_{\mathrm{l}}| =: n_{\mathrm{l}}$, so that $|\lambda_{\mathrm{r}}| = |\mu_{\mathrm{r}}| = n - n_{\mathrm{l}} =: n_{\mathrm{r}}$. We also assume that $\lambda \trianglerighteq \mu$. This assumption implies that $\lambda_{\mathrm{l}} \trianglerighteq \mu_{\mathrm{l}}$ and $\lambda_{\mathrm{r}} \trianglerighteq \mu_{\mathrm{r}}$, which in particular gives

$$(\lambda^{(m)})'_c \geqslant (\mu^{(m)})'_c \geqslant (\mu^{(m)})'_{c+1}$$

so that it is possible to define a multipartition $\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}} \in \mathscr{P}^l_n$ with $(\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}})_{\mathrm{l}} = \lambda_{\mathrm{l}}$ and $(\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}})_{\mathrm{r}} = \mu_{\mathrm{r}}$.

We write $\kappa_{\mathrm{l}} = (\kappa_m, \dots, \kappa_l)$, $\kappa_{\mathrm{r}} = (\kappa_1, \dots, \kappa_m + c)$, $\mathscr{H}_{\mathrm{l}} = \mathscr{H}_{n_{\mathrm{l}}}$ and $\mathscr{H}_{\mathrm{r}} = \mathscr{H}_{n_{\mathrm{r}}}$. For ease of notation, we will assume throughout the following that the Specht modules S_λ, S_μ and $\mathrm{S}_{\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}}}$ are defined using the e-multicharge κ, while $\mathrm{S}_{\lambda_{\mathrm{l}}}$ and $\mathrm{S}_{\mu_{\mathrm{l}}}$ are defined using κ_{l} and $\mathrm{S}_{\lambda_{\mathrm{r}}}$ and $\mathrm{S}_{\mu_{\mathrm{r}}}$ are defined using κ_{r}.

Suppose $\varphi_{\mathrm{l}} \in \mathrm{DHom}_{\mathscr{H}_{\mathrm{l}}}(\mathrm{S}_{\lambda_{\mathrm{l}}}, \mathrm{S}_{\mu_{\mathrm{l}}})$ and $\varphi_{\mathrm{r}} \in \mathrm{DHom}_{\mathscr{H}_{\mathrm{r}}}(\mathrm{S}_{\lambda_{\mathrm{r}}}, \mathrm{S}_{\mu_{\mathrm{r}}})$, and write

$$\varphi_{\mathrm{l}}(z_{\lambda_{\mathrm{l}}}) = \sum_{\mathsf{S}\in\mathrm{Std}_{\lambda_{\mathrm{l}}}(\mu_{\mathrm{l}})} a_{\mathsf{S}} v_{\mathsf{S}}, \qquad \varphi_{\mathrm{r}}(z_{\lambda_{\mathrm{r}}}) = \sum_{\mathsf{T}\in\mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})} b_{\mathsf{T}} v_{\mathsf{T}}$$

with coefficients $a_{\mathsf{S}}, b_{\mathsf{T}} \in \mathbb{F}$. If there is a homomorphism $\varphi : \mathrm{S}_\lambda \to \mathrm{S}_\mu$ satisfying

$$\varphi(z_\lambda) = \sum_{\substack{\mathsf{S}\in\mathrm{Std}_{\lambda_{\mathrm{l}}}(\mu_{\mathrm{l}}) \\ \mathsf{T}\in\mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})}} a_{\mathsf{S}} b_{\mathsf{T}} v_{\mathsf{S}\#\mathsf{T}},$$

then we write $\varphi = \varphi_{\mathrm{l}}\#\varphi_{\mathrm{r}}$, and say that φ is a *product homomorphism*.

Lemma 2.28. *Every product homomorphism $\mathrm{S}_\lambda \to \mathrm{S}_\mu$ factors through $\mathrm{S}_{\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}}}$.*

Proof. Suppose that $\varphi = \varphi_{\mathrm{l}}\#\varphi_{\mathrm{r}}$ is a product homomorphism, and as above write

$$\varphi_{\mathrm{l}}(z_{\lambda_{\mathrm{l}}}) = \sum_{\mathsf{S}\in\mathrm{Std}_{\lambda_{\mathrm{l}}}(\mu_{\mathrm{l}})} a_{\mathsf{S}} v_{\mathsf{S}}, \qquad \varphi_{\mathrm{r}}(z_{\lambda_{\mathrm{r}}}) = \sum_{\mathsf{T}\in\mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})} b_{\mathsf{T}} v_{\mathsf{T}}.$$

Now define

$$\begin{aligned} \varphi_{\mathrm{l}}\#\,\mathrm{id} : \mathrm{S}_{\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}}} &\longrightarrow \mathrm{S}_\mu, & \mathrm{id}\,\#\varphi_{\mathrm{r}} : \mathrm{S}_\lambda &\longrightarrow \mathrm{S}_{\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}}}, \\ z_{\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}}} &\longmapsto \sum_{\mathsf{S}\in\mathrm{Std}_{\lambda_{\mathrm{l}}}(\mu_{\mathrm{l}})} a_{\mathsf{S}} v_{\mathsf{S}\#\mathsf{T}_{\mu_{\mathrm{r}}}}, & z_\lambda &\longmapsto \sum_{\mathsf{T}\in\mathrm{Std}_{\lambda_{\mathrm{r}}}(\mu_{\mathrm{r}})} b_{\mathsf{T}} v_{\mathsf{T}_{\lambda_{\mathrm{l}}}\#\mathsf{T}}. \end{aligned}$$

Then $\varphi_{\mathrm{l}}\#\,\mathrm{id}$ and $\mathrm{id}\,\#\varphi_{\mathrm{r}}$ are both $\mathscr{H}_n$-homomorphisms; this follows from the direct constructions of column removal homomorphisms in the proof of Theorem 2.23 and following the proof of Theorem 2.26. Clearly $(\varphi_{\mathrm{l}}\#\,\mathrm{id}) \circ (\mathrm{id}\,\#\varphi_{\mathrm{r}}) = \varphi$, so φ factors through $\mathrm{S}_{\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}}}$. □

Proposition 2.29. *Assume the hypotheses (on λ and μ) and notation above. Then every $\varphi \in \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ is a sum of product homomorphisms.*

Proof. We may assume that $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$ (since otherwise there are no non-zero homomorphisms $\mathrm{S}_\lambda \to \mathrm{S}_\mu$). So for this proof we write $\alpha := \mathrm{cont}(\lambda)$ and define shift_0

to be the map from $\mathscr{H}_l$ to $\mathscr{H}_\alpha$ obtained by combining the maps $\mathrm{shift}_0 : \mathscr{H}_\beta \to \mathscr{H}_\alpha$ for all $\beta \in Q^+$ of height n_l; similarly, shift_{n_l} denotes the map from $\mathscr{H}_r$ to $\mathscr{H}_\alpha$ obtained by combining the maps $\mathrm{shift}_{n_l} : \mathscr{H}_\beta \to \mathscr{H}_\alpha$ for all $\beta \in Q^+$ of height n_r.

For this proof we make an assumption about the choice of preferred reduced expressions similar to that in the proof of Theorem 2.26. Specifically, we assume that these expressions have been chosen in such a way that if $\mathtt{S} \in \mathrm{Std}_{\lambda_l}(\mu_l)$ and $\mathtt{T} \in \mathrm{Std}_{\lambda_r}(\mu_r)$, then the preferred expression for $w_{\mathtt{S\#T}}$ is just the concatenation of the preferred expression for $w_\mathtt{S}$ with the expression obtained by applying shift_{n_l} to every term in the preferred expression for $w_\mathtt{T}$. Hence $\psi_{\mathtt{S\#T}} = \psi_\mathtt{S}\,\mathrm{shift}_{n_l}(\psi_\mathtt{T})$.

Now we show that every dominated homomorphism $\mathrm{S}_\lambda \to \mathrm{S}_\mu$ is a sum of product homomorphisms. To do this, we first discuss dominated tableaux. Note that the conditions on λ and μ imply that $\mathrm{Std}_\lambda(\mu) = \left\{\mathtt{T}_l\#\mathtt{T}_r \;\middle|\; \mathtt{T}_l \in \mathrm{Std}_{\lambda_l}(\mu_l),\ \mathtt{T}_r \in \mathrm{Std}_{\lambda_r}(\mu_r)\right\}$. Choose a total order $\blacktriangleright$ on $\mathrm{Std}_\lambda(\mu)$ with the property that if $\mathtt{R}, \mathtt{S} \in \mathrm{Std}_{\lambda_l}(\mu_l)$ and $\mathtt{T}, \mathtt{U} \in \mathrm{Std}_{\lambda_r}(\mu_r)$, then

$$\mathtt{R\#T} \blacktriangleright \mathtt{R\#U} \Longleftrightarrow \mathtt{S\#T} \blacktriangleright \mathtt{S\#U} \qquad \text{and} \qquad \mathtt{R\#T} \blacktriangleright \mathtt{S\#T} \Longleftrightarrow \mathtt{R\#U} \blacktriangleright \mathtt{S\#U}.$$

(For example, we could do this by choosing total orders $\blacktriangleright_l, \blacktriangleright_r$ on $\mathrm{Std}_{\lambda_l}(\mu_l), \mathrm{Std}_{\lambda_r}(\mu_r)$ and setting $\mathtt{V} \blacktriangleright \mathtt{W}$ if and only if $\mathtt{V}_l \blacktriangleright_l \mathtt{W}_l$ or ($\mathtt{V}_l = \mathtt{W}_l$ and $\mathtt{V}_r \blacktriangleright_r \mathtt{W}_r$).)

Now suppose $\varphi : \mathrm{S}_\lambda \to \mathrm{S}_\mu$ is a non-zero dominated homomorphism, and write $\varphi(z_\lambda) = \sum_{\mathtt{T} \in \mathrm{Std}_\lambda(\mu)} a_\mathtt{T} v_\mathtt{T}$ with each $a_\mathtt{T} \in \mathbb{F}$. Let $\mathtt{U}$ be the largest tableau (with respect to $\blacktriangleright$) such that $a_\mathtt{U} \neq 0$, and proceed by induction on $\mathtt{U}$.

Claim. Let $\mathcal{U}$ denote the set of tableaux $\mathtt{T} \in \mathrm{Std}_\lambda(\mu)$ such that $\mathtt{T}_r = \mathtt{U}_r$. Then there is an $\mathscr{H}_l$-homomorphism

$$\begin{aligned} \varphi_l^{\mathtt{U}} : \mathrm{S}_{\lambda_l} &\longrightarrow \mathrm{S}_{\mu_l} \\ z_{\lambda_l} &\longmapsto \sum_{T \in \mathcal{U}} a_\mathtt{T} v_{\mathtt{T}_l}. \end{aligned}$$

Proof. First we make an observation, which follows from the construction of

Specht modules and our assumptions on preferred reduced expressions. If $\mathtt{W} \in \mathrm{Std}_\lambda(\mu)$ and $h \in \mathscr{H}_{\mathrm{l}}$, and we write $hv_{\mathtt{W}_{\mathrm{l}}} = \sum_{\mathtt{T}\in\mathrm{Std}(\mu_{\mathrm{l}})} b_{\mathtt{T}} v_{\mathtt{T}}$, then $\mathrm{shift}_0(h)v_{\mathtt{W}} = \sum_{\mathtt{T}\in\mathrm{Std}(\mu_{\mathrm{l}})} b_{\mathtt{T}} v_{\mathtt{T}\#\mathtt{W}_{\mathrm{r}}}$. In particular, $\mathrm{shift}_0(h)v_{\mathtt{W}}$ is a linear combination of basis elements $v_{\mathtt{S}}$ for $\mathtt{S} \in \mathrm{Std}_{\mathrm{lr}}(\mu)$ with $\mathtt{S}_{\mathrm{r}} = \mathtt{W}_{\mathrm{r}}$.

Now take $h \in \mathrm{Ann}(z_{\lambda_{\mathrm{l}}})$. Then $\mathrm{shift}_0(h) \in \mathrm{Ann}(z_\lambda)$, so $\mathrm{shift}_0(h)\sum_{\mathtt{T}\in\mathrm{Std}_\lambda(\mu)} a_{\mathtt{T}} v_{\mathtt{T}} = 0$ (because φ is a homomorphism). If we look just at $\mathrm{shift}_0(h)\sum_{\mathtt{T}\in\mathcal{U}} a_{\mathtt{T}} v_{\mathtt{T}}$, then by the previous paragraph this lies in $\langle v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}_{\mathrm{lr}}(\mu),\ \mathtt{T}_{\mathrm{r}} = \mathtt{U}_{\mathrm{r}}\rangle_{\mathbb{F}}$, while $\mathrm{shift}_0(h)\sum_{\mathtt{T}\notin\mathcal{U}} a_{\mathtt{T}} v_{\mathtt{T}}$ lies in $\langle v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}_{\mathrm{lr}}(\mu),\ \mathtt{T}_{\mathrm{r}} \neq \mathtt{U}_{\mathrm{r}}\rangle_{\mathbb{F}}$. The $v_{\mathtt{T}}$ are linearly independent, and hence

$$\langle v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}_{\mathrm{lr}}(\mu),\ \mathtt{T}_{\mathrm{r}} = \mathtt{U}_{\mathrm{r}}\rangle_{\mathbb{F}} \cap \langle v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}_{\mathrm{lr}}(\mu),\ \mathtt{T}_{\mathrm{r}} \neq \mathtt{U}_{\mathrm{r}}\rangle_{\mathbb{F}} = 0.$$

Hence $\mathrm{shift}_0(h)\sum_{\mathtt{T}\in\mathcal{U}} a_{\mathtt{T}} v_{\mathtt{T}} = 0$.

Define a linear map $\#\mathtt{U}_{\mathrm{r}} : \mathrm{S}_{\mu_{\mathrm{l}}} \to \mathrm{S}_\mu$ by $v_{\mathtt{T}} \mapsto v_{\mathtt{T}\#\mathtt{U}_{\mathrm{r}}}$ for $\mathtt{T} \in \mathrm{Std}(\mu_{\mathrm{l}})$. Then, from above, we have

$$(hm)\#\mathtt{U}_{\mathrm{r}} = h(m\#\mathtt{U}_{\mathrm{r}})$$

for any $h \in \mathscr{H}_{\mathrm{l}}$ and any $m \in \mathrm{S}_{\mu_{\mathrm{l}}}$. So for each $h \in \mathrm{Ann}(z_{\lambda_{\mathrm{l}}})$, we have $h\sum_{\mathtt{T}\in\mathcal{U}} a_{\mathtt{T}} v_{\mathtt{T}_{\mathrm{l}}} = 0$.

We can do essentially the same thing left and right interchanged; that is, if we let $\mathcal{U}' = \{\mathtt{T} \in \mathrm{Std}_\lambda(\mu) \mid \mathtt{T}_{\mathrm{l}} = \mathtt{U}_{\mathrm{l}}\}$, then we have an $\mathscr{H}_{\mathrm{r}}$-homomorphism

$$\begin{aligned}\varphi_{\mathrm{r}}^{\mathtt{U}} : \mathrm{S}_{\lambda_{\mathrm{r}}} &\longrightarrow \mathrm{S}_{\mu_{\mathrm{r}}}\\ z_{\lambda_{\mathrm{r}}} &\longmapsto \sum_{\mathtt{T}\in\mathcal{U}'} a_{\mathtt{T}} v_{\mathtt{T}_{\mathrm{r}}}.\end{aligned}$$

As in the proof of Lemma 2.28, we construct homomorphisms

$$\varphi_{\mathrm{l}}^{\mathtt{U}}\#\,\mathrm{id} : \mathrm{S}_{\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}}} \longrightarrow \mathrm{S}_\mu \quad \text{and} \quad \mathrm{id}\,\#\varphi_{\mathrm{r}}^{\mathtt{U}} : \mathrm{S}_\lambda \longrightarrow \mathrm{S}_{\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}}},$$

whose composition is the product homomorphism $\varphi_{\mathrm{l}}^{\mathtt{U}}\#\varphi_{\mathrm{r}}^{\mathtt{U}} : \mathrm{S}_\lambda \to \mathrm{S}_\mu$. $v_{\mathtt{U}}$ appears with non-zero coefficient (namely $a_{\mathtt{U}}^2$) in $\varphi_{\mathrm{l}}^{\mathtt{U}}\#\varphi_{\mathrm{r}}^{\mathtt{U}}$, and $\mathtt{U}$ is maximal (with respect to the order $\blacktriangleright$) with this property. So if we consider the homomorphism $\xi := \varphi - \frac{1}{a_{\mathtt{U}}}\varphi_{\mathrm{l}}\#\varphi_{\mathrm{r}}$, then (if $\xi \neq 0$) the most dominant tableau occurring with non-zero coefficient in ξ is smaller than $\mathtt{U}$. By induction ξ is a sum of product homomorphisms, and hence so is φ. □

Now we can prove our main result.

Theorem 2.30 (Generalised graded column removal). *Suppose $\lambda, \mu \in \mathscr{P}_n^l$, $c \geqslant 0$ and $1 \leqslant m \leqslant l$ and define $\lambda_{\mathrm{l}}, \lambda_{\mathrm{r}}, \mu_{\mathrm{l}}, \mu_{\mathrm{r}}$ as in Section 2.4. Assume $|\lambda_{\mathrm{l}}(c,m)| = |\mu_{\mathrm{l}}(c,m)| =: n_{\mathrm{l}}$ and $|\lambda_{\mathrm{r}}(c,m)| = |\mu_{\mathrm{r}}(c,m)| =: n_{\mathrm{r}}$ for some fixed $c \geqslant 0$ and $1 \leqslant m \leqslant l$ and define $\mathscr{H}_{\mathrm{l}} = \mathscr{H}_{n_{\mathrm{l}}}$ and $\mathscr{H}_{\mathrm{r}} = \mathscr{H}_{n_{\mathrm{r}}}$.*

1. *For any $\varphi_{\mathrm{l}} \in \mathrm{DHom}_{\mathscr{H}_{\mathrm{l}}}(\mathrm{S}_{\lambda_{\mathrm{l}}}, \mathrm{S}_{\mu_{\mathrm{l}}})$ and $\varphi_{\mathrm{r}} \in \mathrm{DHom}_{\mathscr{H}_{\mathrm{r}}}(\mathrm{S}_{\lambda_{\mathrm{r}}}, \mathrm{S}_{\mu_{\mathrm{r}}})$, there is a product homomorphism $\varphi_{\mathrm{l}}\#\varphi_{\mathrm{r}} \in \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$.*

2. *The map $\varphi_{\mathrm{l}} \otimes \varphi_{\mathrm{r}} \mapsto \varphi_{\mathrm{l}}\#\varphi_{\mathrm{r}}$ defines an isomorphism of graded $\mathbb{F}$-vector spaces*

$$\mathrm{DHom}_{\mathscr{H}_{\mathrm{l}}}(\mathrm{S}_{\lambda_{\mathrm{l}}}, \mathrm{S}_{\mu_{\mathrm{l}}}) \otimes \mathrm{DHom}_{\mathscr{H}_{\mathrm{r}}}(\mathrm{S}_{\lambda_{\mathrm{r}}}, \mathrm{S}_{\mu_{\mathrm{r}}}) \cong \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu).$$

Proof. First suppose $\lambda \ntrianglerighteq \mu$. Then $\mathrm{Std}_\lambda(\mu) = \emptyset$, so $\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) = 0$. Furthermore, we have either $\lambda_{\mathrm{l}} \ntrianglerighteq \mu_{\mathrm{l}}$ or $\lambda_{\mathrm{r}} \ntrianglerighteq \mu_{\mathrm{r}}$, so that either $\mathrm{DHom}_{\mathscr{H}_{\mathrm{l}}}(\mathrm{S}_{\lambda_{\mathrm{l}}}, \mathrm{S}_{\mu_{\mathrm{l}}}) = 0$ or $\mathrm{DHom}_{\mathscr{H}_{\mathrm{r}}}(\mathrm{S}_{\lambda_{\mathrm{r}}}, \mathrm{S}_{\mu_{\mathrm{r}}}) = 0$. So the result follows.

So we can assume that $\lambda \trianglerighteq \mu$, which allows us to define the multipartition $\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}}$ as above. Applying Theorem 2.23 repeatedly, we have

$$\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_{\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}}}) \cong \mathrm{DHom}_{\mathscr{H}_{\mathrm{r}}}(\mathrm{S}_{\lambda_{\mathrm{r}}}, \mathrm{S}_{\mu_{\mathrm{r}}}).$$

Similarly, by Theorem 2.26 applied repeatedly we have

$$\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda_{\mathrm{l}}\#\mu_{\mathrm{r}}}, \mathrm{S}_\mu) \cong \mathrm{DHom}_{\mathscr{H}_{\mathrm{l}}}(\mathrm{S}_{\lambda_{\mathrm{l}}}, \mathrm{S}_{\mu_{\mathrm{l}}}).$$

Combining these isomorphisms, and using the explicit constructions given above, we have an isomorphism of graded vector spaces

$$\mathrm{DHom}_{\mathscr{H}_l}(\mathrm{S}_{\lambda_l}, \mathrm{S}_{\mu_l}) \otimes \mathrm{DHom}_{\mathscr{H}_r}(\mathrm{S}_{\lambda_r}, \mathrm{S}_{\mu_r}) \xrightarrow{\sim} \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda_l\#\mu_r}, \mathrm{S}_{\mu}) \otimes \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda}, \mathrm{S}_{\lambda_l\#\mu_r})$$
$$\varphi_l \otimes \varphi_r \longmapsto (\varphi_l\#\,\mathrm{id}) \otimes (\mathrm{id}\,\#\varphi_r).$$

Composition of homomorphisms yields a map

$$\omega : \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda_l\#\mu_r}, \mathrm{S}_{\mu}) \otimes \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda}, \mathrm{S}_{\lambda_l\#\mu_r}) \longrightarrow \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda}, \mathrm{S}_{\mu})$$

which is homogeneous of degree zero, and by Lemma 2.28 and Proposition 2.29 ω is surjective. So we have a surjective map

$$\mathrm{DHom}_{\mathscr{H}_l}(\mathrm{S}_{\lambda_l}, \mathrm{S}_{\mu_l}) \otimes \mathrm{DHom}_{\mathscr{H}_r}(\mathrm{S}_{\lambda_r}, \mathrm{S}_{\mu_r}) \longrightarrow \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda}, \mathrm{S}_{\mu})$$
$$\varphi_l \otimes \varphi_r \longmapsto \varphi_l\#\varphi_r.$$

This map is easily seen to be injective, and the result follows. □

2.7 Generalised row removal

Now we consider generalised row removal for homomorphisms between column Specht modules. Fix $1 \leqslant m \leqslant l$ and $r \geqslant 0$, and for any $\nu \in \mathscr{P}^l_n$ write $\nu_t = \nu_t(r, m)$, $\nu_b = \nu_b(r, m)$. Suppose $\lambda, \mu \in \mathscr{P}^l_n$ with $|\lambda_t| = |\mu_t| =: n_t$, so that $|\lambda_b| = |\mu_b| = n - n_t =: n_b$. Set $\kappa_t = (\kappa_1, \dots, \kappa_m)$ and $\kappa_b = (\kappa_m - r, \kappa_{m+1}, \dots, \kappa_l)$, and write $\mathscr{H}_t = \mathscr{H}_{n_t}$ and $\mathscr{H}_b = \mathscr{H}_{n_b}$. In what follows we shall take S_λ and S_μ to be defined with respect to the e-multicharge κ, S_{λ_t} and S_{μ_t} with respect to κ_t, and S_{λ_b} and S_{μ_b} with respect to κ_b.

With this notation in place, we can state a generalised row-removal theorem for homomorphisms. This follows from Theorem 2.30 using Theorem 2.18 in the same way that Corollary 2.25 is deduced from Theorem 2.23.

Theorem 2.31 (Generalised graded row removal)**.** *Suppose* $\lambda, \mu \in \mathscr{P}^l_n$, $r \geqslant 0$ *and* $1 \leqslant m \leqslant$

l and define $\lambda_{\mathrm{t}}, \lambda_{\mathrm{b}}, \mu_{\mathrm{t}}, \mu_{\mathrm{b}}, n_{\mathrm{t}}, n_{\mathrm{b}}, \mathscr{H}_{\mathrm{t}}, \mathscr{H}_{\mathrm{b}}$ as above. Then there is an isomorphism of graded $\mathbb{F}$-vector spaces

$$\mathrm{DHom}_{\mathscr{H}_{\mathrm{t}}}(\mathrm{S}_{\lambda_{\mathrm{t}}}, \mathrm{S}_{\mu_{\mathrm{t}}}) \otimes \mathrm{DHom}_{\mathscr{H}_{\mathrm{b}}}(\mathrm{S}_{\lambda_{\mathrm{b}}}, \mathrm{S}_{\mu_{\mathrm{b}}}) \cong \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda}, \mathrm{S}_{\mu}).$$

Our proof of Theorem 2.30 gives a direct construction of the column-removal isomorphism, but a direct construction for row removal seems to be hard to obtain, especially using the standard bases for column Specht modules. The difficulty seems to arise when passing through the isomorphism θ^{λ} from Theorem 2.15, which does not preserve the standard bases.

Example. Take $e = 2$ and $\kappa = (0, 1, 0)$. Let $\lambda = \big((1^2), (2, 1^3), (1)\big)$ and $\mu = \big((1), (3, 1), (3)\big)$, and take $(m, r) = (2, 1)$, so that $\lambda_{\mathrm{t}} = \big((1^2), (2)\big)$, $\lambda_{\mathrm{b}} = \big((1^3), (1)\big)$ and $\mu_{\mathrm{t}} = \mu_{\mathrm{b}} = \big((1), (3)\big)$. Set $\kappa_{\mathrm{t}} = (0, 1)$ and $\kappa_{\mathrm{b}} = (0, 0)$. Then (regardless of the field $\mathbb{F}$) the graded dimensions of $\mathrm{DHom}_{\mathscr{H}_4}(\mathrm{S}_{\lambda_{\mathrm{t}}|\kappa_{\mathrm{t}}}, \mathrm{S}_{\mu_{\mathrm{t}}|\kappa_{\mathrm{t}}})$ and $\mathrm{DHom}_{\mathscr{H}_4}(\mathrm{S}_{\lambda_{\mathrm{b}}|\kappa_{\mathrm{b}}}, \mathrm{S}_{\mu_{\mathrm{b}}|\kappa_{\mathrm{b}}})$ are v and 1 respectively. So by Theorem 2.31 the graded dimension of $\mathrm{DHom}_{\mathscr{H}_8}(\mathrm{S}_{\lambda|\kappa}, \mathrm{S}_{\mu|\kappa})$ is v. The unique (up to scaling) homomorphisms

$$\mathrm{S}_{\lambda_{\mathrm{t}}} \longrightarrow \mathrm{S}_{\mu_{\mathrm{t}}}, \qquad \mathrm{S}_{\lambda_{\mathrm{b}}} \longrightarrow \mathrm{S}_{\mu_{\mathrm{b}}}, \qquad \mathrm{S}_{\lambda} \longrightarrow \mathrm{S}_{\mu}$$

are given by

$$z_{\lambda_{\mathrm{t}}} \longmapsto v_{\mathtt{S}}, \qquad z_{\lambda_{\mathrm{b}}} \longmapsto v_{\mathtt{T}}, \qquad z_{\lambda} \longmapsto v_{\mathtt{U}} + 2v_{\mathtt{V}},$$

where

$$\mathtt{S} = \begin{array}{cc} & \begin{array}{|c|}\hline 3\\\hline\end{array}\\ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline\end{array} & \end{array}, \qquad \mathtt{T} = \begin{array}{cc} & \begin{array}{|c|}\hline 2\\\hline\end{array}\\ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline\end{array} & \end{array},$$

$$\mathtt{U} = \begin{array}{ccc} & & \begin{array}{|c|}\hline 7\\\hline\end{array}\\ & \begin{array}{|c|c|c|}\hline 2&6&8\\\hline 3\\\cline{1-1}\end{array} & \\ \begin{array}{|c|c|c|}\hline 1&4&5\\\hline\end{array} & & \end{array}, \qquad \mathtt{V} = \begin{array}{ccc} & & \begin{array}{|c|}\hline 7\\\hline\end{array}\\ & \begin{array}{|c|c|c|}\hline 4&6&8\\\hline 5\\\cline{1-1}\end{array} & \\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} & & \end{array}.$$

It seems hard to reconcile these homomorphisms when expressed in this form, except perhaps in characteristic 2. (Note that the incompatibility of these expressions is not an artefact of the choice of preferred reduced expressions – the standard basis elements appearing in this example are independent of the choice of reduced expressions.)

In order to obtain an explicit row-removal construction, it seems to be necessary to use a different basis for the Specht module. Suppose we have λ_{b}, λ_{t}, μ_{b} and μ_{t} as above, with $|\mu_{\mathrm{t}}| = n_{\mathrm{t}} = |\lambda_{\mathrm{t}}|$. Partition the set $\{1, \dots, n\}$ into two sets S_{b} and S_{t}, by defining S_{b} to be the set of integers in the bottom part of $\mathtt{T}_\lambda$ and S_{t} the set of integers in the top part; that is,

$$S_{\mathrm{b}} = \{\mathtt{T}_\lambda(s, c, k) \mid (s, c, k) \in [\lambda] \text{ and either } k > m \text{ or } k = m \text{ and } s > r\},$$

$$S_{\mathrm{t}} = \{\mathtt{T}_\lambda(s, c, k) \mid (s, c, k) \in [\lambda] \text{ and either } k < m \text{ or } k = m \text{ and } s \leqslant r\}.$$

Let $\mathrm{lab}_{\mathrm{b}} : \{1, \dots, n_{\mathrm{b}}\} \to S_{\mathrm{b}}$ and $\mathrm{lab}_{\mathrm{t}} : \{1, \dots, n_{\mathrm{t}}\} \to S_{\mathrm{t}}$ be the unique order-preserving bijections.

Now given a μ_{b}-tableau $\mathtt{T}$ and a μ_{t}-tableau $\mathtt{S}$, define a μ-tableau $\mathtt{T}\#_{\mathrm{R}}\mathtt{S}$ by composing $\mathrm{lab}_{\mathrm{b}}$ with $\mathtt{T}$ and $\mathrm{lab}_{\mathrm{t}}$ with $\mathtt{S}$ and 'gluing' in the natural way.

Lemma 2.32. *Suppose λ and μ satisfy the conditions above. If $\mathtt{T} \in \mathrm{Std}_{\lambda_{\mathrm{b}}}(\mu_{\mathrm{b}})$ and $\mathtt{S} \in \mathrm{Std}_{\lambda_{\mathrm{t}}}(\mu_{\mathrm{t}})$, then $\mathtt{T}\#_{\mathrm{R}}\mathtt{S} \in \mathrm{Std}_\lambda(\mu)$.*

Proof. First we show that $\mathtt{T}\#_{\mathrm{R}}\mathtt{S}$ is standard. Suppose A and B are nodes in the same component of $[\mu]$, with B either immediately to the right of A or immediately below A; then we require $\mathtt{T}\#_{\mathrm{R}}\mathtt{S}(B) > \mathtt{T}\#_{\mathrm{R}}\mathtt{S}(A)$. This is clear from the fact that $\mathtt{S}$ and $\mathtt{T}$ are standard and the functions $\mathrm{lab}_{\mathrm{t}}$ and $\mathrm{lab}_{\mathrm{b}}$ are order-preserving, except in the case where $A = (r, b, m)$ and $B = (r+1, b, m)$ for some $1 \leqslant b \leqslant \mu^{(m)}_{r+1}$. So assume we are in this situation.

Let $k = \lambda^{(m)}_{r+1}$. Then the first k columns of $\lambda^{(m)}_{\mathrm{t}}$ all have length r. Since $\mathrm{Std}_{\lambda_{\mathrm{t}}}(\mu_{\mathrm{t}})$ is non-empty we have $\lambda_{\mathrm{t}} \trianglerighteq \mu_{\mathrm{t}}$, and hence the first k columns of $\mu^{(m)}_{\mathrm{t}}$ all have length r also. Hence (since $\mathtt{S}$ is λ_{t}-dominated) $\mathtt{S}$ agrees with $\mathtt{T}_{\lambda_{\mathrm{t}}}$ on these columns. So we have

$\mathtt{T}\#_{\mathrm{R}}\mathtt{S}(A) = \mathrm{lab}_{\mathrm{t}}(\mathtt{T}_{\lambda_{\mathrm{t}}}(A)) = \mathtt{T}_{\lambda}(A)$.

We also have $\lambda_{\mathrm{b}} \trianglerighteq \mu_{\mathrm{b}}$ since $\mathrm{Std}_{\lambda_{\mathrm{b}}}(\mu_{\mathrm{b}}) \neq \emptyset$, so that $k \geqslant \mu^{(m)}_{r+1} \geqslant b$ (and in particular $B \in [\lambda]$). Since $\mathtt{T}$ is λ_{b}-dominated, we have $\mathtt{T}(1,b,1) \geqslant \mathtt{T}_{\lambda_{\mathrm{b}}}(1,b,1)$, so that

$$\mathtt{T}\#_{\mathrm{R}}\mathtt{S}(B) = \mathrm{lab}_{\mathrm{b}}(\mathtt{T}(1,b,1)) \geqslant \mathrm{lab}_{\mathrm{b}}(\mathtt{T}_{\lambda_{\mathrm{b}}}(1,b,1)) = \mathtt{T}_{\lambda}(B).$$

So $\mathtt{T}\#_{\mathrm{R}}\mathtt{S}(A) = \mathtt{T}_{\lambda}(A) < \mathtt{T}_{\lambda}(B) \leqslant \mathtt{T}\#_{\mathrm{R}}\mathtt{S}(B)$, as required.

To see that $\mathtt{T}\#_{\mathrm{R}}\mathtt{S}$ is λ-dominated, it suffices to note that since $\mathtt{S} \in \mathrm{Std}_{\lambda_{\mathrm{t}}}(\mu_{\mathrm{t}})$, every element of S_{t} appears in $\mathrm{lab}_{\mathrm{t}}(\mathtt{S})$ at least as far to the left as it appears in $\mathtt{T}_{\lambda}$, and likewise for $\mathtt{T} \in \mathrm{Std}_{\lambda_{\mathrm{b}}}(\mu_{\mathrm{b}})$ and elements of S_{b}. □

Now we can give a conjectured explicit construction for the generalised row-removal isomorphism for homomorphisms. Recall from Section 2.3 the basis $\{f_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\mu)\}$ for $(\mathrm{S}^{\mu})^{\circledast}$; using Theorem 2.15 and shifting the degree of each $f_{\mathtt{T}}$ by $\mathrm{def}(\mu)$, we can regard $\{f_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\mu)\}$ as a basis for S_{μ}. Note that by the analogue of Lemma 2.19(2) for column Specht modules, any $\varphi \in \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_{\lambda}, \mathrm{S}_{\mu})$ can be written as

$$\varphi(z_{\lambda}) = \sum_{\mathtt{T} \in \mathrm{Std}_{\lambda}(\mu)} a_{\mathtt{T}} f_{\mathtt{T}} \quad \text{for some } a_{\mathtt{T}} \in \mathbb{F}.$$

Conjecture 2.33. *Suppose $\lambda, \mu \in \mathscr{P}^l_n$, $r \geqslant 0$ and $1 \leqslant m \leqslant n$. Define $\lambda_{\mathrm{t}}, \lambda_{\mathrm{b}}, \mu_{\mathrm{t}}, \mu_{\mathrm{b}}, n_{\mathrm{t}}, n_{\mathrm{b}}, \mathscr{H}_{\mathrm{t}}, \mathscr{H}_{\mathrm{b}}$ as above, and assume $|\mu_{\mathrm{t}}| = n_{\mathrm{t}}$. Suppose $\varphi_{\mathrm{t}} \in \mathrm{DHom}_{\mathscr{H}_{\mathrm{t}}}(\mathrm{S}_{\lambda_{\mathrm{t}}}, \mathrm{S}_{\mu_{\mathrm{t}}})$ and $\varphi_{\mathrm{b}} \in \mathrm{DHom}_{\mathscr{H}_{\mathrm{b}}}(\mathrm{S}_{\lambda_{\mathrm{b}}}, \mathrm{S}_{\mu_{\mathrm{b}}})$, and write*

$$\varphi_{\mathrm{b}}(z_{\lambda_{\mathrm{b}}}) = \sum_{\mathtt{T} \in \mathrm{Std}_{\lambda_{\mathrm{b}}}(\mu_{\mathrm{b}})} a_{\mathtt{T}} f_{\mathtt{T}}, \qquad \varphi_{\mathrm{t}}(z_{\lambda_{\mathrm{t}}}) = \sum_{\mathtt{S} \in \mathrm{Std}_{\lambda_{\mathrm{t}}}(\mu_{\mathrm{t}})} b_{\mathtt{S}} f_{\mathtt{S}}$$

with $a_{\mathtt{T}}, b_{\mathtt{S}} \in \mathbb{F}$. Then there is an $\mathscr{H}_n$-homomorphism $\varphi_{\mathrm{b}}\#_{\mathrm{R}}\varphi_{\mathrm{t}} : \mathrm{S}_{\lambda} \to \mathrm{S}_{\mu}$ satisfying

$$\varphi_{\mathrm{b}}\#_{\mathrm{R}}\varphi_{\mathrm{t}}(z_{\lambda}) = \sum_{\substack{\mathtt{T} \in \mathrm{Std}_{\lambda_{\mathrm{b}}}(\mu_{\mathrm{b}}) \\ \mathtt{S} \in \mathrm{Std}_{\lambda_{\mathrm{t}}}(\mu_{\mathrm{t}})}} a_{\mathtt{T}} b_{\mathtt{S}} f_{\mathtt{T}\#_{\mathrm{R}}\mathtt{S}}.$$

Example. Retaining the notation from the last example, we have

$$\mathtt{T}_\lambda = \begin{array}{ccc} & & \begin{array}{|c|}\hline 7\\\hline 8\\\hline\end{array}, \\ & \begin{array}{|c|c|}\hline 2&6\\\hline 3\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array} & \\ \begin{array}{|c|}\hline 1\\\hline\end{array} & & \end{array}$$

so that $S_\mathrm{t} = \{2, 6, 7, 8\}$ and $S_\mathrm{b} = \{1, 3, 4, 5\}$. Taking $\mathtt{S}$, $\mathtt{T}$ and $\mathtt{U}$ as in the last example, we get $\mathtt{T}\#_\mathrm{R}\mathtt{S} = \mathtt{U}$. It is easy to check that

$$f_\mathtt{S} = v_\mathtt{S}, \qquad f_\mathtt{T} = v_\mathtt{T}, \qquad f_\mathtt{U} = v_\mathtt{U} + 2v_\mathtt{V},$$

so the conjecture holds in this case.

Remark. If Conjecture 2.33 is true, then we have a map of graded $\mathbb{F}$-vector spaces

$$\begin{aligned} \mathrm{DHom}_{\mathscr{H}_\mathrm{b}}(\mathrm{S}_{\lambda_\mathrm{b}}, \mathrm{S}_{\mu_\mathrm{b}}) \otimes \mathrm{DHom}_{\mathscr{H}_\mathrm{t}}(\mathrm{S}_{\lambda_\mathrm{t}}, \mathrm{S}_{\mu_\mathrm{t}}) &\longrightarrow \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) \\ \varphi_\mathrm{b} \otimes \varphi_\mathrm{t} &\longmapsto \varphi_\mathrm{b}\#_\mathrm{R}\varphi_\mathrm{t}. \end{aligned}$$

This map is obviously linear, and (since the $f_\mathtt{T}$ are linearly independent) injective. Hence by Theorem 2.31 it is a bijection. So we have an explicit construction for the generalised row-removal isomorphism.

Chapter 3

Decomposable Specht modules

In this chapter, we will investigate Specht modules for the KLR algebra in level 1 with $e = 2$; that is, the Iwahori–Hecke algebra in quantum characteristic 2.

Recall from Theorem 1.44 that if $e \neq 2$, it is known that all Specht modules for the Hecke algebra are indecomposable. When $e = 2$ this is not the case; determining which Specht modules are decomposable is an open and very difficult problem, even for the symmetric group (i.e. when $\mathrm{char}(\mathbb{F}) = 2$). After Murphy's result for hook partitions in [36], no further progress was made until the paper of Dodge and Fayers [16], where they were able to show that many Specht modules indexed by partitions of the form $(a, 3, 1^b)$ are decomposable, giving necessary and sufficient conditions for this.

Here we take a different approach. We would like to extend Murphy's result for the symmetric group to the Hecke algebra. We study Specht modules indexed by hook partitions, for the Hecke algebra, and determine exactly when they are decomposable. This is all studied using the KLR setting outlined in Chapter 1. For the reader's ease, we start by restricting the presentations of the (cyclotomic) KLR algebra and its Specht modules to the relevant case.

3.1 KLR algebras for $l = 1, e = 2$

When $e = 2$ (that is $q = -1$), the Hecke algebra $\mathcal{H} = \mathcal{H}_n = \mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ is isomorphic to the cyclotomic KLR algebra $\mathscr{H}_n^{(0)}$ with the following presentation:

Generators $\{e(i) \mid i \in \{0,1\}^n\} \cup \{y_1, \ldots, y_n\} \cup \{\psi_1, \ldots, \psi_{n-1}\}$.

Relations

$$e(i)e(j) = \delta_{i,j} e(i);$$

$$\sum_{i \in \{0,1\}^n} e(i) = 1;$$

$$y_r e(i) = e(i) y_r;$$

$$\psi_r e(i) = e(s_r i)\psi_r;$$

$$y_r y_s = y_s y_r;$$

$$\psi_r y_s = y_s \psi_r \quad \text{if } s \neq r, r+1;$$

$$\psi_r \psi_s = \psi_s \psi_r \quad \text{if } |r-s| > 1;$$

$$y_r \psi_r e(i) = (\psi_r y_{r+1} - \delta_{i_r, i_{r+1}}) e(i);$$

$$y_{r+1} \psi_r e(i) = (\psi_r y_r + \delta_{i_r, i_{r+1}}) e(i);$$

$$\psi_r^2 e(i) = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ (y_{r+1} - y_r)(y_r - y_{r+1}) e(i) & \text{if } i_r \neq i_{r+1}; \end{cases}$$

$$\psi_r \psi_{r+1} \psi_r e(i) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1}) e(i) & \text{if } i_r = i_{r+1} \text{ or } i_{r+1} = i_{r+2}, \\ (\psi_{r+1}\psi_r\psi_{r+1} + y_r - 2y_{r+1} + y_{r+2}) e(i) & \text{otherwise;} \end{cases}$$

$$y_1 = 0;$$

$$e(i) = 0 \quad \text{if } i_1 = 1.$$

Note that we have taken the usual convention $\kappa = (0)$ here, though in fact we always have $\mathscr{H}_n^{(\kappa_1)} \cong \mathscr{H}_n^{(\kappa_2)}$ for any $\kappa_1, \kappa_2 \in I$. To see this, note that if r is the rotation of the quiver Γ which maps the vertex labelled by j to the vertex labelled by $j+1$ for each $j \in I$, then there is an automorphism of $\mathscr{H}_n$ mapping $e(i) \mapsto e(r(i))$, $y_k \mapsto y_k$,

$\psi_k \mapsto \psi_k$ for all $i \in I^n$ and all admissible k. This automorphism maps the cyclotomic ideal corresponding to an e-multicharge κ to that corresponding to $r(\kappa)$. In view of Brundan and Kleshchev's isomorphism theorem, we identify $\mathscr{H}_n^{(0)}$ with $\mathcal{H} = \mathcal{H}_n$ in this chapter.

Remark. Recall the remarks after we first defined $\mathscr{H}_n$ in Section 1.9 – since $e = 2$, introducing the generators y_r and ψ_s here makes sense; they are the sums of corresponding generators in the algebras $\mathscr{H}_\alpha$. In fact, we can also see the validity of the presentation because of the fact that we are taking a cyclotomic quotient, as discussed at the end of Section 1.9.

3.2 Specht modules for hook partitions

For the remainder of the chapter, we fix $n = a + b$ and $\lambda = (a, 1^b)$. That is, λ is a hook partition of n. Specialising our homogeneous presentation from Section 1.10 to hook partitions, we have

$$\mathrm{S}_\lambda = \langle z_\lambda \mid e(i_\lambda) z_\lambda = z_\lambda,\ y_k z_\lambda = 0\ \forall k,\ \psi_j z_\lambda = 0\ \forall j \neq b+1,\ \psi_1 \psi_2 \dots \psi_{b+1} z_\lambda = 0 \rangle.$$

It's useful to note that in the case of hook partitions, the standard basis $\{ v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ is independent of our choice of reduced expression for $w_{\mathtt{T}}$, since each $w_{\mathtt{T}}$ is fully commutative. We can, for example, appeal to Lemma 1.9 in order to see this.

Example. Let $\lambda = (3, 1^2)$. It is easy to check that $\mathrm{End}_{\mathcal{H}}(\mathrm{S}_\lambda)$ has a basis $\{\mathrm{id}, \varphi\}$ where φ is given by $\varphi(z_\lambda) = \psi_3 \psi_2 \psi_4 \psi_3 z_\lambda = v_{\mathtt{T}^\lambda}$. Moreover, $\varphi^2(z_\lambda) = -2\varphi(z_\lambda)$, so $\mathrm{id} + 1/2\varphi$ and $-1/2\varphi$ are both idempotents so long as $\mathrm{char}(\mathbb{F}) \neq 2$. In particular, S_λ is decomposable if and only if $\mathrm{char}(\mathbb{F}) \neq 2$.

Furthermore, since there are only two non-trivial idempotent endomorphisms (along with the idempotents id and the zero map), we expect S_λ to decompose into a direct sum of two indecomposable summands. If we look at the decomposition matrix for $\mathcal{H}_5$ when $e = 2$, $p \neq 2$ (see for instance, the appendix of [25]) then we see that

S_λ has composition factors $D_{(5)}$ and $D_{(3,2)}$, each appearing once. Now, z_λ and $\varphi(z_\lambda)$ are both annihilated by $y_1, \dots, y_5, \psi_1, \psi_2, \psi_4$, and it is easy to check that $\psi_3\varphi(z_\lambda) = -2\psi_3 z_\lambda$. Thus we see that $\mathrm{im}(\mathrm{id} + 1/2\varphi) \cong D_{(5)}$ and it follows that $\mathrm{im}(-1/2\varphi) \cong D_{(3,2)}$. It is also quite easy to see that $\mathrm{im}(-1/2\varphi)$ is spanned by $\{v_{\mathtt{T}} \mid \mathtt{T} \neq \mathtt{T}_\lambda\}$.

A less direct approach to this example would be to consider the fact that, by [13, Theorem 3.5], $S_{(3,2)}$ is self-dual up to a twist by the sign representation. Since $e = 2$, the sign representation is isomorphic to the trivial representation, so S_λ is in fact self-dual. Now, since S_λ has exactly two composition factors, which are non-isomorphic, it follows that it must be decomposable.

When $\mathrm{char}(\mathbb{F}) = 2$, $D_{(5)}$ appears as a composition factor of S_λ twice, which is why this second argument no longer applies.

Decomposability of Specht modules for hook partitions was solved by Murphy in the case of the symmetric group (i.e. when $\mathrm{char}(\mathbb{F}) = 2$):

Theorem 3.1 [36, Theorem 4.5]. *Suppose* $\mathrm{char}(\mathbb{F}) = 2$. *Then* $S_{(a,1^b)}$ *is indecomposable if and only if n is even or* $a - 1 \equiv b \pmod{2^L}$ *where* $2^{L-1} \leqslant b < 2^L$.

Using this result, we will be able to assume $\mathrm{char}(\mathbb{F}) \neq 2$ where necessary. The following result will also reduce our workload later on.

Theorem 3.2. *Suppose a is odd and b is even. Then* $S_{(a,1^b)}$ *is decomposable if and only if* $S_{(a+1,1^{b+1})}$ *is.*

Proof. For any $r \geqslant 0$ and any i, functors

$$e_i^{(r)} : \mathcal{H}_{n+r}\text{-mod} \longrightarrow \mathcal{H}_n\text{-mod}$$
$$f_i^{(r)} : \mathcal{H}_n\text{-mod} \longrightarrow \mathcal{H}_{n+r}\text{-mod}$$

are introduced in [7, Section 2.2]. These functors are exact, and have the following property: if M is a non-zero module and we let $\varepsilon_i(M) := \max\{r \mid e_i^{(r)} M \neq 0\}$, then:

[7, Lemma 2.12] If D is a simple module, then $e_i^{(\varepsilon_i(D))} D$ is simple.

Since $e_i^{(r)}$ is exact, we have $\varepsilon_i(D) \leqslant \varepsilon_i(M)$ when D is a composition factor of M, and so by the above lemma we deduce that the composition length of $e_i^{(\varepsilon_i(M))}M$ is at most the composition length of M, with equality if and only if $\varepsilon_i(D) = \varepsilon_i(M)$ for all composition factors D of M.

A corresponding result holds with f_i, φ_i in place of e_i, ε_i.

Now consider Specht modules. By [7, Lemma 2.4] and [7, Equations (7)&(8)], $e_i^{(r)}$ and $f_i^{(r)}$ can be interpreted as restriction and induction, respectively, followed by projection onto particular blocks. In view of the block classification for Hecke algebras of type A [31, Theorem 2.11] and the branching rules for induction and restriction of Specht modules ([12, Theorem 7.4] and [3, Proposition 1.9] respectively), we deduce that $\varepsilon_i(\mathrm{S}_\lambda)$ is the number of removable nodes of λ of residue i, and $e_i^{(\varepsilon_i(\mathrm{S}_\lambda))}\mathrm{S}_\lambda$ is the Specht module labelled by the partition obtained by removing these nodes. A corresponding statement holds for f_i and addable nodes.

In particular, when $e = 2$, a is odd and b is even, let $\lambda = (a, 1^b)$ and $\mu = (a+1, 1^{b+1})$. Then $\varepsilon_1(\mathrm{S}_\mu) = \varphi_1(\mathrm{S}_\lambda) = 2$, and $e_1^{(2)}\mathrm{S}_\mu = \mathrm{S}_\lambda$, $f_1^{(2)}\mathrm{S}_\lambda = \mathrm{S}_\mu$.

In view of the above results, this means that S_λ and S_μ have the same composition length and that $e_1^{(2)}D \neq 0$ for every composition factor D of S_μ. Hence (again by exactness) $e_1^{(2)}N \neq 0$ for every submodule N of S_μ. Hence if S_μ is decomposable, then so is S_λ. The same argument the other way round shows that if S_λ is decomposable, then so is S_μ. □

Example. We illustrate the idea behind this proof with $\lambda = (3, 1^2)$ and $\mu = (4, 1^3)$. $\varepsilon_1(\mathrm{S}_\mu) = \varphi_1(\mathrm{S}_\lambda) = 2$, so $e_1^{(2)}\mathrm{S}_\mu \cong \mathrm{S}_\lambda$ and $f_1^{(2)}\mathrm{S}_\lambda \cong \mathrm{S}_\mu$ while $e_1^{(3)}\mathrm{S}_\mu = f_1^{(3)}\mathrm{S}_\lambda = 0$. As seen in the example preceding Theorem 3.1, S_λ has two composition factors when $\mathrm{char}(\mathbb{F}) \neq 2$ and it follows that S_μ does too. It follows that $f_1^{(2)}\mathrm{D}_{(5)} \neq 0 \neq f_1^{(2)}\mathrm{D}_{(3,2)}$ and so $\mathrm{S}_\mu \cong f_1^{(2)}\mathrm{D}_{(5)} \oplus f_1^{(2)}\mathrm{D}_{(3,2)}$.

Conversely, if $\mathrm{char}(\mathbb{F}) = 2$ we've seen that S_λ is indecomposable. If S_μ were decomposable, then applying the functor $e_1^{(2)}$ to each direct summand would yield a decomposition of S_λ, and thus a contradiction.

Suppose f is an $\mathcal{H}$-endomorphism of S_λ. We have $z_\lambda = e(i_\lambda)z_\lambda$, so we have $f(z_\lambda) \in e(i_\lambda)\,S_\lambda$. Now consider the standard basis $\{\,v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$. Lemma 1.31 tells us that $e(i)v_{\mathtt{T}} = \delta_{i,i_{\mathtt{T}}}v_{\mathtt{T}}$ for any $\mathtt{T} \in \mathrm{Std}(\lambda)$. Hence $\{\,v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\} \cap e(i)\,S_\lambda = \{v_{\mathtt{T}} \mid i_{\mathtt{T}} = i\}$. In particular, $f(z_\lambda)$ is a linear combination of elements in

$$\mathscr{D} := \{\,v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\} \cap e(i_\lambda)\,S_\lambda = \{v_{\mathtt{T}} \mid i_{\mathtt{T}} = i_\lambda\}.$$

This is at the core of our approach to understanding $\mathrm{End}_{\mathcal{H}}(S_\lambda)$.

Definition 3.3. When $\lambda = (a, 1^b)$, we define the *arm* to be the set of nodes $\{(1,2),(1,3),\ldots,(1,a)\}$ of λ and the *leg* to be the set of nodes $\{(2,1),(3,1),\ldots,(b+1,1)\}$.

Now, we separate our problem into cases where a and b are odd or even. When b is even, we have $i_\lambda = 0101\ldots01$. If b is odd, however, we have $i_\lambda = 0101\ldots011010\ldots10$, where we have a repetition in the positions $b+1$ and $b+2$.

Lemma 3.4. *Suppose b is even and $v_{\mathtt{T}} \in \mathscr{D}$. Then for all $1 \leqslant i \leqslant \lceil n/2 \rceil - 1$, $2i+1$ appears directly after $2i$ in $\mathtt{T}$. That is, if $2i$ is in the leg of $\mathtt{T}$ then $2i+1$ is directly below it, and if $2i$ is in the arm of $\mathtt{T}$ then $2i+1$ is directly to the right of it.*

Proof. In defining i_λ, we assign all nodes of $[\lambda]$ in which $\mathtt{T}_\lambda$ contains an even entry a 1 and all others a 0. First, we note that since b is even and $v_{\mathtt{T}} \in \mathscr{D}$, the final node in the leg of λ has residue 0. This ensures that if $2i$ is in the leg of $\mathtt{T}$ there must be some entry immediately below it.

By induction on $i > 1$, assume that $2i+1$ appears directly after $2i$ in $\mathtt{T}$, for all $i < k$. Suppose our assertion is false for $i = k$. We assume without loss of generality that $2k$ is in the leg of $\mathtt{T}$ and $2k+1$ is in the arm. Now by induction any even number, $2j < 2k$, is immediately followed by $2j+1$. This forces $2k+1$ to be adjacent to $2j+1$ for some $j < k$, and $v_{\mathtt{T}} \notin \mathscr{D}$. ◻

The fact that entries must stick together in these pairs motivates our next definition.

Definition 3.5. We will call the pair of entries $2i, 2i+1$ for $1 \leqslant i \leqslant \lceil n/2 \rceil - 1$ a *domino*. We will denote the domino by $[2i, 2i+1]$ or D_i. We define a *domino tableau* to be any λ-tableau $\mathtt{T}$ such that $v_{\mathtt{T}} \in \mathscr{D}$. We denote the set of domino tableaux by $\mathrm{Dom}(\lambda)$.

Remark. $\mathscr{D} = \{v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Dom}(\lambda)\}$ is a basis of $e(i_\lambda)\mathrm{S}_\lambda$.

We will now begin by solving the simplest cases, where n is even.

3.3 Decomposability of $\mathrm{S}_{(a,1^b)}$ when n is even

First, we will look at the case where a and b are both even.

Lemma 3.6. *Suppose* $\mathtt{T} \in \mathrm{Std}(\lambda)$ *and* $1 < i < n$. *If* $i, i+1, \ldots, n$ *all lie in the arm of* $\mathtt{T}$ *then* $\psi_i v_{\mathtt{T}} = 0$. *If* i *lies in the leg of* $\mathtt{T}$ *and* $i+1$ *lies in the arm, then* $\psi_i v_{\mathtt{T}} = v_{\mathtt{U}}$, *where* $\mathtt{U}$ *is obtained from* $\mathtt{T}$ *by swapping* i *and* $i+1$.

Proof. First, suppose $i, i+1, \ldots, n$ all lie in the arm of $\mathtt{T}$ for some $1 < i < n$. Then $v_{\mathtt{T}}$ cannot possibly involve ψ_j for any $j > i-2$. It follows that ψ_i commutes with each generator ψ_j appearing in $v_{\mathtt{T}}$ and the result follows from the Specht module relations.

To prove the second part of the lemma, we note that $w_{\mathtt{T}}^{-1}(i) < w_{\mathtt{T}}^{-1}(i+1)$. This is easily seen since $w_{\mathtt{T}}^{-1}(j)$ is the number that occupies the same node in $\mathtt{T}_\lambda$ that j occupies in $\mathtt{T}$. Hence if $s_{i_1}s_{i_2}\ldots s_{i_r}$ is a reduced expression for $w_{\mathtt{T}}$, then $s_i s_{i_1}s_{i_2}\ldots s_{i_r}$ is a reduced expression for $s_i w_{\mathtt{T}}$. So $\psi_i v_{\mathtt{T}} = v_{\mathtt{U}}$. □

Theorem 3.7. *If* a *and* b *are both even, then* $\mathrm{End}_{\mathcal{H}}(\mathrm{S}_\lambda)$ *is one-dimensional. In particular,* S_λ *is indecomposable.*

Proof. Suppose $f \in \mathrm{End}_{\mathcal{H}}(\mathrm{S}_\lambda)$. Then by the above remark,

$$f(z_\lambda) = \sum_{\mathtt{T} \in \mathrm{Dom}(\lambda)} \alpha_{\mathtt{T}} v_{\mathtt{T}} \quad \text{for some } \alpha_{\mathtt{T}} \in \mathbb{F}.$$

Then by Lemma 3.6, acting on the left by ψ_{n-1} annihilates all $v_{\mathtt{T}}$ for tableaux $\mathtt{T}$ which do not have $D_{\frac{n-2}{2}}$ in their leg. Now, for any $\mathtt{T} \in \mathrm{Dom}(\lambda)$ which does have $D_{\frac{n-2}{2}}$ in the leg, Lemma 3.6 gives us that $\psi_{n-1}v_{\mathtt{T}} = v_{s_{n-1}\mathtt{T}} \neq 0$. Since $\psi_{n-1}f(z_\lambda) = 0$, we must have $\alpha_{\mathtt{T}} = 0$ for all $\mathtt{T}$ which have $D_{\frac{n-2}{2}}$ in the leg.

In this way, we act on $f(z_\lambda)$ by ψ_{n+1-2i} for $i = 1, 2, \ldots, (a-2)/2$, to annihilate all $v_{\mathtt{T}}$ for tableaux $\mathtt{T}$ which do not have $D_{\frac{n-2i}{2}}$ in the leg. At each step, we apply Lemma 3.6 to deduce that $\alpha_{\mathtt{T}} = 0$ if $\mathtt{T}$ has $D_{\frac{n-2i}{2}}$ in the leg.

Therefore $f(z_\lambda) = \alpha z_\lambda$ for some $\alpha \in \mathbb{F}$ and the result follows. □

Next, we look at the case where a and b are both odd.

Theorem 3.8. *If a and b are both odd, then $\mathrm{End}_{\mathcal{H}}(\mathrm{S}_\lambda)$ is one-dimensional. In particular, S_λ is indecomposable.*

Proof. The result follows from Theorem 3.7 by application of Theorem 1.41. □

3.4 KLR actions on $\mathscr{D}$ when n is odd

When n is odd, much more work must be done. By Theorem 1.41, we can assume throughout this section that $b < n/2$.

Using Theorem 3.2, we can focus on the case where a is odd and b is even, as it is slightly easier to work with. The case where a is even and b is odd will then follow.

Recall that $\mathscr{D} = \{v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Dom}(\lambda)\}$ is a basis of $e(i_\lambda)\mathrm{S}_\lambda$. At this point we introduce some new notation which is much needed to keep things tidy!

Definition 3.9. We define $\Psi_j := \psi_j\psi_{j+1}\psi_{j-1}\psi_j$. For $3 \leqslant x \leqslant y \leqslant n-2$ two odd integers, we then define:

$$\Psi\overset{y}{\underset{x}{\downarrow}} := \Psi_y\Psi_{y-2}\ldots\Psi_x \quad \text{and} \quad \Psi\overset{y}{\underset{x}{\uparrow}} := \Psi_x\Psi_{x+2}\ldots\Psi_y.$$

If $y < x$ we consider both of the above defined terms to be the identity element of our field.

Remark. Given some $\mathtt{T} \in \mathrm{Dom}(\lambda)$, let $2d$ be the number of entries in the leg of $\mathtt{T}$ which differ from the entries in the corresponding nodes of $\mathtt{T}_\lambda$. Notice that these will consist of the final d dominoes in the leg, since $\mathtt{T} \in \mathrm{Std}(\lambda)$.

Let $j'_1, \ldots, j'_d$ be the odd numbers (in ascending order) in the d dominoes in the leg of $\mathtt{T}$ which differ from the corresponding entries in $\mathtt{T}_\lambda$ and define $j_i := j'_i - 2$ for each i. For example, if $\lambda = (7, 1^6)$ then

$$\mathtt{T}_\lambda = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 8 & 9 & 10 & 11 & 12 & 13 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array} . \quad \text{Let} \quad \mathtt{T} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 4 & 5 & 6 & 7 & 10 & 11 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 8 \\ \cline{1-1} 9 \\ \cline{1-1} 12 \\ \cline{1-1} 13 \\ \cline{1-1} \end{array} .$$

Then $d = 2$ and we see that $j_1 = 7$ and $j_2 = 11$.

Now, we can see that $v_\mathtt{T}$ can be written as the reduced expression

$$\Psi \overset{j_1}{\underset{b+3-2d}{\downarrow}} \Psi \overset{j_2}{\underset{b+5-2d}{\downarrow}} \ldots \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda .$$

We will refer to this as the *normal form* for $v_\mathtt{T}$. Notice that $j_{i+1} > j_i$ for all $i = 1, \ldots, d-1$. It will be useful to note that if $v_\mathtt{T} \in \mathscr{D}$ is in our normal form, then any expression obtained from it by deleting Ψ terms from the left is also an element in $\mathscr{D}$.

Definition 3.10. Let $\mathtt{T} \in \mathrm{Dom}(\lambda)$. We define the length $r(\mathtt{T})$ of $\mathtt{T}$ to be the number of Ψ terms in the normal form of $v_\mathtt{T}$.

In the next three results, we examine the actions of the generators of $\mathcal{H}$ on the elements of $\mathscr{D}$.

Lemma 3.11.

$$e(i_\lambda)\Psi_j = \Psi_j e(i_\lambda) \qquad \textit{for all } j,$$

$$y_k \Psi_j = \Psi_j y_k \qquad \textit{for all } k \geqslant j+3 \textit{ and for all } k \leqslant j-2,$$

$$\psi_k \Psi_j = \Psi_j \psi_k \qquad \textit{for all } k \geqslant j+3 \textit{ and for all } k \leqslant j-3.$$

Proof. Clear from the definition of Ψ_j and the defining relations. □

Proposition 3.12. *Suppose* $\mathtt{T} \in \mathrm{Dom}(\lambda)$. *Then*

$$y_k v_{\mathtt{T}} = 0 \textit{ for all } k; \qquad (A)$$

$$\psi_k v_{\mathtt{T}} = 0 \textit{ for all even } k; \qquad (B)$$

$$\psi_1 v_{\mathtt{T}} = 0. \qquad (C)$$

Proof. Let (A_r) denote the statement that (A) holds for all $\mathtt{T}$ with $r(\mathtt{T}) = r$, and define (B_r) similarly. We first prove (A_r) and (B_r) simultaneously, by induction on r.

First we must show that (A_0) and (B_0) hold. In this case, $v_{\mathtt{T}} = z_\lambda$ and the defining relations give our result immediately.

Now, let $v_{\mathtt{T}} = \Psi \underset{b+3-2d}{\overset{j_1}{\downarrow}} \Psi \underset{b+5-2d}{\overset{j_2}{\downarrow}} \dots \Psi \underset{b+1}{\overset{j_d}{\downarrow}} z_\lambda$ be in normal form for some d, and define

$$v_{\mathtt{T}(2)} := \Psi \underset{b+5-2d}{\overset{j_2}{\downarrow}} \dots \Psi \underset{b+1}{\overset{j_d}{\downarrow}} z_\lambda.$$

We will show that $(A_{r-1})\&(B_{r-1}) \Rightarrow (A_r)$. We split our problem into 5 cases:

1. $k = j_1 + 2$,
2. $k = j_1 + 1$,
3. $k = j_1$,
4. $k = j_1 - 1$,
5. All other k.

We can now solve each case quite simply!

1. $$\begin{aligned} y_{j_1+2} v_{\mathtt{T}} &= \psi_{j_1}\psi_{j_1-1}(y_{j_1+2}\psi_{j_1+1} e(s_{j_1}\cdot i_\lambda))\psi_{j_1} \Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathtt{T}(2)} \\ &= \psi_{j_1}\psi_{j_1-1}(\psi_{j_1+1} y_{j_1+1} + 1)\psi_{j_1} \Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathtt{T}(2)} \\ &= \Psi_{j_1} \underbrace{y_{j_1} \Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathtt{T}(2)}}_{=0 \text{ by } (A_{r-1})} + \psi_{j_1}\psi_{j_1-1}\psi_{j_1} \Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathtt{T}(2)} \end{aligned}$$

$$= (\psi_{j_1-1}\psi_{j_1}\psi_{j_1-1} - y_{j_1-1} + 2y_{j_1} - y_{j_1+1})\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= \psi_{j_1-1}\psi_{j_1}\underbrace{\psi_{j_1-1}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}}_{=0 \text{ by } (B_{r-1})} - \underbrace{y_{j_1-1}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}}_{=0 \text{ by } (A_{r-1})}$$

$$+ 2\underbrace{y_{j_1}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}}_{=0 \text{ by } (A_{r-1})} - \underbrace{y_{j_1+1}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}}_{=0 \text{ by } (A_{r-1})}$$

$$= 0.$$

2.

$$y_{j_1+1}v_{\mathrm{T}} = (y_{j_1+1}\psi_{j_1}e(s_{j_1}\cdot i_\lambda))\psi_{j_1+1}\psi_{j_1-1}\psi_{j_1}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= \psi_{j_1}\psi_{j_1+1}(y_{j_1}\psi_{j_1-1}e(s_{j_1}\cdot i_\lambda))\psi_{j_1}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= \psi_{j_1}\psi_{j_1+1}(\psi_{j_1-1}y_{j_1-1} + 1)\psi_{j_1}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= \Psi_{j_1}\underbrace{y_{j_1-1}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}}_{=0 \text{ by } (A_{r-1})} + \psi_{j_1}\psi_{j_1+1}\psi_{j_1}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= (\psi_{j_1+1}\psi_{j_1}\psi_{j_1+1} + y_{j_1} - 2y_{j_1+1} + y_{j_1+2})\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= 0 \quad \text{by } (A_{r-1}) \text{ and } (B_{r-1}).$$

3.

$$y_{j_1}v_{\mathrm{T}} = (y_{j_1}\psi_{j_1}e(s_{j_1}\cdot i_\lambda))\psi_{j_1+1}\psi_{j_1-1}\psi_{j_1}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= \psi_{j_1}(y_{j_1+1}\psi_{j_1+1}e(s_{j_1}\cdot i_\lambda))\psi_{j_1-1}\psi_{j_1}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= \psi_{j_1}(\psi_{j_1+1}y_{j_1+2} - 1)\psi_{j_1-1}\psi_{j_1}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= \Psi_{j_1}\underbrace{y_{j_1+2}\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}}_{=0 \text{ by } (A_{r-1})}$$

$$- (\psi_{j_1-1}\psi_{j_1}\psi_{j_1-1} - y_{j_1-1} + 2y_{j_1} - y_{j_1+1})\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= 0 \quad \text{by } (A_{r-1}) \text{ and } (B_{r-1}).$$

4. $$\begin{aligned} y_{j_1-1}v_{\mathrm{T}} &= \psi_{j_1}\psi_{j_1+1}(y_{j_1-1}\psi_{j_1-1}e(s_{j_1}\cdot i_\lambda))\psi_{j_1}\Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathrm{T}(2)} \\ &= \psi_{j_1}\psi_{j_1+1}(\psi_{j_1-1}y_{j_1}-1)\psi_{j_1}\Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathrm{T}(2)} \\ &= \Psi_{j_1}\underbrace{y_{j_1+1}\Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathrm{T}(2)}}_{=0 \text{ by } (A_{r-1})} \\ &\quad -(\psi_{j_1+1}\psi_{j_1}\psi_{j_1+1}+y_{j_1}-2y_{j_1+1}+y_{j_1+2})\Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathrm{T}(2)} \\ &= 0 \quad \text{by } (A_{r-1}) \text{ and } (B_{r-1}). \end{aligned}$$

5. Now suppose $k \neq j_1+2, j_1+1, j_1$ or j_1-1. Then

$$\begin{aligned} y_k v_{\mathrm{T}} &= \Psi_{j_1} y_k \Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathrm{T}(2)} \quad \text{by Lemma 3.11} \\ &= 0 \quad \text{by } (A_{r-1}). \end{aligned}$$

Next, we show that $(A_{r-1})\&(B_{r-1}) \Rightarrow (B_r)$. Once again we split this into the following cases:

1. $k = j_1+1$,
2. $k = j_1-1$,
3. All other k.

1. $$\begin{aligned} \psi_{j_1+1}v_{\mathrm{T}} &= (\psi_{j_1+1}\psi_{j_1}\psi_{j_1+1}e(s_{j_1}\cdot i_\lambda))\psi_{j_1-1}\psi_{j_1}\Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathrm{T}(2)} \\ &= \psi_{j_1}\psi_{j_1+1}(\psi_{j_1}\psi_{j_1-1}\psi_{j_1}e(i_\lambda))\Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathrm{T}(2)} \\ &= \psi_{j_1}\psi_{j_1+1}(\psi_{j_1-1}\psi_{j_1}\psi_{j_1-1}-y_{j_1-1}+2y_{j_1}-y_{j_1+1})\Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathrm{T}(2)} \\ &= 0 \quad \text{by } (A_{r-1}) \text{ and } (B_{r-1}). \end{aligned}$$

2. $$\begin{aligned} \psi_{j_1-1}v_{\mathrm{T}} &= (\psi_{j_1-1}\psi_{j_1}\psi_{j_1-1}e(s_{j_1}\cdot i_\lambda))\psi_{j_1+1}\psi_{j_1}\Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathrm{T}(2)} \\ &= \psi_{j_1}\psi_{j_1-1}(\psi_{j_1}\psi_{j_1+1}\psi_{j_1}e(i_\lambda))\Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} v_{\mathrm{T}(2)} \end{aligned}$$

$$= \psi_{j_1}\psi_{j_1-1}(\psi_{j_1+1}\psi_{j_1}\psi_{j_1+1} + y_{j_1} - 2y_{j_1+1} + y_{j_1+2})\Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= 0 \quad \text{by } (A_{r-1}) \text{ and } (B_{r-1}).$$

3. Now suppose k is even but $k \neq j_1 + 1$ or $j_1 - 1$. Then

$$\psi_k v_{\mathrm{T}} = \Psi_{j_1}\psi_k \Psi\overset{j_1-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \quad \text{by Lemma 3.11}$$

$$= 0 \quad \text{by } (B_{r-1}).$$

And so, our results follow.

Now, we prove (C). If $d < b/2$, then Ψ_3 does not occur in v_{T}, and so ψ_1 commutes with all Ψ terms in v_{T} and the result is clear. So suppose $d = b/2$. Then $v_{\mathrm{T}} = \Psi\overset{j_1}{\underset{3}{\downarrow}}\ \Psi\overset{j_2}{\underset{5}{\downarrow}} \dots \Psi\overset{j_d}{\underset{b+1}{\downarrow}}\ z_\lambda$. It's easy to see that

$$\psi_1\psi_2\dots\psi_i\Psi\overset{j_{(i+2d-b+1)/2}}{\underset{i+2}{\downarrow}} = \Psi\overset{j_{(i+2d-b+1)/2}}{\underset{i+4}{\downarrow}}\ \psi_{i+2}\psi_{i+3}(\psi_1\psi_2\dots\psi_{i+2}).$$

Applying this for $i = 1, 3, \dots, b-1$ in turn, we obtain

$$\psi_1 v_{\mathrm{T}} = \Psi\overset{j_1}{\underset{5}{\downarrow}}\ \Psi\overset{j_2}{\underset{7}{\downarrow}} \dots \Psi\overset{j_d}{\underset{b+3}{\downarrow}}\ \psi_3\psi_4\dots\psi_{b+2}\psi_1\psi_2\dots\psi_{b+1}z_\lambda,$$

which is zero in view of the Garnir relation $\psi_1\psi_2\dots\psi_{b+1}z_\lambda = 0$. □

Alternative proof. A shorter but less direct proof of (A) and (B) can be given using the grading on $\mathcal{H}$ and S_λ, closely mimicking the proof of [29, Lemma 4.4]. We notice that every domino tableau has codegree $b/2$, since each domino in the leg contributes precisely $+1$ to the codegree. So $e(i_\lambda)\mathrm{S}_\lambda$ is homogeneous of degree $b/2$. Now, $y_k v_{\mathrm{T}} \in e(i_\lambda)\mathrm{S}_\lambda$, but $\deg(y_k v_{\mathrm{T}}) = \deg(y_k) + \deg(v_{\mathrm{T}}) = 2 + b/2$. Hence $y_k v_{\mathrm{T}} = 0$.

Similarly, if k is even, $\psi_k v_{\mathrm{T}} \in e(s_k i_\lambda)\mathrm{S}_\lambda = 0$, since no standard tableau has residue sequence $s_k i_\lambda$. To see this, we may build up any standard λ-tableau and note that once we have placed $1, 2, \dots, k-1$ of residues $0, 1, \dots, 0$ respectively, the next entry must have residue 1. □

Lemma 3.13. *Suppose j is odd and* $\mathtt{T} \in \mathrm{Dom}(\lambda)$. *Then*

1. $\psi_j \Psi_j v_{\mathtt{T}} = -2\psi_j v_{\mathtt{T}}$,
2. $\psi_j \Psi_{j+2} \Psi_j v_{\mathtt{T}} = \psi_j v_{\mathtt{T}}$,
3. $\Psi_j \psi_{j+2} v_{\mathtt{T}} = 0$,
4. $\psi_j \Psi_{j-2} \Psi_j v_{\mathtt{T}} = \psi_j v_{\mathtt{T}}$,
5. $\Psi_j \psi_{j-2} v_{\mathtt{T}} = 0$.

Proof.

1.
$$\begin{aligned}
\psi_j \Psi_j e(i_\lambda) v_{\mathtt{T}} &= (\psi_j^2 e(s_j \cdot i_\lambda))\psi_{j+1}\psi_{j-1}\psi_j v_{\mathtt{T}} \\
&= (-y_j^2 - y_{j+1}^2 + 2y_j y_{j+1})\psi_{j+1}\psi_{j-1}\psi_j v_{\mathtt{T}} \\
&= -\psi_{j+1} y_j (y_j \psi_{j-1} e(s_j \cdot i_\lambda))\psi_j v_{\mathtt{T}} \\
&\quad - y_{j+1}(y_{j+1}\psi_{j+1} e(s_j \cdot i_\lambda))\psi_{j-1}\psi_j v_{\mathtt{T}} \\
&\quad + 2y_j (y_{j+1}\psi_{j+1} e(s_j \cdot i_\lambda))\psi_{j-1}\psi_j v_{\mathtt{T}} \\
&= -\psi_{j+1} y_j (\psi_{j-1} y_{j-1} + 1)\psi_j v_{\mathtt{T}} \\
&\quad - y_{j+1}(\psi_{j+1} y_{j+2} - 1)\psi_{j-1}\psi_j v_{\mathtt{T}} \\
&\quad + 2y_j (\psi_{j+1} y_{j+2} - 1)\psi_{j-1}\psi_j v_{\mathtt{T}} \\
&= -\psi_{j+1} y_j \psi_j v_{\mathtt{T}} + \psi_{j-1} y_{j+1}\psi_j v_{\mathtt{T}} - 2(y_j \psi_{j-1} e(s_j \cdot i_\lambda))\psi_j v_{\mathtt{T}} \\
&= -\psi_{j+1}\psi_j y_{j+1} v_{\mathtt{T}} + \psi_{j-1}\psi_j y_j v_{\mathtt{T}} - 2(\psi_{j-1} y_{j-1} + 1)\psi_j v_{\mathtt{T}} \\
&= -2\psi_j v_{\mathtt{T}}.
\end{aligned}$$

2–5. We have

$$\begin{aligned}
\psi_j \Psi_{j+2} \Psi_j e(i_\lambda) v_{\mathtt{T}} &= \psi_{j+2}\psi_{j+3}(\psi_j \psi_{j+1}\psi_j e(s_{j+2} \cdot s_j \cdot i_\lambda))\psi_{j+2}\psi_{j+1}\psi_{j-1}\psi_j \\
&= \psi_{j+2}\psi_{j+3}(\psi_{j+1}\psi_j \psi_{j+1} + y_j - 2y_{j+1} + y_{j+2})\cdot \\
&\quad \psi_{j+2}\psi_{j+1}\psi_{j-1}\psi_j v_{\mathtt{T}}
\end{aligned}$$

$$
\begin{aligned}
&= \psi_{j+2}\psi_{j+3}\psi_{j+1}\psi_j(\psi_{j+1}\psi_{j+2}\psi_{j+1}e(s_j\cdot i_\lambda))\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&\quad + \psi_{j+2}\psi_{j+3}\psi_{j+2}\psi_{j+1}(y_j\psi_{j-1}e(s_j\cdot i_\lambda))\psi_j v_{\mathrm{T}}\\
&\quad - 2\psi_{j+2}\psi_{j+3}\psi_{j+2}(y_{j+1}\psi_{j+1}e(s_j\cdot i_\lambda))\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&\quad + \psi_{j+2}\psi_{j+3}(y_{j+2}\psi_{j+2}e(s_j\cdot i_\lambda))\psi_{j+1}\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&= \psi_{j+2}\psi_{j+3}\psi_{j+1}\psi_j\psi_{j+2}\psi_{j+1}\psi_{j+2}\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&\quad + \psi_{j+2}\psi_{j+3}\psi_{j+2}\psi_{j+1}(\psi_{j-1}y_{j-1}+1)\psi_j v_{\mathrm{T}}\\
&\quad - 2\psi_{j+2}\psi_{j+3}\psi_{j+2}(\psi_{j+1}y_{j+2}-1)\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&\quad + \psi_{j+2}\psi_{j+3}\psi_{j+2}y_{j+3}\psi_{j+1}\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&= \Psi_{j+2}\Psi_j\psi_{j+2}v_{\mathrm{T}} + \psi_{j+2}\psi_{j+3}\psi_{j+2}\psi_{j+1}\psi_{j-1}\psi_j \underbrace{y_{j-1}v_{\mathrm{T}}}_{=0}\\
&\quad + (\psi_{j+2}\psi_{j+3}\psi_{j+2}e(s_j\cdot i_\lambda))\psi_{j+1}\psi_j v_{\mathrm{T}}\\
&\quad - 2\psi_{j+2}\psi_{j+3}\psi_{j+2}\psi_{j+1}\psi_{j-1}\psi_j \underbrace{y_{j+2}v_{\mathrm{T}}}_{=0}\\
&\quad + 2(\psi_{j+2}\psi_{j+3}\psi_{j+2}e(s_j\cdot i_\lambda))\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&\quad + \psi_{j+2}\psi_{j+3}\psi_{j+2}\psi_{j+1}\psi_{j-1}\psi_j \underbrace{y_{j+3}v_{\mathrm{T}}}_{=0}\\
&= \Psi_{j+2}\Psi_j\psi_{j+2}v_{\mathrm{T}} + (\psi_{j+3}\psi_{j+2}\psi_{j+3} + y_{j+2} - 2y_{j+3} + y_{j+4})\cdot\\
&\quad \psi_{j+1}\psi_j v_{\mathrm{T}}\\
&\quad + 2(\psi_{j+3}\psi_{j+2}\psi_{j+3} + y_{j+2} - 2y_{j+3} + y_{j+4})\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&= \Psi_{j+2}\Psi_j\psi_{j+2}v_{\mathrm{T}} + \psi_{j+3}\psi_{j+2}\psi_{j+1}\psi_j \underbrace{\psi_{j+3}v_{\mathrm{T}}}_{=0}\\
&\quad + (y_{j+2}\psi_{j+1}e(s_j\cdot i_\lambda))\psi_j v_{\mathrm{T}} + 2\psi_{j+3}\psi_{j+2}\psi_{j-1}\psi_j \underbrace{\psi_{j+3}v_{\mathrm{T}}}_{=0}\\
&= \Psi_{j+2}\Psi_j\psi_{j+2}v_{\mathrm{T}} + (\psi_{j+1}y_{j+1}+1)\psi_j v_{\mathrm{T}}\\
&= \Psi_{j+2}\Psi_j\psi_{j+2}v_{\mathrm{T}} + \psi_j v_{\mathrm{T}} + \psi_{j+1}y_{j+1}\psi_j v_{\mathrm{T}}\\
&= \Psi_{j+2}\Psi_j\psi_{j+2}v_{\mathrm{T}} + \psi_j v_{\mathrm{T}} + \psi_{j+1}\psi_j \underbrace{y_j v_{\mathrm{T}}}_{=0}.
\end{aligned}
$$

We also have

$$\begin{aligned}
\psi_j\Psi_{j-2}\Psi_j e(i_\lambda)v_{\mathrm{T}} &= \psi_{j-2}\psi_{j-3}(\psi_j\psi_{j-1}\psi_j e(s_{j-2}\cdot s_j\cdot i_\lambda))\psi_{j-2}\psi_{j+1}\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&= \psi_{j-2}\psi_{j-3}(\psi_{j-1}\psi_j\psi_{j-1} - y_{j-1} + 2y_j - y_{j+1})\cdot\\
&\quad \psi_{j-2}\psi_{j+1}\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&= \psi_{j-2}\psi_{j-3}\psi_{j-1}\psi_j(\psi_{j-1}\psi_{j-2}\psi_{j-1}e(s_j\cdot i_\lambda))\psi_{j+1}\psi_j v_{\mathrm{T}}\\
&\quad - \psi_{j-2}\psi_{j-3}(y_{j-1}\psi_{j-2}e(s_j\cdot i_\lambda))\psi_{j+1}\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&\quad + 2\psi_{j-2}\psi_{j-3}\psi_{j-2}\psi_{j+1}(y_j\psi_{j-1}e(s_j\cdot i_\lambda))\psi_j v_{\mathrm{T}}\\
&\quad - \psi_{j-2}\psi_{j-3}\psi_{j-2}(y_{j+1}\psi_{j+1}e(s_j\cdot i_\lambda))\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&= \psi_{j-2}\psi_{j-3}\psi_{j-1}\psi_j\psi_{j-2}\psi_{j-1}\psi_{j-2}\psi_{j+1}\psi_j v_{\mathrm{T}}\\
&\quad - \psi_{j-2}\psi_{j-3}\psi_{j-2}y_{j-2}\psi_{j+1}\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&\quad + 2\psi_{j-2}\psi_{j-3}\psi_{j-2}\psi_{j+1}(\psi_{j-1}y_{j-1} + 1)\psi_j v_{\mathrm{T}}\\
&\quad - \psi_{j-2}\psi_{j-3}\psi_{j-2}(\psi_{j+1}y_{j+2} - 1)\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&= \Psi_{j-2}\Psi_j\psi_{j-2}v_{\mathrm{T}} - \psi_{j-2}\psi_{j-3}\psi_{j-2}\psi_{j+1}\psi_{j-1}\psi_j\underbrace{y_{j-2}v_{\mathrm{T}}}_{=0}\\
&\quad + 2(\psi_{j-2}\psi_{j-3}\psi_{j-2}e(s_j\cdot i_\lambda))\psi_{j+1}\psi_j v_{\mathrm{T}}\\
&\quad + (\psi_{j-2}\psi_{j-3}\psi_{j-2}e(s_j\cdot i_\lambda))\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&= \Psi_{j-2}\Psi_j\psi_{j-2}v_{\mathrm{T}}\\
&\quad + 2(\psi_{j-3}\psi_{j-2}\psi_{j-3} - y_{j-3} + 2y_{j-2} - y_{j-1})\psi_{j+1}\psi_j v_{\mathrm{T}}\\
&\quad + (\psi_{j-3}\psi_{j-2}\psi_{j-3} - y_{j-3} + 2y_{j-2} - y_{j-1})\psi_{j-1}\psi_j v_{\mathrm{T}}\\
&= \Psi_{j-2}\Psi_j\psi_{j-2}v_{\mathrm{T}} + 2(0) - (y_{j-1}\psi_{j-1}e(s_j\cdot i_\lambda))\psi_j v_{\mathrm{T}}\\
&= \Psi_{j-2}\Psi_j\psi_{j-2}v_{\mathrm{T}} - (\psi_{j-1}y_j - 1)\psi_j v_{\mathrm{T}}\\
&= \Psi_{j-2}\Psi_j\psi_{j-2}v_{\mathrm{T}} - 0 + \psi_j v_{\mathrm{T}}.
\end{aligned}$$

So we have

$$\psi_j\Psi_{j+2}\Psi_j v_{\mathrm{T}} = \psi_j v_{\mathrm{T}} + \Psi_{j+2}\Psi_j\psi_{j+2}v_{\mathrm{T}} \qquad (*)$$

$$\psi_j\Psi_{j-2}\Psi_j v_{\mathrm{T}} = \psi_j v_{\mathrm{T}} + \Psi_{j-2}\Psi_j\psi_{j-2}v_{\mathrm{T}}. \qquad (**)$$

Now all four statements will follow if we can show that 3 and 5 hold. We will proceed by proving both simultaneously by induction on $r(\mathtt{T})$. That is, we will prove that

$$\Psi_j \psi_{j+2} v_{\mathtt{T}} = 0 \text{ for any odd } j \text{ and } r(\mathtt{T}) = r, \qquad (A_r)$$

$$\Psi_j \psi_{j-2} v_{\mathtt{T}} = 0 \text{ for any odd } j \text{ and } r(\mathtt{T}) = r, \qquad (B_r)$$

by simultaneous induction on r.

First, we prove that (A_r) follows if (A_s) and (B_s) hold for all $s < r$.

(A_0) is clearly true. We have $\Psi_j \psi_{j+2} z_\lambda = \psi_j \psi_{j+1} \psi_{j-1} \psi_{j+2} \psi_j z_\lambda = 0$ since at least one of ψ_j, ψ_{j+2} must annihilate z_λ.

Now let $r > 0$. Suppose $v_{\mathtt{T}} = \Psi \underset{b+3-2d}{\overset{j_1}{\downarrow}} \Psi \underset{b+5-2d}{\overset{j_2}{\downarrow}} \ldots \Psi \underset{b+1}{\overset{j_d}{\downarrow}} z_\lambda$ is in normal form and define $v_{\mathtt{T}'} := \Psi \underset{b+3-2d}{\overset{j_1-2}{\downarrow}} \Psi \underset{b+5-2d}{\overset{j_2}{\downarrow}} \ldots \Psi \underset{b+1}{\overset{j_d}{\downarrow}} z_\lambda$.
If $j_1 \geqslant j+6$ or $j_1 \leqslant j-4$, then we clearly have $\Psi_j \psi_{j+2} v_{\mathtt{T}} = \Psi_{j_1} \Psi_j \psi_{j+2} v_{\mathtt{T}'}$ and our result follows by (A_{r-1}). So we break our proof up for the remaining four possibilities.

(a) Suppose $j_1 = j+4$. We will write $v_{\mathtt{T}(2)} := \Psi \underset{b+5-2d}{\overset{j_2}{\downarrow}} \ldots \Psi \underset{b+1}{\overset{j_d}{\downarrow}} z_\lambda$. If $b+3-2d = j+4$ also, we have

$$\begin{aligned}\Psi_j \psi_{j+2} v_{\mathtt{T}} &= \Psi_j \psi_{j+2} \Psi_{j+4} v_{\mathtt{T}(2)} \\ &= 0,\end{aligned}$$

as we have a ψ_j which commutes with everything to its right, given that the lowest indexed Ψ-term in $v_{\mathtt{T}(2)}$ is Ψ_{j+6}.

If $b+3-2d < j+4$, by $(*)$ we have

$$\begin{aligned}\Psi_j(\psi_{j+2}\Psi_{j+4}\Psi_{j+2})\Psi \underset{b+3-2d}{\overset{j}{\downarrow}} v_{\mathtt{T}(2)} &= \Psi_j(\psi_{j+2} + \Psi_{j+4}\Psi_{j+2}\psi_{j+4})\Psi \underset{b+3-2d}{\overset{j}{\downarrow}} v_{\mathtt{T}(2)} \\ &= 0 \quad \text{by } (A_s) \text{ for some } s < r, \\ &\qquad \text{as } \Psi \underset{b+3-2d}{\overset{j}{\downarrow}} v_{\mathtt{T}(2)} \in \mathscr{D}.\end{aligned}$$

(b) Suppose $j_1 = j+2$. Then we have

$$\begin{aligned}\Psi_j\psi_{j+2}\Psi_{j+2}v_{\mathtt{T}'} &= -2\Psi_j\psi_{j+2}v_{\mathtt{T}'} \text{ by part 1,}\\ &= 0 \quad \text{by } (A_{r-1}), \text{ as } r(\mathtt{T}') = r-1.\end{aligned}$$

(c) Suppose $j_1 = j$. Then we have

$$\begin{aligned}\Psi_j\psi_{j+2}\Psi_j v_{\mathtt{T}'} &= -2\Psi_j\psi_{j+2}v_{\mathtt{T}'} \text{ by part 1,}\\ &= 0 \text{ by } (A_{r-1}).\end{aligned}$$

(d) Suppose $j_1 = j-2$. We will write $v_{\mathtt{T}(3)} := \Psi\overset{j_3}{\underset{b+7-2d}{\downarrow}} \dots \Psi\overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda$. Here, we must divide into further subcases.

i. Suppose $j_2 \geqslant j+4$. Then we have

$$\begin{aligned}\Psi_j\psi_{j+2}v_{\mathtt{T}} &= \Psi_j\psi_{j+2}\Psi\overset{j-2}{\underset{b+3-2d}{\downarrow}}\ \Psi\overset{j_2}{\underset{j+6}{\downarrow}}\Psi_{j+4}\Psi_{j+2}\Psi\overset{j}{\underset{b+5-2d}{\downarrow}} v_{\mathtt{T}(3)}\\ &= \Psi\overset{j_2}{\underset{j+6}{\downarrow}}\ \Psi_j\Psi\overset{j-2}{\underset{b+3-2d}{\downarrow}}(\psi_{j+2}\Psi_{j+4}\Psi_{j+2})\Psi\overset{j}{\underset{b+5-2d}{\downarrow}} v_{\mathtt{T}(3)}\\ &= \Psi\overset{j_2}{\underset{j+6}{\downarrow}}\ \Psi_j\Psi\overset{j-2}{\underset{b+3-2d}{\downarrow}}(\psi_{j+2}+\Psi_{j+4}\Psi_{j+2}\psi_{j+4})\Psi\overset{j}{\underset{b+5-2d}{\downarrow}} v_{\mathtt{T}(3)} \text{ by } (*)\\ &= \Psi\overset{j_2}{\underset{j+6}{\downarrow}}\ \Psi_j\psi_{j+2}\Psi\overset{j-2}{\underset{b+3-2d}{\downarrow}}\ \Psi\overset{j}{\underset{b+5-2d}{\downarrow}}\ v_{\mathtt{T}(3)} + 0\\ &\qquad \text{by } (A_s) \text{ for some } s<r, \text{ as } \Psi\overset{j}{\underset{b+5-2d}{\downarrow}} v_{\mathtt{T}(3)} \in \mathscr{D},\\ &= 0 \text{ by } (A_{s'}) \text{ for some } s'<r, \text{ as } \Psi\overset{j-2}{\underset{b+3-2d}{\downarrow}}\ \Psi\overset{j}{\underset{b+5-2d}{\downarrow}}\ v_{\mathtt{T}(3)} \in \mathscr{D}.\end{aligned}$$

ii. Suppose $j_2 = j+2$. Then we have

$$\begin{aligned}\Psi_j\psi_{j+2}v_{\mathtt{T}} &= \Psi_j\psi_{j+2}\Psi\overset{j-2}{\underset{b+3-2d}{\downarrow}}\ \Psi_{j+2}\Psi\overset{j}{\underset{b+5-2d}{\downarrow}}\ v_{\mathtt{T}(3)}\\ &= \Psi_j\Psi\overset{j-2}{\underset{b+3-2d}{\downarrow}}\ \psi_{j+2}\Psi_{j+2}\Psi\overset{j}{\underset{b+5-2d}{\downarrow}}\ v_{\mathtt{T}(3)}\\ &= -2\Psi_j\Psi\overset{j-2}{\underset{b+3-2d}{\downarrow}}\ \psi_{j+2}\Psi\overset{j}{\underset{b+5-2d}{\downarrow}}\ v_{\mathtt{T}(3)} \text{ by part 1,}\\ &= -2\Psi_j\psi_{j+2}\Psi\overset{j-2}{\underset{b+3-2d}{\downarrow}}\ \Psi\overset{j}{\underset{b+5-2d}{\downarrow}}\ v_{\mathtt{T}(3)}\\ &= 0 \text{ by } (A_s) \text{ for some } s<r, \text{ as } \Psi\overset{j-2}{\underset{b+3-2d}{\downarrow}}\ \Psi\overset{j}{\underset{b+5-2d}{\downarrow}}\ v_{\mathtt{T}(3)} \in \mathscr{D}.\end{aligned}$$

iii. Suppose $j_2 = j$. Then we have

$$\Psi_j \psi_{j+2} v_{\mathrm{T}} = \Psi_j \psi_{j+2} \Psi \underset{b+3-2d}{\overset{j-2}{\downarrow}} \Psi_j \Psi \underset{b+5-2d}{\overset{j-2}{\downarrow}} v_{\mathrm{T}(3)}$$

$$= \underbrace{\psi_j \psi_{j+1} \psi_{j-1} \psi_{j+2} \psi_{j-2}}_{=:\psi_*} (\psi_j \psi_{j-1} \psi_j e(s_{j-2} \cdot s_j \cdot i_\lambda)) \cdot$$

$$\psi_{j-3} \psi_{j-2} \psi_{j+1} \psi_{j-1} \psi_j \underbrace{\Psi \underset{b+3-2d}{\overset{j-4}{\downarrow}} \Psi \underset{b+5-2d}{\overset{j-2}{\downarrow}} v_{\mathrm{T}(3)}}_{=:v_{\mathrm{T}''} \in \mathscr{D}}$$

$$= \psi_* (\psi_{j-1} \psi_j \psi_{j-1} - y_{j-1} + 2y_j - y_{j+1}) \cdot$$

$$\psi_{j-3} \psi_{j-2} \psi_{j+1} \psi_{j-1} \psi_j v_{\mathrm{T}''}$$

$$= \psi_* \psi_{j-1} \psi_j \psi_{j-3} (\psi_{j-1} \psi_{j-2} \psi_{j-1} e(s_j \cdot i_\lambda)) \psi_{j+1} \psi_j v_{\mathrm{T}''}$$

$$- \psi_* \psi_{j-3} (y_{j-1} \psi_{j-2} e(s_j \cdot i_\lambda)) \psi_{j+1} \psi_{j-1} \psi_j v_{\mathrm{T}''}$$

$$+ 2\psi_* \psi_{j-3} \psi_{j-2} \psi_{j+1} (y_j \psi_{j-1} e(s_j \cdot i_\lambda)) \psi_j v_{\mathrm{T}''}$$

$$- \psi_* \psi_{j-3} \psi_{j-2} (y_{j+1} \psi_{j+1} e(s_j \cdot i_\lambda)) \psi_{j-1} \psi_j v_{\mathrm{T}''}$$

$$= \psi_* \psi_{j-1} \psi_j \psi_{j-3} \psi_{j-2} \psi_{j-1} \psi_{j-2} \psi_{j+1} \psi_j v_{\mathrm{T}''}$$

$$- \psi_* \psi_{j-3} \psi_{j-2} y_{j-2} \psi_{j+1} \psi_{j-1} \psi_j v_{\mathrm{T}''}$$

$$+ 2\psi_* \psi_{j-3} \psi_{j-2} \psi_{j+1} (\psi_{j-1} y_{j-1} + 1) \psi_j v_{\mathrm{T}''}$$

$$- \psi_* \psi_{j-3} \psi_{j-2} (\psi_{j+1} y_{j+2} - 1) \psi_{j-1} \psi_j v_{\mathrm{T}''}$$

$$= \psi_* \psi_{j-1} \psi_{j-3} \psi_{j-2} \underbrace{\Psi_j \psi_{j-2} v_{\mathrm{T}''}}_{=0 \text{ by } (B_{r-2})} - 0 + 0$$

$$+ 2 \underbrace{\psi_j \psi_{j+1} \psi_{j-1} \psi_{j+2}}_{=:\psi^*} \psi_{j-2} \psi_{j-3} \psi_{j-2} \psi_{j+1} \psi_j v_{\mathrm{T}''}$$

$$- 0 + \psi^* \psi_{j-2} \psi_{j-3} \psi_{j-2} \psi_{j-1} \psi_j v_{\mathrm{T}''}$$

$$= 2\psi^* (\psi_{j-2} \psi_{j-3} \psi_{j-2} e(s_j \cdot i_\lambda)) \psi_{j+1} \psi_j v_{\mathrm{T}''}$$

$$+ \psi^* (\psi_{j-2} \psi_{j-3} \psi_{j-2} e(s_j \cdot i_\lambda)) \psi_{j-1} \psi_j v_{\mathrm{T}''}$$

$$= 2\psi^* (\psi_{j-3} \psi_{j-2} \psi_{j-3} - y_{j-3} + 2y_{j-2} - y_{j-1}) \psi_{j+1} \psi_j v_{\mathrm{T}''}$$

$$+ \psi^* (\psi_{j-3} \psi_{j-2} \psi_{j-3} - y_{j-3} + 2y_{j-2} - y_{j-1}) \psi_{j-1} \psi_j v_{\mathrm{T}''}$$

$$= 2\psi^* \psi_{j-3} \psi_{j-2} \psi_{j+1} \psi_j \underbrace{\psi_{j-3} v_{\mathrm{T}''}}_{=0 \text{ as } j-3 \text{ is even}} - 0 + 0 - 0$$

$$
\begin{aligned}
&+\psi^*\psi_{j-3}\psi_{j-2}\psi_{j-1}\psi_j \underbrace{\psi_{j-3}v_{\mathrm{T}''}}_{=0} - 0 + 0\\
&-\psi^*(y_{j-1}\psi_{j-1}e(s_j\cdot i_\lambda))\psi_j v_{\mathrm{T}''}\\
&= -\psi^*(\psi_{j-1}y_j - 1)\psi_j v_{\mathrm{T}''}\\
&= -\psi^*\psi_{j-1}\psi_j y_{j+1}v_{\mathrm{T}''} + \psi^*\psi_j v_{\mathrm{T}''}\\
&= -0 + \underbrace{\Psi_j\psi_{j+2}v_{\mathrm{T}''}.}_{=0 \text{ by } (A_{r-2})}
\end{aligned}
$$

Next, we show that (B_r) follows if (A_s) and (B_s) hold for all $s < r$.

For (B_0), we have $\Psi_j\psi_{j-2}z_\lambda = \psi_j\psi_{j+1}\psi_{j-1}\psi_{j-2}\psi_j z_\lambda = 0$ as at least one of ψ_j, ψ_{j-2} must annihilate z_λ.

Now suppose $r > 0$.

If $j_1 \geqslant j+4$ or $j_1 \leqslant j-6$, then we clearly have $\Psi_j\psi_{j-2}v_{\mathrm{T}} = \Psi_{j_1}\Psi_j\psi_{j-2}v_{\mathrm{T}'}$ and our result follows from (B_{r-1}). Once again, we break the proof up for the remaining four possible values of j_1.

(a) Suppose $j_1 = j+2$. If $b+3-2d = j+2$ then we have

$$
\begin{aligned}
\Psi_j\psi_{j-2}v_{\mathrm{T}} &= \Psi_j\psi_{j-2}\Psi_{j+2}v_{\mathrm{T}(2)}\\
&= 0
\end{aligned}
$$

as ψ_{j-2} commutes with everything to its right, since the lowest indexed term in $v_{\mathrm{T}(2)}$ is Ψ_{j+4}.

If $b+3-2d \leqslant j$, we have

$$
\begin{aligned}
\Psi_j\psi_{j-2}v_{\mathrm{T}} &= \Psi_j\psi_{j-2}\Psi_{j+2}\Psi_j\Psi\overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}\\
&= \underbrace{\psi_j\psi_{j+1}\psi_{j-1}\psi_{j-2}\psi_{j+2}\psi_{j+3}}_{=:\psi_*}(\psi_j\psi_{j+1}\psi_j e(s_{j+2}\cdot s_j\cdot i_\lambda))\cdot\\
&\quad \psi_{j+2}\psi_{j+1}\psi_{j-1}\psi_j\Psi\overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}
\end{aligned}
$$

$$
\begin{aligned}
&= \psi_*(\psi_{j+1}\psi_j\psi_{j+1} + y_j - 2y_{j+1} + y_{j+2})\cdot \\
&\quad \psi_{j+2}\psi_{j+1}\psi_{j-1}\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \\
&= \psi_*\psi_{j+1}\psi_j(\psi_{j+1}\psi_{j+2}\psi_{j+1}e(s_j\cdot i_\lambda))\psi_{j-1}\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \\
&\quad + \psi_*\psi_{j+2}\psi_{j+1}(y_j\psi_{j-1}e(s_j\cdot i_\lambda))\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \\
&\quad - 2\psi_*\psi_{j+2}(y_{j+1}\psi_{j+1}e(s_j\cdot i_\lambda))\psi_{j-1}\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \\
&\quad + \psi_*(y_{j+2}\psi_{j+2}e(s_j\cdot i_\lambda))\psi_{j+1}\psi_{j-1}\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \\
&= \psi_*\psi_{j+1}\psi_j\psi_{j+2}\psi_{j+1}\psi_{j+2}\psi_{j-1}\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \\
&\quad + \psi_*\psi_{j+2}\psi_{j+1}(\psi_{j-1}y_{j-1} + 1)\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \\
&\quad - 2\psi_*\psi_{j+2}(\psi_{j+1}y_{j+2} - 1)\psi_{j-1}\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \\
&\quad + \psi_*\psi_{j+2}y_{j+3}\psi_{j+1}\psi_{j-1}\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \\
&= \psi_*\psi_{j+1}\psi_{j+2}\underbrace{\Psi_j\psi_{j+2}\Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)}}_{=0 \text{ by } (A_{r-2})} \\
&\quad + 0 + \psi_j\psi_{j+1}\psi_{j-1}\psi_{j-2}(\psi_{j+2}\psi_{j+3}\psi_{j+2}e(s_j\cdot i_\lambda))\cdot \\
&\quad \psi_{j+1}\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \\
&\quad - 0 + 2\psi_j\psi_{j+1}\psi_{j-1}\psi_{j-2}(\psi_{j+2}\psi_{j+3}\psi_{j+2}e(s_j\cdot i_\lambda))\cdot \\
&\quad \psi_{j-1}\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} + 0 \\
&= (\psi_j\psi_{j+1}\psi_{j-1}\psi_{j-2}(\psi_{j+3}\psi_{j+2}\psi_{j+3} + y_{j+2} - 2y_{j+3} + y_{j+4})\psi_{j+1} \\
&\quad + 2\psi_j\psi_{j+1}\psi_{j-1}\psi_{j-2}(\psi_{j+3}\psi_{j+2}\psi_{j+3} + y_{j+2} - 2y_{j+3} + y_{j+4})\cdot \\
&\quad \psi_{j-1}\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \\
&= 0 + \psi_j\psi_{j+1}\psi_{j-1}\psi_{j-2}(y_{j+2}\psi_{j+1}e(s_j\cdot i_\lambda))\psi_j \Psi \overset{j-2}{\underset{b+3-2d}{\downarrow}} v_{\mathrm{T}(2)} \\
&\quad - 0 + 0 + 0 + 0 - 0 + 0
\end{aligned}
$$

$$= \psi_j \psi_{j+1} \psi_{j-1} \psi_{j-2} (\psi_{j+1} y_{j+1} + 1) \psi_j \Psi \underset{b+3-2d}{\overset{j-2}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= 0 + \Psi_j \psi_{j-2} \Psi \underset{b+3-2d}{\overset{j-2}{\downarrow}} v_{\mathrm{T}(2)}$$

$$= 0 \text{ by } (B_{r-2}).$$

(b) Suppose $j_1 = j$. Then we have

$$\Psi_j \psi_{j-2} v_{\mathrm{T}} = -2\Psi_j \psi_{j-2} v_{\mathrm{T}'} \text{ by part 1,}$$

$$= 0 \text{ by } (B_{s-1}).$$

(c) Suppose $j_1 = j - 2$. Then we have

$$\Psi_j \psi_{j-2} v_{\mathrm{T}} = -2\Psi_j \psi_{j-2} v_{\mathrm{T}'} \text{ by part 1,}$$

$$= 0 \text{ by } (B_{r-1}).$$

(d) Suppose $j_1 = j - 4$. We divide into subcases.

i. Suppose $j_2 \geqslant j + 2$. Then we have

$$\Psi_j \psi_{j-2} v_{\mathrm{T}} = \Psi_j \psi_{j-2} \Psi_{j-4} \Psi \underset{b+3-2d}{\overset{j-6}{\downarrow}} \Psi \underset{j+4}{\overset{j_2}{\downarrow}} \Psi_{j+2} \Psi_j \Psi \underset{b+5-2d}{\overset{j-2}{\downarrow}} v_{\mathrm{T}(3)}$$

$$= \psi_j \psi_{j+1} \psi_{j-1} \psi_{j-2} \Psi_{j-4} \Psi \underset{b+3-2d}{\overset{j-6}{\downarrow}} \Psi \underset{j+4}{\overset{j_2}{\downarrow}} (\psi_j \Psi_{j+2} \Psi_j) \Psi \underset{b+5-2d}{\overset{j-2}{\downarrow}} v_{\mathrm{T}(3)}$$

$$= \psi_j \psi_{j+1} \psi_{j-1} \psi_{j-2} \Psi_{j-4} \Psi \underset{b+3-2d}{\overset{j-6}{\downarrow}} \Psi \underset{j+4}{\overset{j_2}{\downarrow}} (\psi_j + \Psi_{j+2} \Psi_j \psi_{j+2}) \cdot$$

$$\Psi \underset{b+5-2d}{\overset{j-2}{\downarrow}} v_{\mathrm{T}(3)} \text{ by } (*),$$

$$= \Psi_j \psi_{j-2} \Psi \underset{b+3-2d}{\overset{j-4}{\downarrow}} \Psi \underset{j+4}{\overset{j_2}{\downarrow}} \Psi \underset{b+5-2d}{\overset{j-2}{\downarrow}} v_{\mathrm{T}(3)}$$

$$+ 0 \text{ by } (A_s) \text{ for some } s < r, \text{ as } \Psi \underset{b+5-2d}{\overset{j-2}{\downarrow}} v_{\mathrm{T}(3)} \in \mathscr{D},$$

$$= 0 \text{ by } (B_{s'}) \text{ for some } s' < r, \text{ as } \Psi \underset{b+3-2d}{\overset{j-4}{\downarrow}} \Psi \underset{j+4}{\overset{j_2}{\downarrow}} \Psi \underset{b+5-2d}{\overset{j-2}{\downarrow}} v_{\mathrm{T}(3)} \in \mathscr{D}.$$

ii. Suppose $j_2 = j$. Then we have

$$\Psi_j \psi_{j-2} v_{\mathrm{T}} = \Psi_j \psi_{j-2} \Psi \underset{b+3-2d}{\overset{j-4}{\downarrow}} \Psi_j \Psi \underset{b+5-2d}{\overset{j-2}{\downarrow}} v_{\mathrm{T}(3)}$$

$$= -2\Psi_j \psi_{j-2} \Psi \underset{b+3-2d}{\overset{j-4}{\downarrow}} \Psi \underset{b+5-2d}{\overset{j-2}{\downarrow}} v_{\mathrm{T}(3)} \text{ by part 1,}$$

$$= 0 \text{ by } (B_{r-1}), \text{ as } \Psi \overset{j-4}{\underset{b+3-2d}{\downarrow}} \Psi \overset{j-2}{\underset{b+5-2d}{\downarrow}} v_{\mathrm{T}(3)} \in \mathscr{D}.$$

iii. Suppose $j_2 = j-2$. Then we have

$$\Psi_j \psi_{j-2} v_{\mathrm{T}} = \Psi_j \psi_{j-2} \Psi_{j-4} \Psi_{j-2} \underbrace{\Psi \overset{j-6}{\underset{b+3-2d}{\downarrow}} \Psi \overset{j-4}{\underset{b+5-2d}{\downarrow}} v_{\mathrm{T}(3)}}_{=: v_{\mathrm{T}''}}$$

$$= \Psi_j \psi_{j-4} \psi_{j-5} (\psi_{j-2} \psi_{j-3} \psi_{j-2} e(s_{j-4} \cdot s_{j-2} \cdot i_\lambda)) \cdot$$
$$\psi_{j-4} \psi_{j-1} \psi_{j-3} \psi_{j-2} v_{\mathrm{T}''}$$

$$= \Psi_j \psi_{j-4} \psi_{j-5} (\psi_{j-3} \psi_{j-2} \psi_{j-3} - y_{j-3} + 2y_{j-2} - y_{j-1}) \cdot$$
$$\psi_{j-4} \psi_{j-1} \psi_{j-3} \psi_{j-2} v_{\mathrm{T}''}$$

$$= \Psi_j \psi_{j-4} \psi_{j-5} \psi_{j-3} \psi_{j-2} (\psi_{j-3} \psi_{j-4} \psi_{j-3} e(s_{j-2} \cdot i_\lambda)) \cdot$$
$$\psi_{j-1} \psi_{j-2} v_{\mathrm{T}''}$$
$$- \Psi_j \psi_{j-4} \psi_{j-5} (y_{j-3} \psi_{j-4} e(s_{j-2} \cdot i_\lambda)) \psi_{j-1} \psi_{j-3} \psi_{j-2} v_{\mathrm{T}''}$$
$$+ 2\Psi_j \psi_{j-4} \psi_{j-5} \psi_{j-4} \psi_{j-1} (y_{j-2} \psi_{j-3} e(s_{j-2} \cdot i_\lambda)) \psi_{j-2} v_{\mathrm{T}''}$$
$$- \Psi_j \psi_{j-4} \psi_{j-5} \psi_{j-4} (y_{j-1} \psi_{j-1} e(s_{j-2} \cdot i_\lambda)) \psi_{j-3} \psi_{j-2} v_{\mathrm{T}''}$$

$$= \Psi_j \psi_{j-4} \psi_{j-5} \psi_{j-3} \psi_{j-2} \psi_{j-4} \psi_{j-3} \psi_{j-4} \psi_{j-1} \psi_{j-2} v_{\mathrm{T}''}$$
$$- \Psi_j \psi_{j-4} \psi_{j-5} \psi_{j-4} y_{j-4} \psi_{j-1} \psi_{j-3} \psi_{j-2} v_{\mathrm{T}''}$$
$$+ 2\Psi_j \psi_{j-4} \psi_{j-5} \psi_{j-4} \psi_{j-1} (\psi_{j-3} y_{j-3} + 1) \psi_{j-2} v_{\mathrm{T}''}$$
$$- \Psi_j \psi_{j-4} \psi_{j-5} \psi_{j-4} (\psi_{j-1} y_j - 1) \psi_{j-3} \psi_{j-2} v_{\mathrm{T}''}$$

$$= \Psi_j \psi_{j-4} \psi_{j-5} \psi_{j-3} \psi_{j-4} \underbrace{\Psi_{j-2} \psi_{j-4} v_{\mathrm{T}''}}_{=0 \text{ by } (B_{r-2})} - 0$$
$$+ 0 + 2\Psi_j (\psi_{j-4} \psi_{j-5} \psi_{j-4} e(s_{j-2} \cdot i_\lambda)) \psi_{j-1} \psi_{j-2} v_{\mathrm{T}''}$$
$$- 0 + \Psi_j (\psi_{j-4} \psi_{j-5} \psi_{j-4} e(s_{j-2} \cdot i_\lambda)) \psi_{j-3} \psi_{j-2} v_{\mathrm{T}''}$$

$$= 2\Psi_j (\psi_{j-5} \psi_{j-4} \psi_{j-5} - y_{j-5} + 2y_{j-4} - y_{j-3}) \psi_{j-1} \psi_{j-2} v_{\mathrm{T}''}$$
$$+ \Psi_j (\psi_{j-5} \psi_{j-4} \psi_{j-5} - y_{j-5} + 2y_{j-4} - y_{j-3}) \psi_{j-3} \psi_{j-2} v_{\mathrm{T}''}$$

$$= 0 - 0 + 0 - 0 + 0 - 0 + 0 - \Psi_j (y_{j-3} \psi_{j-3} e(s_{j-2} \cdot i_\lambda)) \psi_{j-2} v_{\mathrm{T}''}$$

$$= -\Psi_j (\psi_{j-3} y_{j-2} - 1) \psi_{j-2} v_{\mathrm{T}''}$$

$$= 0 + \Psi_j \psi_{j-2} v_{\mathrm{T}''}$$
$$= 0 \text{ by } (B_{r-2}).$$

Note that both inductive steps are possible because $v_{\mathrm{T}''} \in \mathscr{D}$.

This completes our proof of statements 2–5. □

In the next two results, we are concerned with how the Garnir element $\psi_1\psi_2 \ldots \psi_{b+1}$ acts on elements of $\mathscr{D}$.

Lemma 3.14. *Suppose j is odd with $3 \leqslant j \leqslant n-2$, and $\mathrm{T} \in \mathrm{Dom}(\lambda)$. Then*

1. *For all odd $n-2 \geqslant i \geqslant j+4$, $\psi_1\psi_2 \ldots \psi_j \Psi_i v_{\mathrm{T}} = \Psi_i \psi_1\psi_2 \ldots \psi_j v_{\mathrm{T}}$.*
2. $\psi_1\psi_2 \ldots \psi_j \Psi_{j+2} v_{\mathrm{T}} = \psi_{j+2}\psi_{j+3}\psi_1\psi_2 \ldots \psi_{j+2} v_{\mathrm{T}}$.
3. $\psi_1\psi_2 \ldots \psi_j \Psi_j v_{\mathrm{T}} = -2\psi_1\psi_2 \ldots \psi_j v_{\mathrm{T}}$.
4. $\psi_1\psi_2 \ldots \psi_j \Psi_{j-2} v_{\mathrm{T}} = \Psi_{j-1}\psi_1\psi_2 \ldots \psi_j v_{\mathrm{T}} + \psi_j\psi_{j-1}\psi_1\psi_2 \ldots \psi_{j-2} v_{\mathrm{T}}$.
5. *For all odd $3 \leqslant i \leqslant j-4$,*

$$\psi_1\psi_2 \ldots \psi_j \Psi_i v_{\mathrm{T}} = \Psi_{i+1}\psi_1\psi_2 \ldots \psi_j v_{\mathrm{T}} + \psi_{i+2}\psi_{i+1}\psi_{i+3}\psi_{i+4} \ldots \psi_j \psi_1\psi_2 \ldots \psi_i v_{\mathrm{T}}.$$

Proof. 1 and 2 follow immediately from definitions and the commuting relations between ψ generators. 3 follows immediately from Lemma 3.13. So only statements 4 and 5 require any real work!

4. $$\psi_1\psi_2 \ldots \psi_j \Psi_{j-2} v_{\mathrm{T}} = \underbrace{\psi_1\psi_2 \ldots \psi_{j-4}}_{=:\psi^*} \psi_{j-3}(\psi_{j-2}\psi_{j-1}\psi_{j-2} e(s_j \cdot s_{j-2} \cdot i_\lambda)) \cdot$$
$$\psi_j\psi_{j-1}\psi_{j-3}\psi_{j-2} v_{\mathrm{T}}$$
$$= \psi^* \psi_{j-3}(\psi_{j-1}\psi_{j-2}\psi_{j-1} + y_{j-2} - 2y_{j-1} + y_j) \cdot$$
$$\psi_j\psi_{j-1}\psi_{j-3}\psi_{j-2} v_{\mathrm{T}}$$
$$= \psi_{j-1}\psi^*\psi_{j-3}\psi_{j-2}(\psi_{j-1}\psi_j\psi_{j-1} e(s_{j-2} \cdot i_\lambda))\psi_{j-3}\psi_{j-2} v_{\mathrm{T}}$$

$$
\begin{aligned}
&+\psi_j\psi_{j-1}\psi^*\psi_{j-3}(y_{j-2}\psi_{j-3}e(s_{j-2}\cdot i_\lambda))\psi_{j-2}v_{\mathrm{T}}\\
&-2\psi_j\psi^*\psi_{j-3}(y_{j-1}\psi_{j-1}e(s_{j-2}\cdot i_\lambda))\psi_{j-3}\psi_{j-2}v_{\mathrm{T}}\\
&+\psi^*\psi_{j-3}(y_j\psi_je(s_{j-2}\cdot i_\lambda))\psi_{j-1}\psi_{j-3}\psi_{j-2}v_{\mathrm{T}}\\
=\ &\psi_{j-1}\psi^*\psi_{j-3}\psi_{j-2}(\psi_j\psi_{j-1}\psi_j)\psi_{j-3}\psi_{j-2}v_{\mathrm{T}}\\
&+\psi_j\psi_{j-1}\psi^*\psi_{j-3}(\psi_{j-3}y_{j-3}+1)\psi_{j-2}v_{\mathrm{T}}\\
&-2\psi_j\psi^*\psi_{j-3}(\psi_{j-1}y_j-1)\psi_{j-3}\psi_{j-2}v_{\mathrm{T}}\\
&+\psi^*\psi_{j-3}(\psi_jy_{j+1})\psi_{j-1}\psi_{j-3}\psi_{j-2}v_{\mathrm{T}}\\
=\ &\psi_{j-1}\psi_j\psi^*(\psi_{j-3}\psi_{j-2}\psi_{j-3}e(s_{j-1}\cdot s_j\cdot s_{j-2}\cdot i_\lambda))\cdot\\
&\psi_{j-1}\psi_j\psi_{j-2}v_{\mathrm{T}}\\
&+0+\psi_j\psi_{j-1}\psi^*\psi_{j-3}\psi_{j-2}v_{\mathrm{T}}\\
&-0+2\psi_j\psi^*(\psi_{j-3}^2e(s_{j-2}\cdot i_\lambda))\psi_{j-2}v_{\mathrm{T}}+0\\
=\ &\psi_{j-1}\psi_j\psi^*(\psi_{j-2}\psi_{j-3}\psi_{j-2})\psi_{j-1}\psi_j\psi_{j-2}v_{\mathrm{T}}\\
&+\psi_j\psi_{j-1}\psi^*\psi_{j-3}\psi_{j-2}v_{\mathrm{T}}+0\\
=\ &\psi_{j-1}\psi_j\psi_{j-2}\psi^*\psi_{j-3}(\psi_{j-2}\psi_{j-1}\psi_{j-2}e(s_j\cdot i_\lambda))\psi_jv_{\mathrm{T}}\\
&+\psi_j\psi_{j-1}\psi^*\psi_{j-3}\psi_{j-2}v_{\mathrm{T}}\\
=\ &\psi_{j-1}\psi_j\psi_{j-2}\psi^*\psi_{j-3}(\psi_{j-1}\psi_{j-2}\psi_{j-1})\psi_jv_{\mathrm{T}}\\
&+\psi_j\psi_{j-1}\psi^*\psi_{j-3}\psi_{j-2}v_{\mathrm{T}}\\
=\ &\Psi_{j-1}\psi^*\psi_{j-3}\psi_{j-2}\psi_{j-1}\psi_jv_{\mathrm{T}}+\psi_j\psi_{j-1}\psi^*\psi_{j-3}\psi_{j-2}v_{\mathrm{T}}.
\end{aligned}
$$

5. Let i be odd and $4 \leqslant i \leqslant j-4$. Then

$$
\begin{aligned}
\psi_1\psi_2\dots\psi_j\Psi_iv_{\mathrm{T}} =\ &\underbrace{\psi_1\psi_2\dots\psi_{i-2}}_{\psi_*}\psi_{i-1}\psi_i\psi_{i+1}\psi_{i+2}\Psi_i\psi_{i+3}\underbrace{\psi_{i+4}\dots\psi_j}_{\psi^*}v_{\mathrm{T}}\\
=\ &\psi_*\psi_{i-1}(\psi_i\psi_{i+1}\psi_ie(s_{i+2}\cdot s_i\cdot s_{i+4}\cdot s_{i+6}\cdots s_j\cdot i_\lambda))\cdot\\
&\psi_{i+2}\psi_{i+1}\psi_{i-1}\psi_i\psi_{i+3}\psi^*v_{\mathrm{T}}\\
=\ &\psi_*\psi_{i-1}(\psi_{i+1}\psi_i\psi_{i+1}+y_i-2y_{i+1}+y_{i+2})\cdot
\end{aligned}
$$

$$
\begin{aligned}
&\psi_{i+2}\psi_{i+1}\psi_{i-1}\psi_i\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&= \psi_{i+1}\psi_*\psi_{i-1}\psi_i(\psi_{i+1}\psi_{i+2}\psi_{i+1}e(s_i\cdot s_{i+4}\cdot s_{i+6}\cdots s_j\cdot i_\lambda))\cdot\\
&\quad \psi_{i-1}\psi_i\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&\quad + \psi_{i+2}\psi_{i+1}\psi_*\psi_{i-1}(y_i\psi_{i-1}e(s_i\cdot s_{i+4}\cdot s_{i+6}\cdots s_j\cdot i_\lambda))\cdot\\
&\quad \psi_i\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&\quad - 2\psi_{i+2}\psi_*\psi_{i-1}(y_{i+1}\psi_{i+1}e(s_i\cdot s_{i+4}\cdot s_{i+6}\cdots s_j\cdot i_\lambda))\cdot\\
&\quad \psi_{i-1}\psi_i\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&\quad + \psi_*\psi_{i-1}(y_{i+2}\psi_{i+2}e(s_i\cdot s_{i+4}\cdot s_{i+6}\cdots s_j\cdot i_\lambda))\cdot\\
&\quad \psi_{i+1}\psi_{i-1}\psi_i\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&= \psi_{i+1}\psi_*\psi_{i-1}\psi_i(\psi_{i+2}\psi_{i+1}\psi_{i+2})\psi_{i-1}\psi_i\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&\quad + \psi_{i+2}\psi_{i+1}\psi_*\psi_{i-1}(\psi_{i-1}y_{i-1}+1)\psi_i\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&\quad - 2\psi_{i+2}\psi_*\psi_{i-1}(\psi_{i+1}y_{i+2}-1)\psi_{i-1}\psi_i\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&\quad + \psi_*\psi_{i-1}(\psi_{i+2}y_{i+3})\psi_{i+1}\psi_{i-1}\psi_i\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&= \psi_{i+1}\psi_{i+2}\psi_*(\psi_{i-1}\psi_i\psi_{i-1}e(s_{i+1}\cdot s_{i+2}\cdot s_i\cdot s_{i+4}\cdots s_j\cdot i_\lambda))\cdot\\
&\quad \psi_{i+1}\psi_{i+2}\psi_i\psi_{i+3}\psi^* v_{\mathrm{T}} + 0 + \psi_{i+2}\psi_{i+1}\psi_*\psi_{i-1}\psi_i\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&\quad - 0 + 2\psi_{i+2}\psi_*(\psi_{i-1}^2 e(s_i\cdot s_{i+4}\cdot s_{i+6}\cdots s_j\cdot i_\lambda))\psi_i\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&\quad + \psi_{i+2}\psi_{i+1}\psi_*(\psi_{i-1}^2 e(s_i\cdot s_{i+4}\cdot s_{i+6}\cdots s_j\cdot i_\lambda))\psi_i y_{i+3}\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&= \psi_{i+1}\psi_{i+2}\psi_*(\psi_i\psi_{i-1}\psi_i)\psi_{i+1}\psi_{i+2}\psi_i\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&\quad + \psi_{i+2}\psi_{i+1}\psi_{i+3}\psi^*\psi_*\psi_{i-1}\psi_i v_{\mathrm{T}} + 0 + 0\\
&= \psi_{i+1}\psi_{i+2}\psi_i\psi_*\psi_{i-1}(\psi_i\psi_{i+1}\psi_i e(s_{i+2}\cdot s_{i+4}\cdots s_j\cdot i_\lambda))\cdot\\
&\quad \psi_{i+2}\psi_{i+3}\psi^* v_{\mathrm{T}} + \psi_{i+2}\psi_{i+1}\psi_{i+3}\psi^*\psi_*\psi_{i-1}\psi_i v_{\mathrm{T}}\\
&= \psi_{i+1}\psi_{i+2}\psi_i\psi_*\psi_{i-1}(\psi_{i+1}\psi_i\psi_{i+1})\psi_{i+2}\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&\quad + \psi_{i+2}\psi_{i+1}\psi_{i+3}\psi^*\psi_*\psi_{i-1}\psi_i v_{\mathrm{T}}\\
&= \Psi_{i+1}\psi_*\psi_{i-1}\psi_i\psi_{i+1}\psi_{i+2}\psi_{i+3}\psi^* v_{\mathrm{T}}\\
&\quad + \psi_{i+2}\psi_{i+1}\psi_{i+3}\psi^*\psi_*\psi_{i-1}\psi_i v_{\mathrm{T}}. \qquad \square
\end{aligned}
$$

Proposition 3.15. *Let* $\mathtt{T} \in \mathrm{Dom}(\lambda)$. *Then* $\psi_1\psi_2\ldots\psi_{b+1}v_{\mathtt{T}} = 0$.

Proof. Repeated application of the above lemma yields $\psi_1\psi_2\ldots\psi_{b+1}v_{\mathtt{T}}$ as a sum of expressions ending in $\psi_1\psi_2\ldots\psi_j z_\lambda$ for various odd values of $j \geqslant 3$. In all cases the relations of the Specht module give us our result. □

Having determined the actions of most relators in the presentation of S_λ on each element of $\mathscr{D}$, it remains to calculate the actions of the generators ψ_j when $3 \leqslant j \leqslant n-2$ and j is odd. The rest of this section is devoted to this endeavour. Note that in order to prove the main result of the chapter, the contents of the rest of this section are not necessary, and were thus omitted from [43]. However, to calculate endomorphisms for S_λ (and in particular, to find the endomorphism given in Proposition 3.30) we originally computed the remaining actions, which is why we include them in this thesis.

We begin by looking at when basis vectors $v_{\mathtt{T}} \in \mathscr{D}$ have reduced expressions in which certain terms Ψ_j appear on the left.

Lemma 3.16. *Let* $\mathtt{T} \in \mathrm{Dom}(\lambda)$. *Then*

1. $v_{\mathtt{T}}$ *has a reduced expression with* Ψ_j *on the left (cf. Lemma 3.13 (1)) if and only if* $\mathtt{T}$ *has* $[j-1, j]$ *in the arm and* $[j+1, j+2]$ *in the leg.*

2. $v_{\mathtt{T}}$ *has a reduced expression with* $\Psi_{j+2}\Psi_j$ *on the left (cf. Lemma 3.13 (2)) if and only if* T *has* $[j-1, j]$ *and* $[j+1, j+2]$ *in the arm and* $[j+3, j+4]$ *in the leg.*

3. $v_{\mathtt{T}}$ *has a reduced expression with* $\Psi_{j-2}\Psi_j$ *on the left (cf. Lemma 3.13 (4)) if and only if* T *has* $[j-3, j-2]$ *in the arm and* $[j-1, j]$ *and* $[j+1, j+2]$ *in the leg.*

Proof. 1. First, suppose that Ψ_j can be moved to the left of some reduced expression for $v_{\mathtt{T}}$ using the commuting braid relations only. Then in the normal form for $v_{\mathtt{T}}$ we must have only terms Ψ_k with $k \leqslant j-4$ or $k \geqslant j+4$ appearing further left than Ψ_j in the expression. Ψ_j corresponds to putting the domino $[j+1, j+2]$ in the leg of $\mathtt{T}$ by transposing it with $[j-1, j]$, which moves to the arm. Since, by our hypothesis, Ψ_{j+2} cannot appear to the left of Ψ_j, $[j+1, j+2]$ must remain in the leg and any Ψ_k terms to the left of Ψ_j correspond to placing the dominoes higher up the leg. Therefore we have $k \leqslant j-2$. But since we can't have $k = j-2$, again by our hypothesis, we must have $k \leqslant j-4$. And so, $[j-1, j]$ stays in the arm of $\mathtt{T}$.

Conversely, suppose we have $[j-1, j]$ in the arm of $\mathtt{T}$ and $[j+1, j+2]$ in the leg. Then the normal form for $v_{\mathtt{T}}$ has a Ψ_j which places the domino $[j+1, j+2]$ in the leg, and all Ψ_k terms to the left of this place the dominoes higher in the leg. Since $[j-1, j]$ does not appear in the leg of $\mathtt{T}$, and since $\mathtt{T}$ must be standard, we know that these terms all have $k \leqslant j-4$. Our result follows immediately.

2.& 3. These both follow by applying part 1 twice. □

Corollary 3.17. *Let j be odd and suppose that $\mathtt{T} \in \mathrm{Dom}(\lambda)$ with corresponding $v_{\mathtt{T}} \in \mathscr{D}$. Then*

1. *Suppose $\mathtt{T}$ has the dominoes $[j-1, j]$ and $[j+1, j+2]$ in the arm and $[j+3, j+4]$ in the leg, and let $\mathtt{S}$ be the standard λ-tableau which agrees with $\mathtt{T}$ outside of these three dominoes, but has them permuted so that the domino $[j-1, j]$ lies in the leg (note that $\mathtt{S} \trianglelefteq \mathtt{T}$). Then $\psi_j v_{\mathtt{T}} = \psi_j v_{\mathtt{S}}$.*

2. *Suppose $\mathtt{T}$ has the domino $[j-3, j-2]$ in the arm and the dominoes $[j-1, j]$ and $[j+1, j+2]$ in the leg, and let $\mathtt{S}$ be the standard λ-tableau which agrees with $\mathtt{T}$ outside of these three dominoes, but has them permuted so that the domino $[j+1, j+2]$ lies in the arm (note that $\mathtt{S} \trianglelefteq \mathtt{T}$). Then $\psi_j v_{\mathtt{T}} = \psi_j v_{\mathtt{S}}$.*

Proof. The result follows immediately from Lemmas 3.13 and 3.16. □

We now have some concrete actions of the ψ_j on our elements of $\mathscr{D}$. However, these actions do not necessarily give us a reduced form for $\psi_j v_{\mathtt{T}}$. In fact, it isn't even clear when these may be zero! These are the problems we seek to tackle next.

Lemma 3.18. *Let $\mathtt{T} \in \mathrm{Dom}(\lambda)$ with corresponding $v_{\mathtt{T}} \in \mathscr{D}$ and let $3 \leqslant j \leqslant n-2$ be odd. Then $\mathtt{T}$ satisfies precisely one of the following four conditions with respect to j:*

1. *The domino $[j-1, j]$ is in the arm of $\mathtt{T}$ but $[j+1, j+2]$ is in the leg.*

2. *The dominoes* $[j-1, j]$ *and* $[j+1, j+2]$ *are in the arm of* $\mathtt{T}$.

3. *The domino* $[j-1, j]$ *is in the leg of* $\mathtt{T}$ *but* $[j+1, j+2]$ *is in the arm.*

4. *The dominoes* $[j-1, j]$ *and* $[j+1, j+2]$ *are in the leg of* $\mathtt{T}$.

Proof. Clear. □

We must now work towards computing the ψ_j actions on tableaux $v_{\mathtt{T}} \in \mathscr{D}$ in each case of the above proposition. First we will look at case 2. That is, we want a reduced expression for $\psi_j v_{\mathtt{T}}$ when the dominoes $[j-1, j]$, $[j+1, j+2]$ and $[j+3, j+4]$ all appear in the arm of $\mathtt{T}$ (otherwise we appeal to Corollary 3.17 (1)). We begin by dismissing the "degenerate" situation – if $v_{\mathtt{T}}$ only involves terms Ψ_k for $k \leqslant j-4$ (a trivial example being when $\mathtt{T} = \mathtt{T}_\lambda$) then clearly $\psi_j v_{\mathtt{T}} = 0$. Likewise we may assume $j \neq n-2$, as it does not make sense to talk about the domino $[j+3, j+4]$ here; the case $j = n-2$ is easily dealt with in Proposition 3.27 (2).

Suppose we have Ψ_j appearing in the sequence $\Psi\overset{j_\ell}{\underset{k_\ell}{\downarrow}}$ for some ℓ but not any sequences further left in the normal form of $v_{\mathtt{T}}$. Note that when $\ell = 1$, we have the sequence of dominoes $[2,3], [4,5], \dots, [j+r-1, j+r]$ in the arm of $\mathtt{T}$ for $r \geqslant 4$ if $k_1 = 3$. Otherwise, the domino $[k_1-3, k_1-2]$ is in the leg.

Furthermore, in the sequence $\Psi\overset{j_1}{\underset{k_1}{\downarrow}} \dots \Psi\overset{j_{\ell-1}}{\underset{k_{\ell-1}}{\downarrow}}$ all Ψ terms have subscript at most $j-4$, as $[j-1, j]$ is in the arm. This tells us that ψ_j commutes with all of these terms, so we will introduce the notation $\Psi^* := \Psi\overset{j_1}{\underset{k_1}{\downarrow}} \dots \Psi\overset{j_{\ell-1}}{\underset{k_{\ell-1}}{\downarrow}}$ for the duration of solving of our case 2.

Lemma 3.19. *Let* $v_{\mathtt{T}} \in \mathscr{D}$ *have normal form* $v_{\mathtt{T}} = \Psi\overset{j_1}{\underset{k_1}{\downarrow}} \dots \Psi\overset{j_d}{\underset{k_d}{\downarrow}} z_\lambda$, *for* $d \geqslant 1$. *Suppose* $j \in \{3, \dots, n-2\}$ *is odd and suppose* ℓ *is minimal such that* $k_\ell \leqslant j \leqslant j_\ell$. *Then* $j_{\ell+i-1} \geqslant 4i + j$ *for all* $i = 1, 2, \dots, d-\ell+1$ *if and only if we have* $\psi_j v_{\mathtt{T}} = 0$.

Proof. We shall first prove the *only if* part of the lemma. First note immediately that our condition implies $r \geqslant 4$, so this part of the lemma always applies to tableaux of the form we are interested in. We will prove it by induction on $d-\ell+1$. If $d-\ell+1 = 1$

(that is, $d = \ell$), we have only the condition $r \geqslant 4$, so

$$\begin{aligned}
\psi_j v_{\mathrm{T}} &= \psi_j \Psi^* \Psi \overset{j_\ell}{\underset{j+6}{\downarrow}} \Psi_{j+4} \Psi_{j+2} \Psi_j \Psi \overset{j-2}{\underset{k_\ell}{\downarrow}} z_\lambda \\
&= \psi_j \Psi^* \Psi \overset{j_\ell}{\underset{j+6}{\downarrow}} \Psi_{j+4} \Psi \overset{j-2}{\underset{k_\ell}{\downarrow}} z_\lambda \quad \text{by Lemma 3.13 (2)} \\
&= \psi_j \Psi^* \Psi \overset{j_\ell}{\underset{j+6}{\downarrow}} \Psi \overset{j-2}{\underset{k_\ell}{\downarrow}} \Psi_{j+4} z_\lambda \\
&= 0.
\end{aligned}$$

Note $\Psi \overset{j_\ell}{\underset{j+6}{\downarrow}}$ and $\Psi \overset{j-2}{\underset{k_\ell}{\downarrow}}$ may be empty, but our condition guarantees that we have Ψ_{j+4} there to give our zero.

Now suppose we have $v_{\mathrm{T}} = \Psi \overset{j_1}{\underset{k_1}{\downarrow}} \dots \Psi \overset{j_d}{\underset{k_d}{\downarrow}} z_\lambda$ for some $d > \ell$, with the hypothesised conditions on v_{T}. Then from our base case we have

$$\psi_j v_{\mathrm{T}} = \psi_j \Psi^* \Psi \overset{j_\ell}{\underset{j+6}{\downarrow}} \Psi \overset{j-2}{\underset{k_\ell}{\downarrow}} \Psi_{j+4} \Psi \overset{j_{\ell+1}}{\underset{k_{\ell+1}}{\downarrow}} \dots \Psi \overset{j_d}{\underset{k_d}{\downarrow}} z_\lambda.$$

Now $k_\ell \leqslant j \leqslant j_\ell$ implies that $k_{\ell+1} \leqslant j+4 \leqslant j_{\ell+1}$, so in order to complete our induction we need to show that $j_{\ell+i} \geqslant 4i + (j+4)$ for $i = 1, 2, \dots, d-\ell$. But this follows immediately from our conditions on v_{T}.

Conversely, let t be minimal such that $j_{\ell+t-1} < 4t + j$. If $t = 1$ we have $r \leqslant 2$, so only have the cases $r = 0$ and $r = 2$ to consider. But these are both dealt with using Lemma 3.16 and Corollary 3.17; we see in both cases that $\psi_j v_{\mathrm{T}} \neq 0$, and we are done.

Now suppose $t > 1$. By minimality of t we have $j_{\ell+t-2} \geqslant 4t - 4 + j$ and so $j_{\ell+t-1} \geqslant j_{\ell+t-2} + 2 \geqslant 4t - 2 + j$. So we in fact have $j_{\ell+t-1} = 4t - 2 + j$.

Claim. For each $1 \leqslant i \leqslant t-1$,

$$\psi_j v_{\mathrm{T}} = \psi_j \Psi^* \left(\Psi \overset{j-2}{\underset{k_\ell}{\downarrow}} \Psi \overset{j_\ell}{\underset{k_{\ell+1}}{\downarrow}} \Psi \overset{j_{\ell+1}}{\underset{k_{\ell+2}}{\downarrow}} \dots \Psi \overset{j_{\ell+i-2}}{\underset{k_{\ell+i-1}}{\downarrow}} \right) \Psi \overset{j_{\ell+i-1}}{\underset{4i+j}{\downarrow}} \Psi \overset{j_{\ell+i}}{\underset{k_{\ell+i}}{\downarrow}} \dots \Psi \overset{j_d}{\underset{k_d}{\downarrow}} z_\lambda.$$

Proof. We prove the claim by induction on i. When $i = 1$ we have already seen that $\psi_j v_{\mathrm{T}} = \psi_j \Psi^* \Psi \overset{j-2}{\underset{k_\ell}{\downarrow}} \Psi \overset{j_\ell}{\underset{j+4}{\downarrow}} \Psi \overset{j_{\ell+1}}{\underset{k_{\ell+1}}{\downarrow}} \dots \Psi \overset{j_d}{\underset{k_d}{\downarrow}} z_\lambda$, as claimed.

Now suppose, by induction, that we have

$$\psi_j v_{\mathrm{T}} = \psi_j \Psi^* \left(\Psi \overset{j-2}{\underset{k_\ell}{\downarrow}} \Psi \overset{j_\ell}{\underset{k_{\ell+1}}{\downarrow}} \Psi \overset{j_{\ell+1}}{\underset{k_{\ell+2}}{\downarrow}} \dots \Psi \overset{j_{\ell+i-3}}{\underset{k_{\ell+i-2}}{\downarrow}} \right) \Psi \overset{j_{\ell+i-2}}{\underset{4i+j-4}{\downarrow}} \Psi \overset{j_{\ell+i-1}}{\underset{k_{\ell+i-1}}{\downarrow}} \dots \Psi \overset{j_d}{\underset{k_d}{\downarrow}} z_\lambda.$$

As $i \leqslant t-1$ we have $j_{\ell+i-1} \geqslant 4i+j$ and so

$$\begin{aligned}
\Psi \overset{j_{\ell+i-2}}{\underset{4i+j-4}{\downarrow}} \Psi \overset{j_{\ell+i-1}}{\underset{k_{\ell+i-1}}{\downarrow}} \dots \Psi \overset{j_d}{\underset{k_d}{\downarrow}} z_\lambda &= \Psi \overset{j_{\ell+i-1}}{\underset{4i+j}{\downarrow}} \Psi \overset{j_{\ell+i-2}}{\underset{4i+j-4}{\downarrow}} \Psi_{4i+j-2} \Psi_{4i+j-4} \Psi \overset{4i+j-6}{\underset{k_{\ell+i-1}}{\downarrow}} \\
&\quad \cdot \Psi \overset{j_{\ell+i}}{\underset{k_{\ell+i}}{\downarrow}} \dots \Psi \overset{j_d}{\underset{k_d}{\downarrow}} z_\lambda \\
&= \Psi \overset{j_{\ell+i-1}}{\underset{4i+j}{\downarrow}} \Psi \overset{j_{\ell+i-2}}{\underset{4i+j-4}{\downarrow}} \Psi \overset{4i+j-6}{\underset{k_{\ell+i-1}}{\downarrow}} \Psi \overset{j_{\ell+i}}{\underset{k_{\ell+i}}{\downarrow}} \dots \Psi \overset{j_d}{\underset{k_d}{\downarrow}} z_\lambda \\
&= \Psi \overset{j_{\ell+i-2}}{\underset{k_{\ell+i-1}}{\downarrow}} \Psi \overset{j_{\ell+i-1}}{\underset{4i+j}{\downarrow}} \Psi \overset{j_{\ell+i}}{\underset{k_{\ell+i}}{\downarrow}} \dots \Psi \overset{j_d}{\underset{k_d}{\downarrow}} z_\lambda.
\end{aligned}$$

Now, using the above claim,

$$\begin{aligned}
\psi_j v_{\mathrm{T}} &= \psi_j \Psi^* \left(\Psi \overset{j-2}{\underset{k_\ell}{\downarrow}} \Psi \overset{j_\ell}{\underset{k_{\ell+1}}{\downarrow}} \Psi \overset{j_{\ell+1}}{\underset{k_{\ell+2}}{\downarrow}} \dots \Psi \overset{j_{\ell+t-3}}{\underset{k_{\ell t-2}}{\downarrow}} \right) \Psi \overset{j_{\ell+t-2}}{\underset{4t+j-4}{\downarrow}} \\
&\quad \cdot \Psi \overset{j_{\ell+t-1}}{\underset{k_{\ell+t-1}}{\downarrow}} \dots \Psi \overset{j_d}{\underset{k_d}{\downarrow}} z_\lambda \quad \text{by our claim} \\
&= \psi_j \Psi^* \Psi \overset{j-2}{\underset{k_\ell}{\downarrow}} \Psi \overset{j_\ell}{\underset{k_{\ell+1}}{\downarrow}} \Psi \overset{j_{\ell+1}}{\underset{k_{\ell+2}}{\downarrow}} \dots \Psi \overset{j_{\ell+t-3}}{\underset{k_{\ell+t-2}}{\downarrow}} \Psi \overset{j_{\ell+t-2}}{\underset{k_{\ell+t-1}}{\downarrow}} \Psi \overset{j_{\ell+t}}{\underset{k_{\ell+t}}{\downarrow}} \dots \Psi \overset{j_d}{\underset{k_d}{\downarrow}} z_\lambda \\
&\quad \text{as } j_{\ell+t-1} = 4t-2+j.
\end{aligned}$$

We can easily see that this final term is in our normal form, and is therefore a reduced expression (and, in particular, non-zero). □

Definition 3.20. Suppose

$$\mathtt{T} = \begin{array}{|c|ccccccccccc}
\hline
1 & \cdots & \multicolumn{1}{|c|}{j-1} & \multicolumn{1}{c|}{j} & \multicolumn{1}{c|}{x_1} & \multicolumn{1}{c|}{y_1} & \cdots & \multicolumn{1}{|c|}{x_t} & \multicolumn{1}{c|}{y_t} & \cdots \\
\cline{1-1}\cline{3-6}\cline{8-9}
\multicolumn{1}{c}{\vdots} \\
\cline{1-1}
j_\ell+1 \\ \cline{1-1}
j_\ell+2 \\ \cline{1-1}
\multicolumn{1}{c}{\vdots} \\ \cline{1-1}
j_s+1 \\ \cline{1-1}
j_s+2 \\ \cline{1-1}
j_{s'}+1 \\ \cline{1-1}
j_{s'}+2 \\ \cline{1-1}
\multicolumn{1}{c}{\vdots} \\ \cline{1-1}
j_d+1 \\ \cline{1-1}
j_d+2 \\ \cline{1-1}
\end{array}$$

where the section of the arm $[j-1, j], [x_1, y_1], \ldots, [x_t, y_t]$ contains $t+1$ dominoes and the section of the leg $[j_\ell+1, j_\ell+2], \ldots, [j_{s'}+1, j_{s'}+2]$ contains t dominoes. Here we have $s = \ell + t - 2$ and $s' = s + 1$ for neatness. Then we define $(\mathtt{T})^{t,j}$ to be the standard λ-tableau obtained from $\mathtt{T}$ by moving the domino $[j_{s'}+1, j_{s'}+2]$ to the arm and the domino $[j-1, j]$ to the leg; this can be seen as cyclically permuting the above $2t+1$ dominoes "anticlockwise" by one space but keeping all other entries the same as $\mathtt{T}$. That is,

$$(\mathtt{T})^{t,j} = \begin{array}{|c|ccccccccccc}
\hline
1 & \cdots & \multicolumn{1}{|c|}{x_1} & \multicolumn{1}{c|}{y_1} & \cdots & \multicolumn{1}{|c|}{x_t} & \multicolumn{1}{c|}{y_t} & \multicolumn{1}{c|}{j_{s'}+1} & \multicolumn{1}{c|}{j_{s'}+2} & \cdots \\
\cline{1-1}\cline{3-4}\cline{6-9}
\multicolumn{1}{c}{\vdots} \\
\cline{1-1}
j-1 \\ \cline{1-1}
j \\ \cline{1-1}
j_\ell+1 \\ \cline{1-1}
j_\ell+2 \\ \cline{1-1}
\multicolumn{1}{c}{\vdots} \\ \cline{1-1}
j_s+1 \\ \cline{1-1}
j_s+2 \\ \cline{1-1}
\multicolumn{1}{c}{\vdots} \\ \cline{1-1}
j_d+1 \\ \cline{1-1}
j_d+2 \\ \cline{1-1}
\end{array}$$

with all omitted entries as in $\mathtt{T}$.

Example. Let $\lambda = (7, 1^8)$ and $j = 5$. If

$$\mathtt{T} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 10 \\ \cline{1-1} 11 \\ \cline{1-1} 12 \\ \cline{1-1} 13 \\ \cline{1-1} 14 \\ \cline{1-1} 15 \\ \cline{1-1} \end{array} \quad \text{and } t = 2 \text{ then} \quad (\mathtt{T})^{2,5} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 6 & 7 & 8 & 9 & 12 & 13 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} 10 \\ \cline{1-1} 11 \\ \cline{1-1} 14 \\ \cline{1-1} 15 \\ \cline{1-1} \end{array} .$$

Note that in this example $v_{\mathtt{T}} = \Psi\overset{9}{\underset{5}{\downarrow}}\,\Psi\overset{11}{\underset{7}{\downarrow}}\,\Psi\overset{13}{\underset{9}{\downarrow}}\,z_\lambda$ and $v_{(\mathtt{T})^{2,5}} = \Psi\overset{9}{\underset{7}{\downarrow}}\,\Psi\overset{13}{\underset{9}{\downarrow}}\,z_\lambda$.

Corollary 3.21. *Suppose* $\mathtt{T} \in \mathrm{Dom}(\lambda)$ *and the dominoes* $[j-1, j], [j+1, j+2]$ *and* $[j+3, j+4]$ *are all in the arm of* $\mathtt{T}$. *Let* $v_{\mathtt{T}} = \Psi\overset{j_1}{\underset{k_1}{\downarrow}} \dots \Psi\overset{j_d}{\underset{k_d}{\downarrow}}\, z_\lambda$. *That is, for some* ℓ *the arm of* $\mathtt{T}$ *has dominoes* $[k_\ell - 1, k_\ell], [k_\ell + 1, k_\ell + 2], \dots, [j-1, j], [j+1, j+2], \dots, [j+r-1, j+r]$, *where* $r \geqslant 4$ *and if* $k_\ell \neq 3$ *the domino* $[k_\ell - 3, k_\ell - 2]$ *is in the leg of* $\mathtt{T}$. *Suppose* $\psi_j v_{\mathtt{T}} \neq 0$ *and let* t *be minimal such that* $j_{\ell+t-1} < 4t + j$. *Then* $\psi_j v_{\mathtt{T}} = \psi_j v_{(\mathtt{T})^{t,j}}$, *which is in reduced form.*

Remark. Note that, as we saw in the proof of Lemma 3.19, our minimality condition on t ensures $(j_{s'} =) j_{\ell+t-1} = 4t - 2 + j$.

Also, since the corollary is telling us that $\psi_j v_{(\mathtt{T})^{t,j}}$ is in reduced form, we know that $y_2 \geqslant j + r$. If not, the rotation action would be moving the domino $[j_{s'} + 1, j_{s'} + 2]$ somewhere into the middle of the sequence $[j-1, j], \dots, [j+r-1, j+r]$. But $j + r = j_\ell \leqslant j_{s'} < j_{s'} + 2$, so $(\mathtt{T})^{t,j}$ would not be standard, and therefore our corresponding $(v_{\mathtt{T}})^t$ would not be in reduced form.

Finally, note that for tableaux of the case 2 form but with $[j+3, j+4]$ is in the leg, our result agrees with this lemma, with $t = 1$.

Proof. From the proof of Lemma 3.19 we have

$$\psi_j(v_{\mathtt{T}}) = \psi_j \Psi^* \Psi \overset{j-2}{\underset{k_\ell}{\downarrow}}\; \Psi \overset{j_\ell}{\underset{k_{\ell+1}}{\downarrow}}\; \Psi \overset{j_{\ell+1}}{\underset{k_{\ell+2}}{\downarrow}} \dots \Psi \overset{j_{\ell+t-3}}{\underset{k_{\ell+t-2}}{\downarrow}}\; \Psi \overset{j_{\ell+t-2}}{\underset{k_{\ell+t-1}}{\downarrow}}\; \Psi \overset{j_{\ell+t}}{\underset{k_{\ell+t}}{\downarrow}} \dots \Psi \overset{j_d}{\underset{k_d}{\downarrow}}\, z_\lambda$$

which is in reduced form. But this is precisely the result we need. To see why $t+1$ of the dominoes in the arm are changed, we notice that the dominoes moved are all dominoes from $[j-1, j]$ to $[j_{\ell+t-1}+1, j_{\ell+t-1}+2] = [j+4t-1, j+4t]$. There are in total $2t+1$ of these dominoes, and t of them are in the leg. □

Next, we look at the final remaining unsolved case, case 4 from Lemma 3.18. This time we are interested in finding a reduced expression for $\psi_j v_{\mathtt{T}}$, when the dominoes $[j-3, j-2], [j-1, j]$ and $[j+1, j+2]$ all appear in the leg of $\mathtt{T}$, else we may appeal to Corollary 3.17 (2). Again, we begin by dismissing the "degenerate" case – if $v_{\mathtt{T}}$ only involves terms Ψ_k for $k \geqslant j+4$ (once again, $\mathtt{T} = \mathtt{T}_\lambda$ is a trivial example of this) then clearly $\psi_j v_{\mathtt{T}} = 0$. We may assume $j \neq 3$, where it does not make sense to consider the domino $[j-3, j-2]$; the case $j = 3$ is easily dealt with in Proposition 3.27 (4).

Definition 3.22. Let $\mathtt{T} \in \mathrm{Dom}(\lambda)$. Then $v_{\mathtt{T}}$ can be written in the following form, which we will call ***reverse normal form***, or ***RNF***:

$$v_{\mathtt{T}} = \Psi\overset{k_1}{\underset{j_1}{\uparrow}} \Psi\overset{k_2}{\underset{j_2}{\uparrow}} \dots \Psi\overset{k_d}{\underset{j_d}{\uparrow}} z_\lambda$$

with $j_{i+1} < j_i$ and $k_{i+1} = k_i - 2$ for all $i = 1, 2, \dots, d$. Once again, $k_d = b+1$.

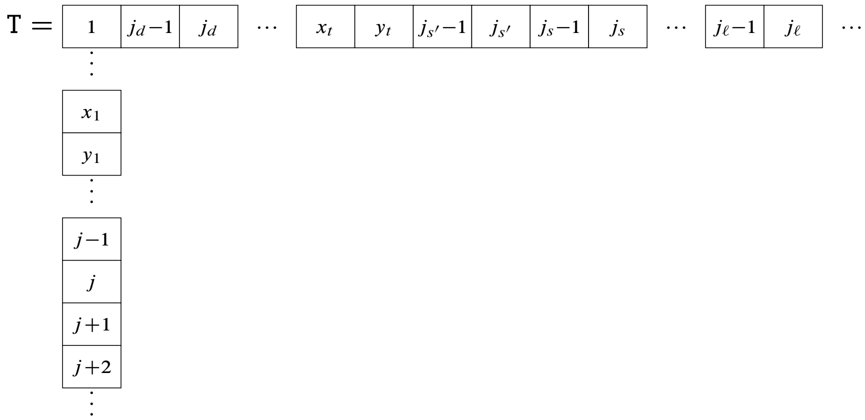
Remark. The existence of the RNF of $v_{\mathtt{T}}$ can be realised as placing dominoes in the arm of $\mathtt{T}$, as opposed to our normal form which acted to place the dominoes in the leg. Explicitly, the RNF is placing the dominoes $[j_d - 1, j_d], [j_{d-1} - 1, j_{d-1}], \dots, [j_1 - 1, j_1]$ at the start of the arm, while any further dominoes the same as in $\mathtt{T}_\lambda$ obviously do not have any impact on the expression.

Suppose ℓ is minimal such that Ψ_j appears in the sequence $\Psi\overset{k_\ell}{\underset{j_\ell}{\uparrow}}$ in the RNF of $v_{\mathtt{T}}$. Having the dominoes $[j-3, j-2], [j-1, j]$ and $[j+1, j+2]$ in the leg of $\mathtt{T}$ is then equivalent to the conditions $j_\ell \leqslant j-4$ and $k_\ell \geqslant j$. Then we have

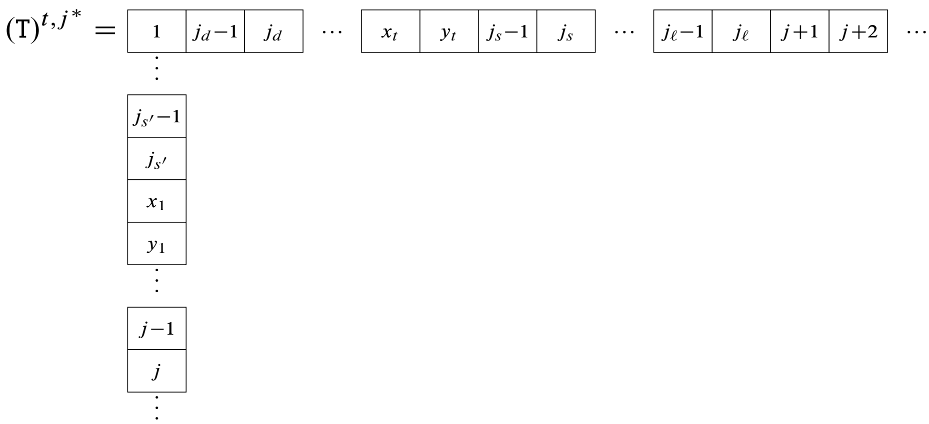
Lemma 3.23. *Let $v_{\mathtt{T}} \in \mathscr{D}$ have RNF $v_{\mathtt{T}} = \Psi\overset{k_1}{\underset{j_1}{\uparrow}} \Psi\overset{k_2}{\underset{j_2}{\uparrow}} \dots \Psi\overset{k_d}{\underset{j_d}{\uparrow}} z_\lambda$, for $d \geqslant 1$, with $j_\ell \leqslant j-4$ and $k_\ell \geqslant j$. Then $j_{\ell+i-1} \leqslant j - 4i$ for all $i = 1, 2, \dots, d-\ell+1$ if and only if $\psi_j v_{\mathtt{T}} = 0$.*

Proof. With some minor tweaking of notation and indices, the proof follows the proof of Lemma 3.19 almost identically. □

Definition 3.24. Suppose



where the section of the leg $[x_1, y_1], \ldots, [j-1, j], [j+1, j+2]$ contains $t+1$ dominoes and the section of the arm $[j_{s'}-1, j_{s'}], [j_s-1, j_s], \ldots, [j_\ell-1, j_\ell]$ contains t dominoes. Here, as in Definition 3.20 we have $s = \ell + t - 2$ and $s' = s + 1$ for neatness. Then we define $(\mathtt{T})^{t,j^*}$ to be the standard λ-tableau obtained from $\mathtt{T}$ by moving the domino $[j+1, j+2]$ to the arm and the domino $[j_{s'}-1, j_{s'}]$ to the leg; this can be seen as cyclically permuting these $2t+1$ dominoes "anticlockwise" by one space but keeping all other entries the same as they are in $\mathtt{T}$. That is,



with all omitted entries as in $\mathtt{T}$.

Example. Let $\lambda = (7, 1^8)$ and $j = 13$. If

$$\mathtt{T} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 10 \\ \cline{1-1} 11 \\ \cline{1-1} 12 \\ \cline{1-1} 13 \\ \cline{1-1} 14 \\ \cline{1-1} 15 \\ \cline{1-1} \end{array} \quad \text{and } t = 2 \text{ then} \quad (\mathtt{T})^{2,13^*} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 4 & 5 & 8 & 9 & 14 & 15 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 6 \\ \cline{1-1} 7 \\ \cline{1-1} 10 \\ \cline{1-1} 11 \\ \cline{1-1} 12 \\ \cline{1-1} 13 \\ \cline{1-1} \end{array}.$$

Note that in this example $v_{\mathtt{T}} = \Psi\overset{9}{\underset{5}{\downarrow}} \Psi\overset{11}{\underset{7}{\downarrow}} \Psi\overset{13}{\underset{9}{\downarrow}} z_\lambda = \Psi\overset{13}{\underset{9}{\uparrow}} \Psi\overset{11}{\underset{7}{\uparrow}} \Psi\overset{9}{\underset{5}{\uparrow}} z_\lambda$ and $v_{(\mathtt{T})^{2,13^*}} = \Psi\overset{11}{\underset{9}{\uparrow}} \Psi\overset{9}{\underset{5}{\uparrow}} z_\lambda$.

Corollary 3.25. *Suppose $T \in \mathrm{Dom}(\lambda)$ and the dominoes $[j-3, j-2], [j-1, j]$ and $[j+1, j+2]$ are all in the leg of* T. *Let $v_{\mathtt{T}} = \Psi\overset{k_1}{\underset{j_1}{\uparrow}} \dots \Psi\overset{k_d}{\underset{j_d}{\uparrow}} z_\lambda$. That is, for some x the leg of* T *has consecutive dominoes $[x, x+1], [x+2, x+3], \dots, [j-3, j-2], [j-1, j], \dots, [j+r-1, j+r]$, where $r \geqslant 2$. Suppose $\psi_j v_{\mathtt{T}} \neq 0$ and let t be minimal such that $j_{\ell+t-1} > j - 4t$. Then $\psi_j v_{\mathtt{T}} = \psi_j v_{(\mathtt{T})^{t,j^*}}$, which is in reduced form.*

Proof. Analogously to Corollary 3.21, this follows immediately from Lemma 3.23. □

Remark. For a given tableau T in the form of Lemma 3.18 (2) or Lemma 3.18 (4), we have an equivalent way of calculating directly from T exactly when a certain ψ_j acts on $v_{\mathtt{T}}$ to give zero or a "rotation" as in Definition 3.20 or Definition 3.24, and what t should be for the latter. Suppose T has the dominoes $[j-1, j], [j+1, j+2]$ and $[j+3, j+4]$ in the arm, and we wish to act on $v_{\mathtt{T}}$ by ψ_j. Examine, the dominoes containing $j, j+2, j+4, \dots$ in the natural order. Count how many are in the arm and how many are in the leg as we go along. If at any stage we have counted $t+1$ dominoes in the arm and t in the leg, for some t, we stop. We have found the t in Corollary 3.21 and we can perform the corresponding rotation. If we reach the final domino (that is, the domino $[n-1, n]$) without ever satisfying this condition, $\psi_j v_{\mathtt{T}} = 0$.

In the case of Lemma 3.18 (4), we have an analogous algorithm. Suppose we have a tableau T with the dominoes $[j-3, j-2], [j-1, j]$ and $[j+1, j+2]$ in the leg. This time, we count the dominoes containing $j+2, j, j-2, \dots$ and stop if at some stage we have counted $t+1$ dominoes in the leg and t dominoes in the arm. This gives us the t seen in Corollary 3.25 and we can find the necessary tableau. If we reach the domino $[2,3]$ and our condition has not been satisfied at any point, we once again have 0.

Proposition 3.26. *The algorithms in the previous remarks yield the correct reduced form for* $\psi_j v_{\mathrm{T}}$.

Proof. We prove this for the first algorithm; the second may be proved analogously. Our proof is mainly just a translation between the language of the algorithm and the language of Corollary 3.21. Let ℓ be minimal such that $k_\ell \leqslant j \leqslant j_\ell$; equivalently, the first domino in the leg our algorithm counts is $[j_\ell+1, j_\ell+2]$. Note that for each i, we have that $[j_{\ell+i-1}+1, j_{\ell+i-1}+2]$ is the ith domino we count in the leg. When counting the ith leg domino during the running of our algorithm, we must have counted at least $i+2$ arm dominoes, else the algorithm would terminate. Therefore we have, in total, counted at least $2i+2$ dominoes, starting with $[j-1, j]$ and thus $j_{\ell+i-1} \geqslant j+4i$. If our algorithm gets to the final domino without ever satisfying the condition that we have counted one more domino in the arm than in the leg, it is clear that this inequality holds for all i and Lemma 3.19 tells us that $\psi_j v_{\mathrm{T}} = 0$. Otherwise, counting the tth domino in the leg, where t satisfies the desired property, we have counted exactly $2t+1$ dominoes, and therefore $j_{\ell+t-1} = j+4t-2$. Appealing to Corollary 3.21 completes the proof. Note that when $t=1$, this agrees with Corollary 3.17. □

In particular the following proposition gives actions of ψ_j on v_{T} in each case of Lemma 3.18.

Proposition 3.27.

1. *Suppose the domino* $[j-1, j]$ *is in the arm of* T *but* $[j+1, j+2]$ *is in the leg. Then* $v_{\mathrm{T}} = \Psi_j v_{\mathrm{S}}$, *where* v_{S} *is the tableau obtained from* T *by transposing the dominoes* $[j-1, j]$ *and* $[j+1, j+2]$, *and* $\psi_j v_{\mathrm{T}} = -2\psi_j v_{\mathrm{S}}$, *which is in reduced form.*

2. *Suppose the dominoes* $[j-1, j]$ *and* $[j+1, j+2]$ *are in the arm of* T.

   If $j = n-2$, *then* $\psi_{n-2} v_T = 0$.

If for every $k \in \{j+2, j+4, \ldots, n\}$, more of the dominoes $[j+1, j+2], \ldots, [k-1, k]$ lie in the arm of $\mathtt{T}$ *than in the leg, then $\psi_j v_{\mathtt{T}} = 0$. Otherwise, $\psi_j v_T = \psi_j v_{\mathtt{U}}$, where* $\mathtt{U}$ *is given in Proposition 3.26 and $\psi_j v_{\mathtt{U}}$ is in reduced form.*

3. *Suppose the domino $[j-1, j]$ is in the leg of* $\mathtt{T}$ *but $[j+1, j+2]$ is in the arm. Then $\psi_j v_T$ is already in reduced form.*

4. *Suppose the dominoes $[j-1, j]$ and $[j+1, j+2]$ are in the leg of* $\mathtt{T}$*.*

   If $j = 3$ then $\psi_3 v_T = 0$.

   If for every $k \in \{j, j-2, \ldots, 3\}$, more of the dominoes $[j-1, j], \ldots, [k-1, k]$ lie in the leg of $\mathtt{T}$ *than in the arm, then $\psi_j v_{\mathtt{T}} = 0$. Otherwise, $\psi_j v_T = \psi_j v_{\mathtt{U}}$, where* $\mathtt{U}$ *is given in Proposition 3.26 and $\psi_j v_{\mathtt{U}}$ is in reduced form.*

Proof. Case 3 falls out naturally as j is in the leg and $j+1$ is in the arm. Cases 1, 2 (if $[j+3, j+4]$ is in the leg of $\mathtt{T}$) and 4 (if $[j-3, j-2]$ is in the arm of $\mathtt{T}$) follow from Lemma 3.16 and Corollary 3.17. The remaining parts of cases 2 and 4 are handled by Proposition 3.26. □

We have now given all actions of the KLR generators on elements of $\mathscr{D}$. This information was crucial in our discovery of the endomorphism f given in Proposition 3.30.

Lemma 3.28. $\dim(\mathrm{End}_{\mathcal{H}}(S_\lambda)) \leqslant 1 + b/2$.

Proof. Define M to be a matrix whose columns are indexed by $\mathtt{T} \in \mathrm{Dom}(\lambda)$ (in increasing order with respect to $\trianglerighteq$), and whose rows are indexed by pairs $(j, \mathtt{T})$ for $3 \leqslant j \leqslant n-2$ odd, $j \neq b+1$ and $\mathtt{T} \in \mathrm{Dom}(\lambda)$. The entry of M in position $((j, \mathtt{T}), \mathtt{S})$ is the coefficient of $v_{s_j \mathtt{T}}$ when $\psi_j v_{\mathtt{S}}$ is written as a linear combination of elements of $\mathscr{D}$. Thus we may consider $\mathrm{End}_{\mathcal{H}}(\mathrm{S}_\lambda)$ as the nullspace of M, and we have only defined M up to reordering of rows.

First we note that every action of ψ_j on $v_{\mathtt{T}} \in \mathscr{D}$ yields a linear combination of basis vectors indexed by tableaux which are dominated by $s_j \mathtt{T}$, unless $\psi_j v_{\mathtt{T}}$ is already a reduced ψ-expression (equivalently, $\mathtt{T}$ has the domino $[j-1, j]$ in the leg and the domino $[j+1, j+2]$ in the arm). So, for $j \neq b+1$, ψ_j gives rise to a family of relations whose $\trianglerighteq$-maximal term is some tableau with the domino $[j-1, j]$ in the leg and the domino $[j+1, j+2]$ in the arm. In particular, if we take a submatrix M_j of M by only choosing rows with first index j, we have an upper triangular square matrix; M_j has a 1 on the diagonal whenever the entry corresponds to a tableau with the domino $[j-1, j]$ in the leg and the domino $[j+1, j+2]$ in the arm.

But every tableau in $\mathrm{Dom}(\lambda)$ is of the above form for some odd $j \neq b+1$, except some of those with $[b, b+1]$ in the leg and $[b+2, b+3]$ in the arm, and also the tableau $\mathtt{T}^\lambda$. In the former case, the tableau can still be viewed as being in the above form for some $j \neq b+1$ unless the leg dominoes come in two consecutive strings, the first ending in $[b, b+1]$ and the second ending in $[n-1, n]$. So in fact, we only have $b/2$ choices for the position of $[b, b+1]$ in the leg and the rest of the tableau is completely determined by this choice, if we would like the tableau not to correspond to a 1 on the diagonal of some submatrix M_j. So we have that the row rank of A is at least $|\mathrm{Dom}(\lambda)| - b/2 - 1$, and thus the nullity of M is at most $b/2 + 1$. □

Example. Let $\lambda = (5, 1^4)$. We have the following tableaux in $\mathrm{Dom}(\lambda)$:

$$\mathtt{T}_\lambda = \begin{array}{|c|c|c|c|c|}\hline 1&6&7&8&9\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}, \quad \mathtt{S} = \begin{array}{|c|c|c|c|c|}\hline 1&4&5&8&9\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 6\\\cline{1-1} 7\\\cline{1-1}\end{array}, \quad \mathtt{T} = \begin{array}{|c|c|c|c|c|}\hline 1&4&5&6&7\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 8\\\cline{1-1} 9\\\cline{1-1}\end{array},$$

$$\mathtt{U} = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&8&9\\\hline 4\\\cline{1-1} 5\\\cline{1-1} 6\\\cline{1-1} 7\\\cline{1-1}\end{array}, \quad \mathtt{V} = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&6&7\\\hline 4\\\cline{1-1} 5\\\cline{1-1} 8\\\cline{1-1} 9\\\cline{1-1}\end{array}, \quad \mathtt{T}^\lambda = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 6\\\cline{1-1} 7\\\cline{1-1} 8\\\cline{1-1} 9\\\cline{1-1}\end{array}.$$

We want to consider the actions of ψ_3 and ψ_7 on these six tableaux. The actions, easily deduced from Lemma 3.13, are as follows:

$$\psi_3 v_{\mathtt{T}_\lambda} = 0, \qquad \psi_7 v_{\mathtt{T}_\lambda} = 0,$$

$$\psi_3 v_{\mathtt{S}} = v_{s_3\mathtt{S}}, \qquad \psi_7 v_{\mathtt{S}} = v_{s_7\mathtt{S}},$$

$$\psi_3 v_{\mathtt{T}} = v_{s_3\mathtt{T}}, \qquad \psi_7 v_{\mathtt{T}} = -2v_{s_7\mathtt{S}},$$

$$\psi_3 v_{\mathtt{U}} = -2v_{s_3\mathtt{S}}, \qquad \psi_7 v_{\mathtt{U}} = v_{s_7\mathtt{U}},$$

$$\psi_3 v_{\mathtt{V}} = -2v_{s_3\mathtt{T}}, \qquad \psi_7 v_{\mathtt{V}} = -2v_{s_7\mathtt{U}},$$

$$\psi_3 v_{\mathtt{T}^\lambda} = v_{s_3\mathtt{T}}, \qquad \psi_7 v_{\mathtt{T}^\lambda} = v_{s_7\mathtt{U}}.$$

So the matrix A in the above proof is as follows.

	$\mathtt{T}_\lambda$	S	T	U	V	$\mathtt{T}^\lambda$
$\mathtt{T}_\lambda$	·	·	·	·	·	·
S	·	1	·	−2	·	·
T	·	·	1	·	−2	1
U	·	·	·	·	·	·
V	·	·	·	·	·	·
$\mathtt{T}^\lambda$	·	·	·	·	·	·
$\mathtt{T}_\lambda$	·	·	·	·	·	·
S	·	1	−2	·	·	·
T	·	·	·	·	·	·
U	·	·	·	1	−2	1
V	·	·	·	·	·	·
$\mathtt{T}^\lambda$	·	·	·	·	·	·

Note that the top half of the matrix corresponds to ψ_3 (or $j = 3$ if we are indexing the rows as in the proof) and the bottom half to ψ_7. It is clear that, as in the proof, the matrix has row rank at least 3, as we having leading terms "on the diagonal" in columns corresponding to S, T and U.

3.5 Decomposability of $S_{(a,1^b)}$ when n is odd

We can now begin calculating $\mathcal{H}$-endomorphisms of S_λ. We now know that $f \in \mathrm{End}_{\mathcal{H}}(S_\lambda)$ if and only if

$$f(z_\lambda) = \sum_{\mathtt{T}\in\mathrm{Dom}(\lambda)} \alpha_{\mathtt{T}} v_{\mathtt{T}} \quad \text{for some } \alpha_{\mathtt{T}} \in \mathbb{F}$$

with $\psi_j f(z_\lambda) = 0$ for all odd $j \neq b+1$ with $3 \leqslant j \leqslant n-2$.

Definition 3.29. Let i, j be odd integers with $3 \leqslant i \leqslant b+1 < j \leqslant n$. We will denote by $\mathtt{T}_{i,j}$ the tableau with dominoes $\{[2,3],[4,5],\ldots,[b,b+1],[j-1,j]\} \setminus \{[i-1,i]\}$ in the leg.

Example. If $\lambda = (5, 1^4)$ then $\mathtt{T}_{5,9} = \begin{array}{|c|c|c|c|c|}\hline 1 & 4 & 5 & 6 & 7 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 8 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array}$ and $\mathtt{T}_{3,7} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 8 & 9 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}$.

Remark. We observe that the normal form for $v_{\mathtt{T}_{i,j}}$ is $\Psi \overset{b-1}{\underset{i}{\uparrow}} \Psi \overset{j-2}{\underset{b+1}{\downarrow}} z_\lambda$.

Proposition 3.30. *Suppose a is odd and b is even. Then there exists an $\mathcal{H}$-endomorphism f of S_λ given by*

$$f(z_\lambda) = \sum_{\substack{3 \leqslant i \leqslant b+1 \\ b+3 \leqslant j \leqslant n \\ i,j \text{ odd}}} \frac{i-1}{2} \cdot \frac{n+2-j}{2} v_{\mathtt{T}_{i,j}}.$$

Proof. All we need to show is that $\psi_k f(z_\lambda) = 0$ for all odd $k \neq b+1$ with $3 \leqslant k \leqslant n-2$. We will rely extensively on our previous results regarding the actions of ψ generators on tableaux.

First, notice that $\psi_3 v_{\mathtt{T}_{i,j}} = 0$ for all $i \geqslant 7$. So

$$\begin{aligned} \psi_3 f(z_\lambda) &= \psi_3 \left(\sum_j 2 \cdot \frac{n+2-j}{2} v_{\mathtt{T}_{5,j}} + \frac{n+2-j}{2} v_{\mathtt{T}_{3,j}} \right) \\ &= \sum_j \frac{n+2-j}{2} (2\psi_3 \cdot v_{\mathtt{T}_{5,j}} - 2\psi_3 \cdot v_{\mathtt{T}_{5,j}}) \\ &= 0. \end{aligned}$$

Next, suppose $5 \leqslant k \leqslant b-1$. We notice that $\psi_k v_{\mathtt{T}_{i,j}} = 0$ for all $i \leqslant k-4$ and for all $i \geqslant k+4$. So

$$\begin{aligned} \psi_k f(z_\lambda) = \psi_k \Bigg(\sum_j \frac{k+1}{2} \cdot \frac{n+2-j}{2} v_{\mathtt{T}_{k+2,j}} + \frac{k-1}{2} \cdot \frac{n+2-j}{2} v_{\mathtt{T}_{k,j}} \\ + \frac{k-3}{2} \cdot \frac{n+2-j}{2} v_{\mathtt{T}_{k-2,j}} \Bigg) \end{aligned}$$

$$= \sum_j \frac{n+2-j}{2}\left(\frac{k+1}{2} - 2\cdot\frac{k-1}{2} + \frac{k-3}{2}\right)\psi_k v_{\mathrm{T}_{k+2,j}}$$
$$= 0.$$

Now, for $b+3 \leqslant k \leqslant n-4$, we notice that $\psi_k v_{\mathrm{T}_{i,j}} = 0$ for all $j \leqslant k-2$ and for all $j \geqslant k+6$. So

$$\begin{aligned}\psi_k f(z_\lambda) &= \psi_k\left(\sum_i \frac{i-1}{2}\cdot\frac{n+2-k}{2}v_{\mathrm{T}_{i,k}} + \frac{i-1}{2}\cdot\frac{n-k}{2}v_{\mathrm{T}_{i,k+2}}\right.\\ &\quad\left. + \frac{i-1}{2}\cdot\frac{n-k-2}{2}v_{\mathrm{T}_{i,k+4}}\right)\\ &= \sum_i \frac{i-1}{2}\left(\frac{n+2-k}{2} - 2\cdot\frac{n-k}{2} + \frac{n-k-2}{2}\right)\psi_k v_{\mathrm{T}_{i,k}}\\ &= 0.\end{aligned}$$

Finally, we notice that $\psi_{n-2} v_{\mathrm{T}_{i,j}} = 0$ unless $j = n-2$ or n. So

$$\begin{aligned}\psi_{n-2} f(z_\lambda) &= \psi_{n-2}\left(\sum_i \frac{i-1}{2}\cdot 2\cdot v_{\mathrm{T}_{i,n-2}} + \frac{i-1}{2}v_{\mathrm{T}_{i,n}}\right)\\ &= \sum_i (i-1)\psi_{n-2}v_{\mathrm{T}_{i,n-2}} - 2\cdot\frac{i-1}{2}\psi_{n-2}v_{\mathrm{T}_{i,n-2}}\\ &= 0.\end{aligned}$$

□

Example. If $\lambda = (5, 1^4)$, then our endomorphism is given by

$$f(z_\lambda) = 2v_{\mathrm{T}_{3,7}} + 4v_{\mathrm{T}_{5,7}} + v_{\mathrm{T}_{3,9}} + 2v_{\mathrm{T}_{5,9}} = 2\Psi_3\Psi_5 z_\lambda + 4\Psi_5 z_\lambda + \Psi_3\Psi_7\Psi_5 z_\lambda + 2\Psi_7\Psi_5 z_\lambda.$$

Remark. This endomorphism allows us to tackle our decomposability question. In particular, S_λ can be decomposed into a direct sum of the generalised eigenspaces of f. That is $E_x = \{v \in \mathrm{S}_\lambda \mid (f - xI)^n v = 0 \text{ for some } n \in \mathbb{N}\}$ for each eigenvalue x of f, and

$$\mathrm{S}_\lambda = \bigoplus_{x \text{ an eigenvalue of } f} E_x.$$

From the definition of E_x it is clear that it is a non-zero $\mathcal{H}$-module whenever x is an eigenvalue of f. The existence of two distinct eigenvalues of f would ensure that we have at least two non-trivial summands in the decomposition above, and we would be done.

The following lemma will be used repeatedly in further proofs.

Lemma 3.31. *Suppose $x_1 \geqslant y_1 \geqslant 3$ and $x_2 \geqslant y_2 \geqslant 3$ are all odd numbers. Suppose also that $X \in e(i_\lambda)\,\mathrm{S}_\lambda$. Then we have the following cancellation relations:*

1. *If $x_1 \geqslant x_2 \geqslant y_1$ we have*

$$\Psi\underset{y_1}{\overset{x_1}{\downarrow}}\,\Psi\underset{y_2}{\overset{x_2}{\downarrow}}\,X = \Psi\underset{y_1}{\overset{x_2-4}{\downarrow}}\,\Psi\underset{y_2}{\overset{x_1}{\downarrow}}\,X.$$

2. *If $x_2 \geqslant y_1 \geqslant y_2$ we have*

$$\Psi\underset{y_1}{\overset{x_1}{\downarrow}}\,\Psi\underset{y_2}{\overset{x_2}{\downarrow}}\,X = \Psi\underset{y_2}{\overset{x_1}{\downarrow}}\,\Psi\underset{y_1+4}{\overset{x_2}{\downarrow}}\,X.$$

Proof. 1.

$$\begin{aligned}\Psi\underset{y_1}{\overset{x_1}{\downarrow}}\,\Psi\underset{y_2}{\overset{x_2}{\downarrow}}\,X &= \Psi\underset{x_2+2}{\overset{x_1}{\downarrow}}\,\Psi_{x_2}\Psi_{x_2-2}\Psi\underset{y_1}{\overset{x_2-4}{\downarrow}}\,\Psi\underset{y_2}{\overset{x_2}{\downarrow}}\,X\\ &= \Psi\underset{x_2+2}{\overset{x_1}{\downarrow}}\,\Psi_{x_2}\Psi_{x_2-2}\Psi_{x_2}\Psi\underset{y_1}{\overset{x_2-4}{\downarrow}}\,\Psi\underset{y_2}{\overset{x_2-2}{\downarrow}}\,X\\ &= \Psi\underset{x_2+2}{\overset{x_1}{\downarrow}}\,\Psi_{x_2}\Psi\underset{y_1}{\overset{x_2-4}{\downarrow}}\,\Psi\underset{y_2}{\overset{x_2-2}{\downarrow}}\,X\\ &= \Psi\underset{y_1}{\overset{x_2-4}{\downarrow}}\,\Psi\underset{y_2}{\overset{x_1}{\downarrow}}\,X.\end{aligned}$$

2. The proof proceeds similarly to the previous case. □

Now, we work towards computing the eigenvalues of f. It is clear that f acts on $e(i_\lambda)\,\mathrm{S}_\lambda$; $f(v_{\mathtt{T}}) \in e(i_\lambda)\,\mathrm{S}_\lambda$ whenever $\mathtt{T} \in \mathrm{Dom}(\lambda)$ by the nature of our actions of ψ generators on elements of $\mathscr{D}$. We will show that the action of f on $e(i_\lambda)\,\mathrm{S}_\lambda$ is triangular. Take $\mathtt{T} \in \mathrm{Dom}(\lambda)$, and write $v_{\mathtt{T}}$ in normal form:

$$v_{\mathtt{T}} = \Psi\underset{b+3-2d}{\overset{j_1}{\downarrow}}\,\Psi\underset{b+5-2d}{\overset{j_2}{\downarrow}}\,\dots\Psi\underset{b+1}{\overset{j_d}{\downarrow}}\,z_\lambda.$$

Then we want to look at

$$\begin{aligned} f(v_{\mathrm{T}}) &= \Psi \overset{j_1}{\underset{b+3-2d}{\downarrow}} \Psi \overset{j_2}{\underset{b+5-2d}{\downarrow}} \dots \Psi \overset{j_d}{\underset{b+1}{\downarrow}} \cdot f(z_\lambda) \\ &= \sum_{\substack{3 \leqslant i \leqslant b+1 \\ b+3 \leqslant j \leqslant n \\ i,j \text{ odd}}} \frac{i-1}{2} \cdot \frac{n+2-j}{2} \Psi \overset{j_1}{\underset{b+3-2d}{\downarrow}} \Psi \overset{j_2}{\underset{b+5-2d}{\downarrow}} \dots \Psi \overset{j_d}{\underset{b+1}{\downarrow}} \cdot \Psi \overset{b-1}{\underset{i}{\uparrow}} \Psi \overset{j-2}{\underset{b+1}{\downarrow}} z_\lambda. \end{aligned}$$

We begin by looking at the simplified case where $d = 1$.

Lemma 3.32. *Let $3 \leqslant i \leqslant b+1 < j \leqslant n$ and $j_d \geqslant b+1$. Then*

$$\Psi \overset{j_d}{\underset{b+1}{\downarrow}} \cdot \Psi \overset{b-1}{\underset{i}{\uparrow}} \Psi \overset{j-2}{\underset{b+1}{\downarrow}} z_\lambda = \begin{cases} -2\Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda & \text{if } i = b+1 \text{ and } j = b+3, \\ \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda & \text{if } i = b+1 \text{ and } j = b+5, \\ 0 & \text{if } i = b+1 \text{ and } j \geqslant b+7, \\ \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda & \text{if } i = b-1 \text{ and } j = b+3, \\ \Psi \overset{b-3}{\underset{i}{\uparrow}} \Psi \overset{j_d}{\underset{b-1}{\downarrow}} \Psi \overset{j-2}{\underset{b+1}{\downarrow}} z_\lambda & \text{if } i \leqslant b-1 \text{ and } j_d \leqslant j-4, \\ 0 & \text{if } i < b-1 \text{ and } j_d \geqslant j-2 \text{ and } j = b+3, \\ \Psi \overset{b-3}{\underset{i}{\uparrow}} \Psi \overset{j-6}{\underset{b-1}{\downarrow}} \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda & \text{if } i \leqslant b-1 \text{ and } j_d \geqslant j-2 \text{ and } j \geqslant b+5. \end{cases}$$

Proof. First suppose $i = b+1$. If $j = b+3$ we have

$$\Psi \overset{j_d}{\underset{b+1}{\downarrow}} \cdot \Psi_{b+1} z_\lambda = -2\Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda.$$

If $j = b+5$ we have

$$\Psi \overset{j_d}{\underset{b+1}{\downarrow}} \cdot \Psi_{b+3}\Psi_{b+1} z_\lambda = \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda.$$

If $j \geqslant b+7$ we have

$$\Psi \overset{j_d}{\underset{b+1}{\downarrow}} \cdot \Psi \overset{j-2}{\underset{b+1}{\downarrow}} z_\lambda = \Psi \overset{j_d}{\underset{b+1}{\downarrow}} \cdot \Psi \overset{j-2}{\underset{b+5}{\downarrow}} z_\lambda = 0.$$

If $i = b-1$ and $j = b+3$ we have

$$\Psi \overset{j_d}{\underset{b+1}{\downarrow}} \cdot \Psi_{b-1}\Psi_{b+1} z_\lambda = \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda .$$

Next, suppose $i \leqslant b-1$. If $j_d \leqslant j-4$ we already have an expression in reduced form and the commuting relations alone put it into our normal form to give the stated result.

So let $j_d \geqslant j-2$. Suppose $i < b-1$ and $j = b+3$. Then we have

$$\begin{aligned}
\Psi \overset{j_d}{\underset{b+1}{\downarrow}} \cdot \Psi \overset{b-1}{\underset{i}{\uparrow}} \Psi_{b+1} z_\lambda &= \Psi \overset{j_d}{\underset{b+3}{\downarrow}} \cdot \Psi \overset{b-3}{\underset{i}{\uparrow}} \Psi_{b+1}\Psi_{b-1}\Psi_{b+1} z_\lambda \\
&= \Psi \overset{j_d}{\underset{b+1}{\downarrow}} \cdot \Psi \overset{b-3}{\underset{i}{\uparrow}} z_\lambda \\
&= 0.
\end{aligned}$$

Finally, let $j \geqslant b+5$. Then we have

$$\begin{aligned}
\Psi \overset{j_d}{\underset{b+1}{\downarrow}} \cdot \Psi \overset{b-1}{\underset{i}{\uparrow}} \Psi \overset{j-2}{\underset{b+1}{\downarrow}} z_\lambda &= \Psi \overset{b-3}{\underset{i}{\uparrow}} \Psi \overset{j_d}{\underset{b-1}{\downarrow}} \Psi \overset{j-2}{\underset{b+1}{\downarrow}} z_\lambda \\
&= \Psi \overset{b-3}{\underset{i}{\uparrow}} \Psi \overset{j-6}{\underset{b-1}{\downarrow}} \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda
\end{aligned}$$

which is reduced and in our normal form. □

Proposition 3.33. *Suppose* $v_T = \Psi \overset{j_1}{\underset{b+3-2d}{\downarrow}} \Psi \overset{j_2}{\underset{b+5-2d}{\downarrow}} \dots \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda \in \mathscr{D}$ *is in reduced normal form, i and j are odd with $3 \leqslant i \leqslant b+1 < j \leqslant n$ and let*

$$(*) = \Psi \overset{j_1}{\underset{b+3-2d}{\downarrow}} \Psi \overset{j_2}{\underset{b+5-2d}{\downarrow}} \dots \Psi \overset{j_d}{\underset{b+1}{\downarrow}} \cdot \Psi \overset{b-1}{\underset{i}{\uparrow}} \Psi \overset{j-2}{\underset{b+1}{\downarrow}} z_\lambda .$$

Then $(*)$ is a scalar multiple of either v_T or some longer Ψ-expression. In particular, $(*)$ is a scalar multiple of v_T in precisely the following cases:

$(*) = v_T$ *if*

- $i + j = 2b + 6$, $i \geqslant b + 3 - 2d$, $j_v \geqslant j - 4 - 4(d - v)$ *for all* v;

- $i + j = 2b + 2$, $i \geqslant b + 1 - 2d$ *and* $j_v \geqslant j - 2 - 4(d - v)$ *for all* v.

$(*) = -2v_T$ *if*

- $i + j = 2b + 4$, $i \geqslant b + 3 - 2d$ *and* $j_v \geqslant j - 2 - 4(d - v)$ *for all* v.

Proof. We will use the previous lemma and work down the cases in the order they appear above. We will always look to put expressions into reduced normal form.

1. Let $d > 0$. When $i = b + 1$, we can clearly see that we get $(*) = -2v_T$ when $j = b + 3$, $(*) = v_T$ when $j = b + 5$ and $(*) = 0$ otherwise. It is also clear that when $i = b - 1$ and $j = b + 3$ we have $(*) = v_T$, so in all further cases we will ignore this combination.

2. If $i \leqslant b - 1$ and $j_d \leqslant j - 4$, we must split into two subcases.

   (a) First suppose $i \leqslant b + 1 - 2d$. Then we have

$$\Psi \underset{b+3-2d}{\overset{j_1}{\downarrow}} \Psi \underset{b+5-2d}{\overset{j_2}{\downarrow}} \ldots \Psi \underset{b+1}{\overset{j_d}{\downarrow}} \cdot \Psi \underset{i}{\overset{b-1}{\uparrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}} z_\lambda$$
$$= \Psi \underset{b+3-2d}{\overset{j_1}{\downarrow}} \Psi \underset{b+5-2d}{\overset{j_2}{\downarrow}} \ldots \Psi \underset{b-1}{\overset{j_{d-1}}{\downarrow}} \cdot \Psi \underset{i}{\overset{b-3}{\uparrow}} \Psi \underset{b-1}{\overset{j_d}{\downarrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}} z_\lambda$$
$$= \Psi \underset{i}{\overset{b-1-2d}{\uparrow}} \Psi \underset{b+1-2d}{\overset{j_1}{\downarrow}} \Psi \underset{b+3-2d}{\overset{j_2}{\downarrow}} \ldots \Psi \underset{b-1}{\overset{j_d}{\downarrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}} z_\lambda.$$

   The above expression is reduced and longer than v_T.

   (b) If $i \geqslant b + 3 - 2d$, say $i = k_s - 2 = b - 1 - 2d + 2s$ for some $s \geqslant 2$, we have

$$(*) = \underbrace{\Psi \underset{b+3-2d}{\overset{j_1}{\downarrow}} \ldots \Psi \underset{b-1-2(d-s)}{\overset{j_{s-1}}{\downarrow}}}_{=:\Psi^*} \cdot \Psi \underset{b-1-2(d-s)}{\overset{j_s}{\downarrow}} \Psi \underset{b+1-2(d-s)}{\overset{j_{s+1}}{\downarrow}} \ldots \Psi \underset{b-1}{\overset{j_d}{\downarrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}} z_\lambda.$$

Claim. Suppose for some $s-1 \leqslant u \leqslant d-1$ we have $j_v \geqslant b+3-2(d+s-2v)$ for all $s-1 \leqslant v \leqslant u$. Then the above expression is equal to

$$\Psi^* \, \Psi \overset{j_s}{\underset{b+1-2(d-s)}{\downarrow}} \ldots \Psi \overset{j_u}{\underset{b+1+2(d-u)}{\downarrow}} \Psi \overset{j_{u+1}}{\underset{b+7-2(d+s-2u)}{\downarrow}} \Psi \overset{j_{u+2}}{\underset{b+3-2(d-u)}{\downarrow}} \ldots \Psi \overset{j_d}{\underset{b-1}{\downarrow}} \Psi \overset{j-2}{\underset{b+1}{\downarrow}} z_\lambda .$$

If for the maximal such u we have $u \leqslant d-2$, the expression above is reduced and longer than v_{T}.

Assuming the claim to be true, we need to look at what happens if the condition in the claim holds for $u = d-1$. In this instance, by the claim we have

$$\begin{aligned}
(*) &= \Psi^* \, \Psi \overset{j_s}{\underset{b+1-2(d-s)}{\downarrow}} \ldots \Psi \overset{j_{d-1}}{\underset{b-1}{\downarrow}} \Psi \overset{j_d}{\underset{b+3+2(d-s)}{\downarrow}} \Psi \overset{j-2}{\underset{b+1}{\downarrow}} z_\lambda \\
&= \begin{cases}
-2v_{\mathrm{T}} & \text{if } j = b+5+2(d-s) \text{ and } j_d \geqslant b+3+2(d-s), \\
v_{\mathrm{T}} & \text{if } j = b+7+2(d-s) \text{ and } j_d \geqslant b+3+2(d-s), \\
0 & \text{if } j \geqslant b+9+2(d-s) \text{ and } j_d \geqslant b+3+2(d-s), \\
v_{\mathrm{S}} & \text{otherwise, where } v_{\mathrm{S}} \text{ is some expression longer than } v_{\mathrm{T}}.
\end{cases}
\end{aligned}$$

Note that the first case above never actually occurs here, by the condition that $j_d \leqslant j-4$. We can see that we get $(*) = v_{\mathrm{T}}$ precisely when $j_v \geqslant b+3-2(d+s-2v)$ for all $s-1 \leqslant v \leqslant d-1$ and $j_d = b+3+2(d-s) = j-4$.

Proof of claim. We prove the claim by induction on u. When $u = s-1$, we have that $j_{s-1} \geqslant b-1-2(d-s)$ (which we already knew a priori) and $j_s = b+1-2(d-s)$. Then

$$\Psi^* \cdot \Psi \overset{b+1-2(d-s)}{\underset{b-1-2(d-s)}{\downarrow}} \Psi \overset{j_{s+1}}{\underset{b+1-2(d-s)}{\downarrow}} \ldots \Psi \overset{j_d}{\underset{b-1}{\downarrow}} \Psi \overset{j-2}{\underset{b+1}{\downarrow}} z_\lambda = \Psi^* \cdot \Psi \overset{j_{s+1}}{\underset{b+1-2(d-s)}{\downarrow}} \ldots \Psi \overset{j_d}{\underset{b-1}{\downarrow}} \Psi \overset{j-2}{\underset{b+1}{\downarrow}} z_\lambda$$

and the claim holds.

Suppose the claim is true for some $s-1 \leqslant u \leqslant d-2$, and that $j_v \geqslant b+3-2(d+$

$s-2v$) for all $s-1 \leqslant v \leqslant u+1$. Then by induction, we have

$$\Psi^* \cdot \Psi \underset{b-1-2(d-s)}{\overset{b+1-2(d-s)}{\downarrow}} \Psi \underset{b+1-2(d-s)}{\overset{j_{s+1}}{\downarrow}} \dots \Psi \underset{b-1}{\overset{j_d}{\downarrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}} z_\lambda$$

$$= \Psi^* \underbrace{\Psi \underset{b+1-2(d-s)}{\overset{j_s}{\downarrow}} \dots \Psi \underset{b+1+2(d-u)}{\overset{j_u}{\downarrow}}}_{=:\Psi^\dagger} \Psi \underset{b+7-2(d+s-2u)}{\overset{j_{u+1}}{\downarrow}} \Psi \underset{b+3-2(d-u)}{\overset{j_{u+2}}{\downarrow}} \dots \Psi \underset{b-1}{\overset{j_d}{\downarrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}} z_\lambda$$

$$= \Psi^*\Psi^\dagger \Psi \underset{b+7-2(d+s-2u)}{\overset{j_{u+1}}{\downarrow}} \Psi \underset{b+11-2(d+s-2u)}{\overset{j_{u+2}}{\downarrow}} \Psi \underset{b+3-2(d-u)}{\overset{b+5-2(d+s-2u)}{\downarrow}} \Psi \underset{b+5-2(d-u)}{\overset{j_{u+3}}{\downarrow}} \dots \Psi \underset{b-1}{\overset{j_d}{\downarrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}} z_\lambda$$

since $j_{u+1} \geqslant b+7-2(d+s-2u)$ by hypothesis

$$= \Psi^*\Psi^\dagger \Psi \underset{b+3-2(d-u)}{\overset{j_{u+1}}{\downarrow}} \Psi \underset{b+11-2(d+s-2u)}{\overset{j_{u+2}}{\downarrow}} \Psi \underset{b+5-2(d-u)}{\overset{j_{u+3}}{\downarrow}} \dots \Psi \underset{b-1}{\overset{j_d}{\downarrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}} z_\lambda$$

and the claim is proved.

3. Next, we look at the final case, $i \leqslant b-1$, $j_d \geqslant j-2$ and $j \geqslant b+5$. We have that

$$(*) = \Psi \underset{b+3-2d}{\overset{j_1}{\downarrow}} \Psi \underset{b+5-2d}{\overset{j_2}{\downarrow}} \dots \Psi \underset{b-1}{\overset{j_{d-1}}{\downarrow}} \cdot \Psi \underset{i}{\overset{b-3}{\uparrow}} \Psi \underset{b-1}{\overset{j-6}{\downarrow}} \Psi \underset{b+1}{\overset{j_d}{\downarrow}} z_\lambda$$

Once again, we split into subcases.

(a) First suppose $i \leqslant b+1-2d$. Then

$$(*) = \Psi \underset{i}{\overset{b-1-2d}{\uparrow}} \Psi \underset{b+1-2d}{\overset{j_1}{\downarrow}} \Psi \underset{b+3-2d}{\overset{j_2}{\downarrow}} \dots \Psi \underset{b-3}{\overset{j_{d-1}}{\downarrow}} \Psi \underset{b-1}{\overset{j-6}{\downarrow}} \Psi \underset{b+1}{\overset{j_d}{\downarrow}} z_\lambda.$$

Claim. Suppose for some $0 \leqslant u \leqslant d-1$ we have $j < j_{d-v} + 4(v+1)$ for all $0 \leqslant v \leqslant u$ and $j \geqslant b+3+2v$ for all $0 \leqslant v \leqslant u$. Then

$$(*) = \Psi \underset{i}{\overset{b-1-2d}{\uparrow}} \Psi \underset{b+1-2d}{\overset{j_1}{\downarrow}} \Psi \underset{b+3-2d}{\overset{j_2}{\downarrow}} \dots \Psi \underset{b-3-2u}{\overset{j_{d-u-1}}{\downarrow}} \Psi \underset{b-1-2u}{\overset{j-6-4u}{\downarrow}} \Psi \underset{b+1-2u}{\overset{j_{d-u}}{\downarrow}} \dots \Psi \underset{b+1}{\overset{j_d}{\downarrow}} z_\lambda$$

and this expression is of length $2(u+1)$ less than the length of $(*)$. Furthermore, if we take the maximal such u and have $u \leqslant d-2$ and $j \geqslant b+5+2u$, it is reduced. If $j = b+3+2u$, the expression is equal to 0.

Proof. We prove the claim by induction on u. If $u = 0$, the result follows immediately from Lemma 3.32. Now suppose the claim holds for some $0 \leqslant u \leqslant d-2$, and that $j < j_{d-v} + 4(v+1)$ for all $0 \leqslant v \leqslant u+1$, but $j \geqslant j_{d-u-2} + 4(u+3)$ (if $u \leqslant d-3$). Then by induction, we have

$$
\begin{aligned}
(*) &= \underbrace{\Psi \overset{b-1-2d}{\underset{i}{\uparrow}} \Psi \overset{j_1}{\underset{b+1-2d}{\downarrow}} \Psi \overset{j_2}{\underset{b+3-2d}{\downarrow}} \dots \Psi \overset{j_{d-u-2}}{\underset{b-5-2u}{\downarrow}}}_{=:\Psi^*} \Psi \overset{j_{d-u-1}}{\underset{b-3-2u}{\downarrow}} \Psi \overset{j-6-4u}{\underset{b-1-2u}{\downarrow}} \Psi \overset{j_{d-u}}{\underset{b+1-2u}{\downarrow}} \dots \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda \\
&= \Psi^* \Psi \overset{j_{d-u-1}}{\underset{j-4-4u}{\downarrow}} \Psi_{j-6-4u} \Psi_{j-8-4u} \Psi_{j-6-4u} \Psi \overset{j-10-4u}{\underset{b-3-2u}{\downarrow}} \Psi \overset{j-8-4u}{\underset{b-1-2u}{\downarrow}} \Psi \overset{j_{d-u}}{\underset{b+1-2u}{\downarrow}} \dots \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda \\
&= \Psi^* \Psi \overset{j-10-4u}{\underset{b-3-2u}{\downarrow}} \Psi \overset{j_{d-u-1}}{\underset{b-1-2u}{\downarrow}} \Psi \overset{j_{d-u}}{\underset{b+1-2u}{\downarrow}} \dots \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda,
\end{aligned}
$$

which is the claimed expression. In the induction step, 2 Ψ terms have been deleted, which proves the length part of the claim. It is clear that if $j < b+5+2u$ then the expression in the claim is 0 and likewise that when $u \leqslant d-2$ (and $j \geqslant b+5+2u$), we have a reduced expression.

Now, let u be maximal under the conditions in the claim. First, suppose that $u \leqslant d-2$. By the claim, we can assume that $j \geqslant b+5+2u$. This implies that $\Psi \overset{j-2}{\underset{b+1}{\downarrow}}$ has length at least $u+2$. Similarly, $i \leqslant b+1-2d$ and $u \leqslant d-2$ imply $\Psi \overset{b-1}{\underset{i}{\uparrow}}$ also has length at least $u+2$. So by the claim, once $(*)$ has been written in a reduced form, it has length at least 2 more than v_T.

But what if $u = d-1$? The above claim tells us that

$$
(*) = \Psi \overset{b-1-2d}{\underset{i}{\uparrow}} \Psi \overset{j-2-4d}{\underset{b+1-2d}{\downarrow}} v_\mathrm{T}.
$$

This is zero unless $j \geqslant b+3+2d$, in which case we have a (reduced) longer expression than v_T, or $i \geqslant b+1-2d$. Note that in the latter case, we in fact have $i = b+1-2d$ because of the conditions on the subcase we are looking at. Looking at this case, we assume $j < b+3+2d$, since $j \geqslant b+3+2d$ yields an expression longer than v_T. Under these conditions, we have $(*) = v_T$.

(b) Finally, suppose that $i \geqslant b+3-2d$. Say $i = k_s - 2 = b-1-2(d-s)$ for some $s \geqslant 2$. Then

$$(*) = \Psi \underset{b+3-2d}{\overset{j_1}{\downarrow}} \dots \Psi \underset{b-1-2(d-s)}{\overset{j_{s-1}}{\downarrow}} \cdot\Psi \underset{b-1-2(d-s)}{\overset{j_s}{\downarrow}} \Psi \underset{b+1-2(d-s)}{\overset{j_{s+1}}{\downarrow}} \dots \Psi \underset{b-1}{\overset{j_d}{\downarrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}} z_\lambda.$$

Claim 1. Suppose we have $-1 \leqslant u \leqslant d-s-1$ with $j_{s+v} \geqslant b+3-2(d-s)+4v$ for all $-1 \leqslant v \leqslant u$. Let

$$\Psi^* := \Psi \underset{b+3-2d}{\overset{j_1}{\downarrow}} \dots \Psi \underset{b+1-2(d-(s+u))}{\overset{j_{s+u}}{\downarrow}}.$$

Then

$$(*) = \Psi^* \Psi \underset{b+7-2(d-s-2u)}{\overset{j_{s+u+1}}{\downarrow}} \cdot\Psi \underset{b+3-2(d-(s+u))}{\overset{j_{s+u+2}}{\downarrow}} \Psi \underset{b+5-2(d-(s+u))}{\overset{j_{s+u+3}}{\downarrow}} \dots \Psi \underset{b-1}{\overset{j_d}{\downarrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}} z_\lambda.$$

The above claim is proved by a simple but tedious induction, in the spirit of previous claims in this proof. Now first suppose we have $u = d-s-1$ satisfying the conditions in the claim, but also $j_d \geqslant b+3+2(d-s)$. Then

$$(*) = \Psi \underset{b+3-2d}{\overset{j_1}{\downarrow}} \dots \Psi \underset{b-1}{\overset{j_{d-1}}{\downarrow}} \Psi \underset{b+3+2(d-s)}{\overset{j_d}{\downarrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}} z_\lambda$$
$$= \begin{cases} -2T' & \text{if } j = b+5+2(d-s), \\ T' & \text{if } j = b+3+2(d-s) \text{ or } j = b+7+2(d-s), \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise, take u maximal, satisfying the conditions of Claim 1. We have

$$(*) = \Psi^* \cdot \Psi \underset{b+3-2(d-(s+u))}{\overset{j_{s+u+2}}{\downarrow}} \Psi \underset{b+5-2(d-(s+u))}{\overset{j_{s+u+3}}{\downarrow}} \dots \Psi \underset{b-1}{\overset{j_d}{\downarrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}} z_\lambda,$$

by the claim. Note that for these conditions on u to hold, we have $j_{s+u} = b+3-2(d-s)+4u$ and $j_{s+u+1} = b+5-2(d-s)+4u$.

Claim 2. Let $0 \leqslant r \leqslant d-s-u-2$ be such that $j_{d-v} \geqslant j-2-4v$ for all $0 \leqslant v \leqslant r$. Then

$$(*) = \Psi^*\Psi \overset{j_{s+u+2}}{\underset{b+3-2(d-(s+u))}{\downarrow}} \ldots \Psi \overset{j_{d-r-1}}{\underset{b-3-2r}{\downarrow}} \Psi \overset{j-6-4r}{\underset{b-1-2r}{\downarrow}} \Psi \overset{j_{d-r}}{\underset{b+1-2r}{\downarrow}} \Psi \overset{j_{d-r+1}}{\underset{b+3-2r}{\downarrow}} \ldots \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda.$$

If r is maximal such that $j_{d-v} \geqslant j-2-4v$ for all $0 \leqslant v \leqslant r$ and $r \leqslant d-s-u-3$, then this expression is reduced.

Again, this claim can be proved by induction as with the previous claims. Note that the above term is zero unless $j \geqslant b+5+2r$.

Whenever $r \leqslant d-s-u-3$, the reduced expression above is longer than v_{T}. To see why, note that we have the condition $j_{s+u+1} = b+5-2(d-s-2u)$ from Claim 1. Since $j_{i+1} \geqslant j_i + 2$, this yields $j_{d-r-1} \geqslant b+1+2(u-r)$. Now, we have assumed that $j_{d-r-1} < j-6-4r$, so we can combine these inequalities to yield $j \geqslant b+9+2(u+r)$.

We now have enough information to compare lengths. To leave this reduced form, we first deleted $2u+4$ Ψ terms from $(*)$ to arrive at the result from Claim 1. Next, we deleted $2r+2$ Ψ terms to arrive at the result of Claim 2. So in total, we have deleted $2(r+u+3) =: \delta$ Ψ terms from $(*)$ to leave a reduced expression.

Now, how many Ψ terms did we append to v_{T} in the definition of $(*)$? Call the number of terms appended α. Since $i = b-1-2(d-s)$, $\Psi \overset{b-1}{\underset{i}{\uparrow}}$ is a product of $d-s+1$ Ψ terms. Since $j \geqslant b+9+2(u+r)$, $\Psi \overset{j-2}{\underset{b+1}{\downarrow}}$ has length at least $4+u+r$. So, $\alpha \geqslant d+u+r-s+5$. By the definition of r, we have that $d-s \geqslant r+u+2$, so $\alpha \geqslant 2(u+r+3)+1 > \delta$, and we are done.

Now suppose $r = d-s-u-2$ satisfies the conditions of Claim 2, and we are left with a reduced expression. The claim tells us that the reduced expression is

$$(*) = \Psi^*\Psi \overset{j+2-4(d-(s+u))}{\underset{b+3-2(d-(s+u))}{\downarrow}} \Psi \overset{j_{s+u+2}}{\underset{b+5-2(d-(s+u))}{\downarrow}} \dots \Psi \overset{j_d}{\underset{b+1}{\downarrow}} z_\lambda.$$

Now, using the fact that $j_{s+u} = b+3-2(d-s)+4u$, we see that for this expression to be reduced we have $j \geqslant b+3+2(d-s)$. Now, arguing as above, we have $\delta = 2(r+u+3) = 2(d-s+1)$, the length of $\Psi \overset{b-1}{\underset{i}{\uparrow}}$ is once again $d-s+1$ and the length of $\Psi \overset{j-2}{\underset{b+1}{\downarrow}}$ is at least $d-s+1$. Hence $\alpha \geqslant \delta$, with equality precisely when $j = b+3+2(d-s)$, in which case we have $(*) = v_{\mathrm{T}}$.

Now suppose $r = d-s-u-2$ satisfies the conditions of Claim 2, but we are not left with a reduced expression. Then

$$(*) = \Psi^*\Psi \overset{j+2-4(d-(s+u))}{\underset{b+3-2(d-(s+u))}{\downarrow}} \underbrace{\Psi \overset{j_{s+u+2}}{\underset{b+5-2(d-(s+u))}{\downarrow}} \dots \Psi \overset{j_d}{\underset{b+1}{\downarrow}}}_{=:\Psi^{**}} z_\lambda$$

which is zero unless $j \geqslant b+1+2(d-(s+u))$, and we have the following:

> **Claim 3.** Let $-1 \leqslant x \leqslant s+u-1$ be such that $j_{s+u-v} \geqslant j+2-4(d+v-(s+u))$ for all $-1 \leqslant v \leqslant x$. Then
>
> $$(*) = \Psi \overset{j_1}{\underset{b+3-2d}{\downarrow}} \dots \Psi \overset{j_{s+u-x-1}}{\underset{b-1-2(d+x-(s+u))}{\downarrow}} \Psi \overset{j-2-4(d+x-(s+u))}{\underset{b+1-2(d+x-(s+u))}{\downarrow}} \Psi \overset{j_{s+u-x}}{\underset{b+3-2(d+x-(s+u))}{\downarrow}} \dots$$
> $$\Psi \overset{j_{s+u}}{\underset{b+3-2(d-(s+u))}{\downarrow}} \Psi^{**} z_\lambda.$$
>
> Note that if $x \leqslant s+u-2$, this term is zero unless $j \geqslant b+3+2(d+x-(s+u))$.

Take x to be the maximal such that the conditions in Claim 3 are met. First, suppose $x \leqslant s+u-2$. Then $j_{s+u-x} \geqslant j+2-4(d+x-(s+u))$ and $j_{s+u-x-1} < j-2-4(d+x-(s+u))$. When $x \leqslant u-1$, we have our assumption in using Claim 1 that $j_{s+u-x-1} \geqslant b-1-2(d-s)+4(u-x)$. Comparing these inequalities yields $j \geqslant b+3+2(d-s)$.

Similarly if $x \geqslant u$, $j_{s+u-x-1} \geqslant b-1-2(d+x-(s+u))$ can be read off from the expression in Claim 3. But this yields $j \geqslant b+3+2((d-s)+(x-u)) \geqslant b+3+2(d-s)$. Now in either case,

$$\begin{aligned} b+3-2(d-s)+4u &= j_{s+u} \text{ by the comment after Claim 1,} \\ &\geqslant j+2-4(d-(s+u)) \text{ by the conditions in Claim 3,} \\ &\geqslant b+5-2(d-s)+4u. \end{aligned}$$

We have a contradiction, and so if $j_{s+u} \geqslant j+2-4(d-(s+u))$ but $j_1 < j+6-4d$, we must have $(*)=0$.

Now suppose $x = s+u-1$. Then we have $j \geqslant b-1+2d$, or else $(*)=0$ and we're done. So

$$\begin{aligned} b+3-2d+2s+4u &= j_{s+u} \text{ by the comment after Claim 1,} \\ &\geqslant j+2-4(d-(s+u)) \text{ by the conditions in Claim 3,} \\ &\geqslant b+1-2d+4(s+u). \end{aligned}$$

This implies $s=1$, and so $i = b+1-2d$. But this breaks the initial conditions of the subcase we are in, so we again have a contradiction. So in fact we never get terms that look like the expression in Claim 3; Ψ^* remains intact in the final reduced expression for $(*)$, if it is non-zero.

If we collect the cases where $(*)$ is equal to a scalar multiple of v_{T}, we get the following list:

$(*) = v_{\mathrm{T}}$ if

- $i = b+1$, $j = b+5$, $d > 0$ – from case 1;
- $b+3-2d \leqslant i \leqslant b-1$, $j = 2b+6-i$, $j_d = 2b+2-i$, and $j_v \geqslant j-4-4(d-v)$ for all v – from case 2(b);

- $b+3-2d \leqslant i \leqslant b-1$, $j = 2b+6-i$ and $j_v \geqslant j-4-4(d-v)$ for all v – from case 3(b);

- $i = b-1$, $j = b+3$, $d > 0$ – from case 1;
- $i = b+1-2d$, $j = b+1+2d \geqslant b+5$ and $j_v \geqslant j-2-4(d-v)$ for all v – from case 3(a);
- $b+3-2d \leqslant i$, $j = 2b+2-i \geqslant b+5$, $j_d \geqslant j-2$ and $j_v \geqslant j-2-4(d-v)$ for all v – from case 3(b);
- $b+3-2d \leqslant i$, $j = 2b+2-i \geqslant b+5$, $j_d \geqslant j-2$ and $j_v \geqslant j-4(d-v)$ for all v – from case 3(b).

These conditions can be written compactly as the first and second conditions in the statement of the proposition.

$(*) = -2v_{\mathtt{T}}$ if

- $i = b+1$, $j = b+3$, $d > 0$ – from case 1;
- $b+3-2d \leqslant i \leqslant b-1$, $j = 2b+4-i \geqslant b+5$, and $j_v \geqslant j-2-4(d-v)$ for all v – from case 3(b).

These two conditions can be written compactly as the final condition in the statement of the proposition. □

The above result immediately leads to the following crucial fact.

Corollary 3.34. *Order $\mathscr{D}$ so that $v_{\mathtt{U}}$ comes after $v_{\mathtt{T}}$ whenever $r(\mathtt{U}) > r(\mathtt{T})$. With respect to this ordering, the action of f on $e(i_\lambda)\,S_\lambda$ is lower triangular. In particular, for each $\mathtt{T} \in \mathrm{Dom}(\lambda)$, the coefficient of $v_{\mathtt{T}}$ in $f(v_{\mathtt{T}})$ is an eigenvalue of f.*

Proposition 3.35. *f has the eigenvalues*

$$-\frac{d}{2}(n-2d+1) \quad \textit{for } d = 0, 1, \ldots, b/2.$$

Proof. Fix some $d \in \{0, 1, \ldots, b/2\}$. Let

$$v_{\mathrm{T}} = \Psi \underset{b+3-2d}{\overset{n-2d}{\downarrow}} \ldots \Psi \underset{b-1}{\overset{n-4}{\downarrow}} \Psi \underset{b+1}{\overset{n-2}{\downarrow}} z_\lambda.$$

Using the three bullet points in Proposition 3.33, we will compute the eigenvalue $-\frac{d}{2}(n-2d+1)$ as the coefficient of v_{T} in $f(v_{\mathrm{T}})$. First, note that by choice of $\mathtt{T}$ the inequality on j_d for each bullet point is the strongest. So to check when the family of inequalities at the end of each bullet point holds, it suffices to only verify the inequality on j_d.

If $i + j = 2b + 6$ and $i \geqslant b + 3 - 2d$, then we claim that the inequalities in the first bullet point are always satisfied by v_{T}. For this, we need $j_d \geqslant 2b + 2 - i$. Using that $d \leqslant b/2$ and $n > 2b$ we also get that $2b + 2 - i \leqslant b - 1 + 2d \leqslant 2b - 1 \leqslant n - 2$. So the inequalities always hold in the case of the first bullet point.

Now, if $i + j = 2b + 2$ and $i \geqslant b + 1 - 2d$, we claim that the inequalities in the second bullet point are always satisfied by v_{T}. To see this, we must show that $j_d \geqslant 2b - i$. We have $2b - i \leqslant b - 1 + 2d \leqslant 2b - 1 \leqslant n - 2$ and so the inequalities always hold in the case of the second bullet point.

Finally, $i + j = 2b + 4$ and $i \geqslant b + 3 - 2d$, then we claim that the inequalities in the third bullet point are always satisfied by v_{T}. We need $j_d \geqslant 2b + 2 - i$ but have $2b + 2 - i \leqslant b - 1 + 2d \leqslant 2b - 1 \leqslant n - 2$ and we are done.

So now we only need to verify which pairs (i, j) satisfy the first two conditions in each bullet point. For the first bullet point, we have the pairs $(b + 3 - 2d, b + 3 + 2d), (b + 5 - 2d, b + 1 + 2d), \ldots, (b + 1, b + 5)$. For the second, we have the pairs $(b + 1 - 2d, b + 1 + 2d), (b + 3 - 2d, b - 1 - 2d), \ldots, (b - 1, b + 3)$. For the third, we have the pairs $(b + 3 - 2d, b + 1 + 2d), (b + 5 - 2d, b - 1 + 2d), \ldots, (b + 1, b + 3)$.

Recall that the coefficient of $\Psi \underset{i}{\overset{b-1}{\uparrow}} \Psi \underset{b+1}{\overset{j-2}{\downarrow}}$ in $f(z_\lambda)$ is $\frac{i-1}{2} \cdot \frac{n+2-j}{2}$. Hence the coefficient of v_{T} in $f(v_{\mathrm{T}})$ is

$$\frac{1}{4} \sum_{r=0}^{d-1} (b-2r)(a-3-2r) + (b-2-2r)(a-1-2r) - 2(b-2r)(a-1-2r)$$

$$= \frac{1}{4}\sum_{r=0}^{d-1} 8r - 2(n-1)$$
$$= -\frac{1}{2}d(n-1) + 2\sum_{r=0}^{d-1} r$$
$$= -\frac{d}{2}(n-2d+1).$$ □

Remark. The sequence of eigenvalues given above is

$$0, -\frac{(n-1)}{2}, -(n-3), -\frac{3}{2}(n-5), \ldots, -\frac{b}{4}(a+1).$$

If we write $a = 2r+1$ and $b = 2s$ then this sequence can be rewritten as

$$0, -(r+s), -2(r+s-1), -3(r+s-2), \ldots, -s(r+1).$$

Lemma 3.36. *If $p \geqslant n$, then $\{1, f, f^2, \ldots, f^{b/2}\}$ is a basis of $\mathrm{End}_{\mathcal{H}}(S_\lambda)$.*

Proof. Since $p \geqslant n$, all $b/2+1$ eigenvalues of f are distinct. Thus, we know that the minimal polynomial of f has $b/2+1$ distinct linear factors, so $\{1, f, f^2, \ldots, f^{b/2}\}$ are linearly independent. □

Remark. Note also that our proof shows that we have an exhaustive list of eigenvalues of f, since we would otherwise have more that $b/2+1$ linearly independent maps in $\mathrm{End}_{\mathcal{H}}(S_\lambda)$, contradicting Lemma 3.28. We also see that the upper bound on the dimension of $\mathrm{End}_{\mathcal{H}}(S_\lambda)$ given in Lemma 3.28 is obtained when $p \geqslant n$.

Theorem 3.37. *Suppose $\mathrm{char}(\mathbb{F}) \neq 2$. Then $S_{(a,1^b)}$ is decomposable if either $b \geqslant 4$ or $b = 2$ with $\mathrm{char}(\mathbb{F}) \nmid \frac{n-1}{2}$.*

Proof. By the remark after Proposition 3.30, it suffices to show that f has at least two distinct eigenvalues. When $s \geqslant 2$, 0, $-\frac{(n-1)}{2}$ and $-(n-3)$ are three eigenvalues of f; if $S_{(a,1^b)}$ were indecomposable, these would be equal. Since $p \neq 2$, this is impossible, and we have the desired result.

When $b = 2$ and $\text{char}(\mathbb{F}) \nmid \frac{n-1}{2}$, we have the distinct eigenvalues 0 and $-\frac{(n-1)}{2}$ and we are done. □

It remains to resolve the case $a = 2r + 1$, $b = 2$ when $\text{char}(\mathbb{F}) \mid \frac{n-1}{2}$. We have

$$f(z_\lambda) = r \cdot v_{\mathrm{T}_{3,b+3}} + (r-1) \cdot v_{\mathrm{T}_{3,b+5}} + \cdots + v_{\mathrm{T}_{3,n}} = \sum_{c=1}^{r} c \cdot \Psi \overset{3+2(r-c)}{\underset{3}{\downarrow}} z_\lambda.$$

When $b = 2$ and $\text{char}(\mathbb{F}) \mid \frac{n-1}{2}$, we will prove that S_λ is indecomposable by showing that $\mathrm{End}_{\mathcal{H}}(\mathrm{S}_\lambda)$ has no non-trivial idempotents.

Lemma 3.38. *Suppose a is odd. Then $\{I, f\}$ is a basis of $\mathrm{End}_{\mathcal{H}}(\mathrm{S}_{(a,1^2)})$, where I is the identity map on $\mathrm{S}_{(a,1^2)}$.*

Proof. Suppose we have $g \in \mathrm{S}_{(a,1^2)} \setminus \langle I, f \rangle_{\mathbb{F}}$. Since the coefficient of $v_{\mathrm{T}_{3,n}}$ in f is 1, we can add multiples of I and f to assume without loss of generality that

$$g(z_\lambda) = \sum_{j=2}^{(n-3)/2} \alpha_j v_{\mathrm{T}_{3,2j+1}} = \sum_{j=2}^{(n-3)/2} \alpha_j \Psi \overset{2j-1}{\underset{3}{\downarrow}} z_\lambda.$$

We will show that applying the relations $\psi_{n-2k} g(z_\lambda) = 0$ for $k = 1, 2, \ldots, (n-5)/2$ yields $\alpha_{(n-2k-1)/2} = 0$. It then follows that g is the zero map, a contradiction.

Suppose, by induction on k, we have

$$g(z_\lambda) = \sum_{j=2}^{(n-2k-1)/2} \alpha_j \Psi \overset{2j-1}{\underset{3}{\downarrow}} z_\lambda.$$

Then, acting on $g(z_\lambda)$ by ψ_{n-2k} yields $\alpha_{(n-2k-1)/2} \psi_{n-2k} \Psi \overset{n-2k-2}{\underset{3}{\downarrow}} z_\lambda = 0$ and we are done. □

In order to find idempotents, we would like to know how to compose elements of our basis. This amounts to the following lemma.

Lemma 3.39. *Let $a = 2r + 1$ and $b = 2$. Then $f^2(z_\lambda) = -(r+1) f(z_\lambda)$.*

Proof. Notice that $\Psi_3 \cdot \Psi \overset{3+2(r-c)}{\underset{3}{\downarrow}} z_\lambda = 0$ for all $c \leqslant r-2$. So

$$\begin{aligned} f^2(z_\lambda) &= \sum_{c=1}^{r} c \cdot \Psi \overset{3+2(r-c)}{\underset{3}{\downarrow}} f(z_\lambda) \\ &= \sum_{c=1}^{r} c \cdot \Psi \overset{3+2(r-c)}{\underset{3}{\downarrow}} (r-1\Psi_5\Psi_3 z_\lambda + r\Psi_3 z_\lambda) \\ &= \sum_{c=1}^{r} c \cdot \Psi \overset{3+2(r-c)}{\underset{3}{\downarrow}} (-(r+1)z_\lambda) \\ &= -(r+1)f(z_\lambda). \end{aligned}$$ □

Lemma 3.40. *Suppose $a = 2r+1$ and $\mathrm{char}(\mathbb{F}) \mid \frac{n-1}{2}$. Then the only idempotents in $\mathrm{End}_{\mathcal{H}}(S_{(a,1^2)})$ are 0 and I, and hence $S_{(a,1^2)}$ is indecomposable.*

Proof. Let $\alpha, \beta \in \mathbb{F}$. Using Lemma 3.39, we have $f^2(z_\lambda) = 0$ and therefore

$$(\alpha I + \beta f)^2 = \alpha^2 I + 2\alpha\beta f.$$

So $\alpha I + \beta f$ is an idempotent if and only $\alpha^2 = \alpha$ and $2\alpha\beta = \beta$.

Whether $\alpha = 0$ or $\alpha = 1$, we must have $\beta = 0$. The result follows. □

With the aid of Murphy's result (Theorem 3.1), we have now completely determined decomposability of the Specht modules $S_{(a,1^b)}$. We summarise our result in the following theorem.

Theorem 3.41. *Suppose* $\mathrm{char}(\mathbb{F}) \neq 2$. *Then $S_{(a,1^b)}$ is indecomposable if and only if n is even, or $b = 2$ or 3 and* $\mathrm{char}(\mathbb{F}) \mid \lceil \frac{a}{2} \rceil$.

Chapter 4

Graded decomposition numbers for two-part partitions

In this chapter, we will study the graded decomposition numbers for $\mathcal{H} = \mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$. This problem is extremely difficult in full generality, and we here restrict our attention to the case of two-part partitions. We make use of homogeneous homomorphisms between Specht modules to calculate these decomposition numbers. In the final section, we also investigate some exact sequences of these homomorphisms.

Recall from Section 1.12 that the ungraded (classical) decomposition number $[\mathrm{S}_\lambda : \mathrm{D}_\mu]$ is defined to be the number of times D_μ appears as a composition factor of S_λ, while the graded decomposition number $[\mathrm{S}_\lambda : \mathrm{D}_\mu]_v$ also records the graded shift of each copy of D_μ in S_λ.

We will now seek to determine all graded decomposition numbers $[\mathrm{S}_\lambda : \mathrm{D}_\mu]_v$, where λ and μ are both two-part partitions.

Remark. The ungraded decomposition numbers here are known, with a relatively simple formula proved by Gordon James. Let $p = \mathrm{char}(\mathbb{F})$ and let e be the quantum characteristic of $\mathcal{H}$. The function $f_{e,p} : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ is defined as

$$f_{e,p}(x, y) = \begin{cases} 1 & \text{if } \lfloor \frac{y}{e} \rfloor \preceq_p \lfloor \frac{x+1}{e} \rfloor \text{ and } e \mid y \text{ or } e \mid x - y + 1, \\ 0 & \text{otherwise,} \end{cases}$$

where we say $a \preceq_p b$ if the p-adic expansions of a and b are

$$a = a_0 + a_1 p + a_2 p^2 + \cdots + a_r p^r \quad \text{and} \quad b = b_0 + b_1 p + b_2 p^2 + \cdots + b_t p^t$$

respectively, with $a_i = 0$ or $a_i = b_i$ for all i. Then if $\lambda = (n - m, m)$ and $\mu = (n - r, r)$,

$$[\mathrm{S}_\lambda : \mathrm{D}_\mu] = f_{e,p}(n - 2r, m - r).$$

Example. Let $e = p = 2$ and $n = 6$. We will use James's result to calculate the decomposition numbers for all two-part partitions of 6. First, since (6), (5, 1) and (4, 2) are all 2-regular, we know that the three corresponding Specht modules have simple heads $\mathrm{D}_{(6)}$, $\mathrm{D}_{(5,1)}$ and $\mathrm{D}_{(4,2)}$ respectively, giving 1s down the leading diagonal of the two-part component of the decomposition matrix. Next, we see that $d_{(5,1)(6)} = f_{2,2}(6, 1) = 1$ since the condition $0 \preceq_2 3$ holds trivially and the condition "$e \mid x - y + 1$" is "$2 \mid 6$". Similarly, $d_{(4,2)(5,1)} = d_{(3,3)(4,2)} = 1$ as each involves evaluating $f_{2,2}(x, 1)$ for some even x; the condition $0 \preceq_2 \lfloor \frac{x+1}{e} \rfloor$ is always satisfied trivially. Next $d_{(4,2)(6)} = f_{2,2}(6, 2) = 1$ as $1 \not\preceq_2 3$ and "$e \mid y$" becomes "$2 \mid 2$". $d_{(3,3)(6)} = f_{2,2}(6, 3) = 1$ as $1 \preceq_2 3$ and "$e \mid x - y + 1$" becomes "$2 \mid 4$". Finally, $d_{(3,3)(5,1)} = f_{2,2}(4, 2) = 0$ as $1 \not\preceq_2 2$. This yields the following submatrix of the decomposition matrix of $\mathcal{H}$ when $e = p = 2$ (which is just $\mathbb{F}\mathfrak{S}_6$).

	(6)	(5, 1)	(4, 2)
(6)	1	·	·
(5, 1)	1	1	·
(4, 2)	1	1	1
(3, 3)	1	·	1

Definition 4.1. We define $D_2^{e,p}(v)$ to be the submatrix of the graded decomposition

matrix for $\mathcal{H}$ which corresponds to two-part partitions.

Theorem 4.2. *Each column of $D_2^{e,p}(v)$ for $p > 0$ is a sum of columns of $D_2^{e,0}$.*

Proof. Since we know for any p that every entry of $D_2^{e,p}(v)|_{v=1}$ is 0 or 1, the ungraded adjustment matrix (for any p) must also have entries 0 or 1. By Theorem 1.49, entries of the graded adjustment matrix have non-negative coefficients and are symmetric in v, v^{-1}. So the graded adjustment matrix also consists of just 0s and 1s. □

Remark. Using this theorem, it is sufficient to calculate graded decomposition numbers for $p = 0$. To calculate those for $p > 0$, there is a unique choice of entries obeying the above result. The author is extremely grateful to Sinéad Lyle, to whom this result must be attributed, for pointing it out.

We now give a presentation for the Specht modules (in the KLR setting) for two-part partitions, which we will use extensively.

Definition 4.3. For $\lambda = (n-m, m)$, Section 1.10 gives us the presentation

$$\mathrm{S}_\lambda = \left\langle z_\lambda \;\middle|\; \begin{array}{c} \psi_j z_\lambda = 0 \;\forall j = 1, 3, \dots, 2m-1 \text{ or } j = 2m+1, 2m+2, \dots, n-1, \\ \psi_j \psi_{j+1} z_\lambda = 0 \;\forall j = 1, 3, \dots, 2m-1, \;\; \psi_j \psi_{j-1} z_\lambda = 0 \;\forall j = 3, 5, \dots, 2m-1, \\ y_k z_\lambda = 0 \;\forall k, \quad e(i_\lambda) z_\lambda = z_\lambda \end{array} \right\rangle.$$

Remark. For any two-part partition, λ, we have unique reduced expressions for standard λ-tableaux, up to applying the commuting relations on the ψ generators. So we get a well-defined basis $\{v_\mathtt{T} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ of S_λ without having to worry about fixing any reduced expressions for elements of $\mathfrak{S}_n$.

4.1 Decomposition Numbers when $e = 2$

We here look at the case $e = 2$. In some ways, we expect this to be the hardest case to work with, partly because of the more involved relations on the generators ψ_i. We will split this into the two subcases where n is either even or odd. The parity here makes a difference to the block structure of $\mathcal{H}$; when n is even all Specht modules (for two-part partitions) lie in the same block, but when n is odd they are split between two different blocks, determined by the parity of m. However, we see that when n is odd the result is extremely simple.

Theorem 4.4. *If n is odd, then $[\mathrm{S}_\lambda : \mathrm{D}_\mu]_v = [\mathrm{S}_\lambda : \mathrm{D}_\mu]$ for any two-part partitions λ and μ of n.*

Proof. James's result tells us that when $p = 0$, $[\mathrm{S}_\lambda : \mathrm{D}_\mu] = \delta_{\lambda,\mu}$. Application of Theorem 4.2 completes the proof. □

For the remainder of this section, we let n be even. All Specht modules lie in the same block so we would like to consider them simultaneously. We will start by calculating $[\mathrm{S}_\lambda : \mathrm{D}_{(n)}]_v$ for all two part partitions $\lambda = (n-m, m)$. We obviously have $[\mathrm{S}_{(n)} : \mathrm{D}_{(n)}]_v = 1$.

As in our work with hook partitions, we can see the following result.

Lemma 4.5. *Let $\lambda = (n-m, m)$. Then $e(i_{(n)})\,\mathrm{S}_\lambda$ has a homogeneous basis indexed by standard "domino tableaux". That is, a basis $\mathscr{D} = \{v_\mathtt{T} \mid \mathtt{T} \in \mathrm{Dom}(\lambda)\}$ indexed by the set $\mathrm{Dom}(\lambda)$ of standard tableaux where the entries i and $i+1$ appear consecutively in the same row, for all even $2 \leqslant i \leqslant n-2$.*

Remark. If m is odd, then every element of $\mathscr{D}$ is homogeneous of degree 1, as the entry n must be placed at the end of the second row. If m is even, every element of $\mathscr{D}$ is homogeneous of degree 0.

Lemma 4.6. *Let $\mathtt{T}$ be a standard $(n-m, m)$-tableau with residue sequence $i_{(n)}$. Then $y_k v_\mathtt{T} = 0$ for any k.*

Proof. The proof is morally the same as that of [29, Lemma 4.4], and thus Proposition 3.12. We note that

$$y_k v_\mathtt{T} = y_k e(i_{(n)}) v_\mathtt{T} = e(i_{(n)}) y_k v_\mathtt{T} \in e(i_{(n)})\,\mathrm{S}_{(n-m,m)}$$

and so $y_k v_{\mathrm{T}}$ is a linear combination of elements in $\mathscr{D}$. These are all homogeneous of degree 1 or 0, depending on whether m is odd or even, respectively. But then v_{T} also has this same homogeneous degree, and so $y_k v_{\mathrm{T}}$ must have degree 3 or 2, respectively. So in fact we must have $y_k v_{\mathrm{T}} = 0$. □

Lemma 4.7.

$$y_j \cdot \psi_{j-1}\psi_{j-2}\dots\psi_{2m} z_{(n-m,m)} = \begin{cases} 0 & \textit{for all even } j \geqslant 2m, \\ \psi_{j-2}\psi_{j-3}\dots\psi_{2m} z_{(n-m,m)} & \textit{for all odd } j \geqslant 2m. \end{cases}$$

Proof. We will prove this by induction on j. When $j = 2m$ the result follows trivially from the relations in the Specht module (y_{2m} annihilates the generator $z_{(n-m,m)}$). Now let $j > 2m$ be even. Then

$$\begin{aligned} &y_j \cdot \psi_{j-1}\psi_{j-2}\dots\psi_{2m} z_{(n-m,m)} \\ &= (y_j\psi_{j-1} e(s_{j-2}\dots s_{2m}\cdot i_{(n-m,m)}))\psi_{j-2}\dots\psi_{2m} z_{(n-m,m)} \\ &= \psi_{j-1} y_{j-1}\psi_{j-2}\psi_{j-3}\dots\psi_{2m} z_{(n-m,m)} \\ &= 0 \quad \text{by induction.} \end{aligned}$$

Finally, let $j > 2m$ be odd. Then

$$\begin{aligned} &y_j \cdot \psi_{j-1}\psi_{j-2}\dots\psi_{2m} z_{(n-m,m)} \\ &= (\psi_{j-1} y_{j-1} + 1)\psi_{j-2}\psi_{j-3}\dots\psi_{2m} z_{(n-m,m)} \\ &= \psi_{j-2}\psi_{j-3}\dots\psi_{2m} z_{(n-m,m)} \quad \text{by induction.} \end{aligned}$$

□

Lemma 4.8. *The map* $f_0 : \mathrm{S}_{(n)} \to \mathrm{S}_{(n-1,1)}$ *defined by* $f_0(z_{(n)}) = \psi_{n-1}\psi_{n-2}\dots\psi_2 z_{(n-1,1)}$ *defines a degree* 1 $\mathcal{H}$*-homomorphism.*

Proof. First, note that $f_0(z_{(n)}) = v_{\mathrm{T}^{(n-1,1)}}$. $\mathrm{T}^{(n-1,1)}$ has residue sequence $i_{(n)}$ because n is even. Likewise, the previous lemma gives us $y_k f_0(z_{(n)}) = 0$ for all k. All that remains is to check that $\psi_j v_{\mathrm{T}^{(n-1,1)}} = 0$ for all j.

Firstly, $\psi_1 \cdot \psi_{n-1} \ldots \psi_2 z_{(n-1,1)} = \psi_{n-1} \ldots \psi_3(\psi_1\psi_2 z_{(n-1,1)}) = 0$.

Next, if $2 \leqslant j \leqslant n-2$ is even, we have

$$\begin{aligned}
\psi_j \cdot \psi_{n-1} \ldots \psi_2 z_{(n-1,1)} &= \psi_{n-1} \ldots \psi_{j+2}(\psi_j \psi_{j+1} \psi_j e(s_{j-1} \ldots s_2 \cdot i_{(n-1,1)}))\cdot \\
&\quad \psi_{j-1}\psi_{j-2} \ldots \psi_2 z_{(n-1,1)} \\
&= \psi_{n-1} \ldots \psi_{j+2}(\psi_{j+1}\psi_j\psi_{j+1})\psi_{j-1}\psi_{j-2} \ldots \psi_2 z_{(n-1,1)} \\
&= 0.
\end{aligned}$$

Next, if $2 \leqslant j \leqslant n-2$ is odd, we have

$$\begin{aligned}
\psi_j \cdot \psi_{n-1} \ldots \psi_2 z_{(n-1,1)} &= \psi_{n-1} \ldots \psi_{j+2}(\psi_j \psi_{j+1} \psi_j e(s_{j-1} \ldots s_2 \cdot i_{(n-1,1)})\cdot \\
&\quad \psi_{j-1}\psi_{j-2} \ldots \psi_2 z_{(n-1,1)} \\
&= \psi_{n-1} \ldots \psi_{j+2}(\psi_{j+1}\psi_j\psi_{j+1} + y_j - 2y_{j+1} + y_{j+2})\cdot \\
&\quad \psi_{j-1} \ldots \psi_2 z_{(n-1,1)} \\
&= 0,
\end{aligned}$$

as ψ_{j+1}, ψ_{j+2}, y_{j+1} and y_{j+2} commute through to the right hand side in the four summands, respectively.

Finally,

$$\begin{aligned}
\psi_{n-1} \cdot \psi_{n-1} \ldots \psi_2 z_{(n-1,1)} &= (\psi_{n-1}^2 e(s_{n-2} \ldots s_2 \cdot i_{(n-1,1)}))\psi_{n-2} \ldots \psi_2 z_{(n-1,1)} \\
&= (-y_{n-1}^2 - y_n^2 + 2y_{n-1}y_n)\psi_{n-2} \ldots \psi_2 z_{(n-1,1)} \\
&= -y_{n-1}(y_{n-1}\psi_{n-2} e(s_{n-3} \ldots s_2 \cdot i_{(n-1,1)}))\psi_{n-3} \ldots \psi_2 z_{(n-1,1)} \\
&= -y_{n-1}\psi_{n-3} \ldots \psi_2 z_{(n-1,1)} \quad \text{by Lemma 4.7,} \\
&= 0.
\end{aligned}$$

□

We immediately have the following result.

Corollary 4.9. $[\mathrm{S}_{(n-1,1)} : \mathrm{D}_{(n)}]_v = v$, *regardless of the characteristic of* $\mathbb{F}$.

Remark. If $p = 0$, we have filled in the only two non-zero entries of the first column of the decomposition matrix (that is, the column corresponding to $[\mathrm{S}_\lambda : \mathrm{D}_{(n)}]_v$).

Lemma 4.8 and Theorem 2.23 immediately imply the following:

Corollary 4.10. $f_m : \mathrm{S}_{(n-m,m)} \to \mathrm{S}_{(n-m-1,m+1)}$ *with*

$f_m(z_{(n-m,m)}) = \psi_{n-1}\psi_{n-2}\ldots\psi_{2m+2}z_{(n-m-1,m+1)}$ *defines a degree 1 $\mathcal{H}$-homomorphism for all* $m \leqslant n/2-1$.

Proof. This can be seen easily by looking at the tableau given by $s_{n-1}\ldots s_{2m+2}\mathrm{T}_{(n-m-1,m+1)}$, which is

1	3	$\cdots$	$2m-1$	$2m+1$	$2m+2$	$\cdots$	$n-1$
2	4	$\cdots$	$2m$	n			

and considering the homomorphism column removal in Theorem 2.23. □

Remark. $f_{n/2-1}$ is surjective, as our construction clearly gives this homomorphism as mapping generator to generator. Note also that the homomorphisms f_i for $i = 0, 1, \ldots, n/2-1$ explicitly give the one-node Carter–Payne homomorphisms found in [32].

The following is a simple lemma in a general (ungraded) setting which will be useful to us.

Lemma 4.11. *Let M and N be (left) R-modules for some ring R, and suppose $f : M \to N$ is a non-zero homomorphism, and that* $\mathrm{hd}(M)$ *is simple. Then* $\mathrm{hd}(\mathrm{im}(f)) \cong \mathrm{hd}(M)$.

Proof. Let $I = \mathrm{im}(f)$ and $K = \ker(f)$. Then

$$\begin{aligned}
\mathrm{hd}(I) &\cong \mathrm{hd}(M/K) \quad \text{by the first isomorphism theorem,}\\
&\cong (M/K)/\,\mathrm{rad}(M/K) \quad \text{by definition,}\\
&\cong (M/K)/((\mathrm{rad}(M)+K)/K)
\end{aligned}$$

$$
\begin{aligned}
&\cong M/(\operatorname{rad}(M)+K) \quad \text{by the third isomorphism theorem,}\\
&\cong (M/\operatorname{rad}(M))/((\operatorname{rad}(M)+K)/\operatorname{rad}(M)) \quad \text{by the third isomorphism theorem,}\\
&\cong \operatorname{hd}(M)/((\operatorname{rad}(M)+K)/\operatorname{rad}(M)).
\end{aligned}
$$

But $I \neq 0$, so $\operatorname{hd}(I) \neq 0$, and since $\operatorname{hd}(M)$ is simple its only non-zero quotient is itself, so $((\operatorname{rad}(M)+K)/\operatorname{rad}(M)) = 0$ and $\operatorname{hd}(I) \cong \operatorname{hd}(M)$. □

Corollary 4.12. $[\mathrm{S}_{(n-m-1,m+1)} : \mathrm{D}_{(n-m,m)}] = v$ *for all* $0 \leqslant m \leqslant n/2 - 1$.

Proof. We have a non-zero degree 1 homomorphism $f_m : \mathrm{S}_{(n-m,m)} \to \mathrm{S}_{(n-m-1,m+1)}$ and we know that $\mathrm{D}_{(n-m,m)} = \operatorname{hd}(\mathrm{S}_{(n-m,m)})$. Therefore

$$\mathrm{D}_{(n-m,m)}\langle 1\rangle \cong \operatorname{hd}(\operatorname{im}(f_m))$$

and the result follows. □

Remark. If $p = 0$, we have filled in all decomposition numbers (for two-part partitions). Thus, by Theorem 4.2 we can calculate the graded decomposition numbers for two-part partitions for any p. Explicitly, we can replace the function $f_{e,p}$ in James's formula with a graded version $f^v_{e,p} : \mathbb{N} \times \mathbb{N} \to \{0, 1, v\}$ given by

$$f^v_{e,p}(x,y) = \begin{cases} 1 & \text{if } \lfloor \frac{y}{e} \rfloor \preceq_p \lfloor \frac{x+1}{e} \rfloor \text{ and } e \mid y, \\ v & \text{if } \lfloor \frac{y}{e} \rfloor \preceq_p \lfloor \frac{x+1}{e} \rfloor \text{ and } e \mid x-y+1, \\ 0 & \text{otherwise.} \end{cases}$$

Example. It is easy to verify that $D_2^{2,2}$ is the following.

	(6)	(5, 1)	(4, 2)
(6)	1	·	·
(5, 1)	v	1	·
(4, 2)	1	v	1
(3, 3)	v	·	v

4.2 Exact sequences of homomorphisms between Specht modules

Next, we build on the work of the previous section and investigate the homomorphisms we have found when $e = 2$. In particular, our main result will be the construction of an exact sequence of homomorphisms between Specht modules when $e = 2$ and n is even, for any p.

Lemma 4.13. *Let $j \geqslant 2m + 4$. Then*

$$\psi_j \cdot \psi_{j+1}\psi_j \ldots \psi_{2m+4} z_{(n-m-2,m+2)} = \begin{cases} 0 & \text{if } j \text{ is even,} \\ \psi_{j-2}\psi_{j-3} \ldots \psi_{2m+4} z_{(n-m-2,m+2)} & \text{if } j \text{ is odd.} \end{cases}$$

Proof. First, suppose j is even. Then

$$\begin{aligned}
&\psi_j \cdot \psi_{j+1}\psi_j \ldots \psi_{2m+4} z_{(n-m-2,m+2)} \\
&= (\psi_j \psi_{j+1}\psi_j e(s_{j-1} \cdot s_{j-2} \cdots s_{2m+4} \cdot i_{(n-m-2,m+2)}))\psi_{j-1} \ldots \psi_{2m+4} z_{(n-m-2,m+2)} \\
&= \psi_{j+1}\psi_j\psi_{j+1}\psi_{j-1} \ldots \psi_{2m+4} z_{(n-m-2,m+2)} \\
&= 0.
\end{aligned}$$

Now suppose j is odd. Then

$$\begin{aligned}
&\psi_j \cdot \psi_{j+1}\psi_j \ldots \psi_{2m+4} z_{(n-m-2,m+2)} \\
&= (\psi_j \psi_{j+1}\psi_j e(s_{j-1} \cdot s_{j-2} \cdots s_{2m+4} \cdot i_{(n-m-2,m+2)}))\psi_{j-1} \ldots \psi_{2m+4} z_{(n-m-2,m+2)} \\
&= (\psi_{j+1}\psi_j\psi_{j+1} + y_j - 2y_{j+1} + y_{j+2})\psi_{j-1} \ldots \psi_{2m+4} z_{(n-m-2,m+2)} \\
&= 0 + y_j\psi_{j-1}\psi_{j-2} \ldots \psi_{2m+4} z_{(n-m-2,m+2)} - 0 + 0 \\
&= \psi_{j-2}\psi_{j-3} \ldots \psi_{2m+4} z_{(n-m-2,m+2)} \quad \text{by Lemma 4.7.}
\end{aligned}$$

□

Lemma 4.14. (A_j) *$y_j \cdot \psi_{j-1}\psi_{j-2} \ldots \psi_{2m+2}\psi_j\psi_{j-1} \ldots \psi_{2m+4} z_{(n-m-2,m+2)} = 0$ for all even $j \geqslant 2m + 2$.*

(B_j) $\psi_j \cdot \psi_{j-1}\psi_{j-2}\dots\psi_{2m+2}\psi_j\psi_{j-1}\dots\psi_{2m+4}z_{(n-m-2,m+2)} = 0$ *for all* $j \geqslant 2m+3$.

Proof. We will prove these by simultaneous induction. First, we prove (A_{2m+2}) and (B_{2m+3}). (A_{2m+2}) holds trivially, as $\psi_{2m+3}\psi_{2m+2}z_{(n-m-2,m+2)} = 0$ is a relation in $S_{(n-m-2,m+2)}$. (B_{2m+3}) also holds trivially, as the statement becomes $y_{2m+2}z_{(n-m-2,m+2)} = 0$, which is a relation in $S_{(n-m-2,m+2)}$.

Next, we will show that if $j > 2m+2$ is even, $(A_{j-2})\&(B_{j-3}) \Rightarrow (A_j)$.

$$\begin{aligned}
&y_j \cdot \psi_{j-1}\psi_{j-2}\dots\psi_{2m+2}\psi_j\psi_{j-1}\dots\psi_{2m+4}z_{(n-m-2,m+2)}\\
&= (y_j\psi_{j-1})\psi_{j-2}\dots\psi_{2m+2}\psi_j\psi_{j-1}\dots\psi_{2m+4}z_{(n-m-2,m+2)}\\
&= (\psi_{j-1}y_{j-1})\psi_{j-2}\dots\psi_{2m+2}\psi_j\psi_{j-1}\dots\psi_{2m+4}z_{(n-m-2,m+2)}\\
&= \psi_{j-1}(y_{j-1}\psi_{j-2})\psi_{j-3}\dots\psi_{2m+2}\psi_j\psi_{j-1}\dots\psi_{2m+4}z_{(n-m-2,m+2)}\\
&= \psi_{j-1}(\psi_{j-2}y_{j-2}+1)\psi_{j-3}\dots\psi_{2m+2}\psi_j\psi_{j-1}\dots\psi_{2m+4}z_{(n-m-2,m+2)}\\
&= \psi_{j-1}\psi_{j-2}\psi_j\psi_{j-1}(y_{j-2}\psi_{j-3}\dots\psi_{2m+2}\psi_{j-2}\dots\psi_{2m+4}z_{(n-m-2,m+2)})\\
&\quad + \psi_{j-3}\dots\psi_{2m+2}(\psi_{j-1}\psi_j\psi_{j-1}\psi_{j-2}\dots\psi_{2m+4}z_{(n-m-2,m+2)})\\
&= 0 + \psi_{j-3}\dots\psi_{2m+2}\psi_{j-3}\psi_{j-4}\dots\psi_{2m+4}z_{(n-m-2,m+2)}\\
&\quad \text{by } (A_{j-2}) \text{ and Lemma 4.13, respectively}\\
&= 0 \quad \text{by } (B_{j-3}).
\end{aligned}$$

Next we will show that if $j > 2m+3$ is even, $(B_{j-1}) \Rightarrow (B_j)$. In this case, we have

$$\begin{aligned}
&\psi_j \cdot \psi_{j-1}\psi_{j-2}\dots\psi_{2m+2}\psi_j\psi_{j-1}\dots\psi_{2m+4}z_{(n-m-2,m+2)}\\
&= (\psi_j\psi_{j-1}\psi_j)\psi_{j-2}\dots\psi_{2m+2}\psi_{j-1}\dots\psi_{2m+4}z_{(n-m-2,m+2)}\\
&= (\psi_{j-1}\psi_j\psi_{j-1})\psi_{j-2}\dots\psi_{2m+2}\psi_{j-1}\dots\psi_{2m+4}z_{(n-m-2,m+2)}\\
&= 0 \quad \text{by } (B_{j-1}).
\end{aligned}$$

Finally, we show that if $j > 2m+3$ is odd, $(A_{j-1}), (B_{j-1}) \& (B_{j-2}) \Rightarrow (B_j)$. Then

$$\psi_j \cdot \psi_{j-1}\psi_{j-2}\dots\psi_{2m+2}\psi_j\psi_{j-1}\dots\psi_{2m+4}z_{(n-m-2,m+2)}$$

$$= (\psi_j\psi_{j-1}\psi_j)\psi_{j-2}\dots\psi_{2m+2}\psi_{j-1}\dots\psi_{2m+4}z_{(n-m-2,m+2)}$$

$$= (\psi_{j-1}\psi_j\psi_{j-1} - y_{j-1} + 2y_j - y_{j+1})\psi_{j-2}\dots\psi_{2m+2}\psi_{j-1}\dots\psi_{2m+4}z_{(n-m-2,m+2)}$$

$= 2\psi_{j-2}\dots\psi_{2m+2}(y_j\psi_{j-1})\psi_{j-2}\dots\psi_{2m+4}z_{(n-m-2,m+2)}$ where the first summand is 0 by (B_{j-1}), the second by (A_{j-1}), and the fourth as y_{j+1} commutes through all ψs,

$$= 2\psi_{j-2}\dots\psi_{2m+2}(\psi_{j-1}y_{j-1} + 1)\psi_{j-2}\dots\psi_{2m+4}z_{(n-m-2,m+2)}$$

$= 0$ by Lemma 4.7 and (B_{j-2}). □

Theorem 4.15. *The sequence*

$$0 \xrightarrow{f_{-1}} S_{(n)} \xrightarrow{f_0} S_{(n-1,1)} \xrightarrow{f_1} \cdots \xrightarrow{f_{n/2-2}} S_{(n/2+1,n/2-1)} \xrightarrow{f_{n/2-1}} S_{(n/2,n/2)} \xrightarrow{f_{n/2}} 0$$

is an exact sequence.

Proof. We begin by showing that $f_{m+1} \circ f_m = 0$ for any $m \geqslant -1$. Exactness at the end of the sequence follows by surjectivity of $f_{n/2-1}$, so we can assume that $m \leqslant n/2-2$. It suffices to show that $f_{m+1}(f_m(z_{(n-m,m)})) = 0$. We will show this by induction on m. When $m = -1$, the result is obvious. So assume $0 \leqslant m \leqslant n/2-2$.

$$f_{m+1}(f_m(z_{(n-m,m)})) = f_{m+1}(\psi_{n-1}\psi_{n-2}\dots\psi_{2m+2}z_{(n-m-1,m+1)})$$

$$= \psi_{n-1}\psi_{n-2}\dots\psi_{2m+2}(f_{m+1}(z_{(n-m-1,m+1)}))$$

$$= \psi_{n-1}\psi_{n-2}\dots\psi_{2m+2} \cdot \psi_{n-1}\psi_{n-2}\dots\psi_{2m+4}z_{(n-m-2,m+2)}$$

$= 0$ by Lemma 4.14.

Hence we know that $\mathrm{im}(f_m) \subseteq \ker(f_{m+1})$ for all $m \geqslant 0$. We will use a dimension counting argument to complete the proof. First recall that the hook length formula (see

[24], for example) yields

$$\dim(\mathrm{S}_{(n-m,m)}) = \binom{n}{m}\frac{n-2m+1}{n-m+1}.$$

Now, looking at

$$f_m(z_{(n-m,m)}) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 3 & \cdots & 2m-1 & 2m+1 & 2m+2 & \cdots & n-1 \\ \hline 2 & 4 & \cdots & 2m & n & & & \\ \hline \end{array}$$

it is clear that

$$\dim(\mathrm{im}(f_m)) \geqslant \dim(\mathrm{S}_{(n-m-1,m)}) = \binom{n-1}{m}\frac{n-2m}{n-m}.$$

Similarly,

$$\dim(\mathrm{S}_{(n-m-1,m+1)}) = \binom{n}{m+1}\frac{n-2m-1}{n-m}$$

and

$$\dim(\mathrm{im}(f_{m+1})) \geqslant \dim(\mathrm{S}_{(n-m-2,m+1)}) = \binom{n-1}{m+1}\frac{n-2m-2}{n-m-1}.$$

So we have

$$\begin{aligned}
\dim(\ker(f_{m+1})) &\leqslant \binom{n}{m+1}\frac{n-2m-1}{n-m} - \binom{n-1}{m+1}\frac{n-2m-2}{n-m-1} \\
&= \binom{n-1}{m+1}\left(\frac{n(n-2m-1)}{(n-m-1)(n-m)} - \frac{(n-2m-2)}{(n-m-1)}\right) \\
&= \binom{n-1}{m+1}\frac{n(n-2m-1)-(n-2m-2)(n-m)}{(n-m-1)(n-m)} \\
&= \binom{n-1}{m+1}\frac{(n-2m)(m+1)}{(n-m-1)(n-m)} \\
&= \binom{n-1}{m}\frac{n-2m}{n-m} \\
&\leqslant \dim(\mathrm{im}(f_m)),
\end{aligned}$$

which completes the proof. □

Corollary 4.16. *If m is even and $\mathrm{S}_{(n-m,m)}$ and $\mathrm{S}_{(n-m-1,m+1)}$ both have $\underline{\mathrm{S}}_{(n)}$ (where the underlining represents forgetting all information of the grading, as in [29]) as a composition factor, then*

$$\mathrm{S}_{(n)} \subseteq \mathrm{im}(f_m).$$

Proof. We will argue by induction on m. Firstly, note that from the James formula we know that $\underline{\mathrm{S}}_{(n)}$ is a composition factor of $\mathrm{S}_{(n-m,m)}$ if and only if it is a composition factor of $\mathrm{S}_{(n-m-1,m+1)}$, when m is even. Now, when $m = 0$, the result is clear. So suppose $m \neq 0$. If $\mathrm{S}_{(n-m+2,m-2)}$ and $\mathrm{S}_{(n-m+1,m-1)}$ both have $\underline{\mathrm{S}}_{(n)}$ as a composition factor, then the induction hypothesis, along with exactness, yields our result. If neither $\mathrm{S}_{(n-m+2,m-2)}$ nor $\mathrm{S}_{(n-m+1,m-1)}$ have $\underline{\mathrm{S}}_{(n)}$ as a composition factor, we know that $\mathrm{im}(f_{m-1})$ cannot contain $\underline{\mathrm{S}}_{(n)}$, so $\ker(f_m)$ can't. The result follows. □

Chapter 5

The branching rule and dominated homomorphisms for $e = 2$

In this chapter, we wish to prove an analogue of Theorem 2.7 when $e = 2$, which in turn would yield a column removal result as in Theorem 2.30 for the whole of $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$, not just $\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$. Unfortunately we were unsuccessful in this venture, but made some progress towards it, and include our ideas in this final chapter. We end the chapter with Conjecture 5.15, where we boldly hypothesise that in the most general scenario, with $\lambda, \mu \in \mathscr{P}_n^l$, $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) = \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ when λ is *regular*. In practice, however, we attempt to prove a level 1 version of this in Section 5.2.

Remark. In light of the example after Theorem 2.23, we clearly need some extra conditions for any analogous result, even in level 1. Indeed, the example showed that we cannot even expect a single column removal result to hold without some extra conditions.

5.1 The branching rule

Our approach to proving that all homomorphisms are dominated (under certain combinatorial conditions) will make great use of (a generalisation of) the branching rule given in [10, Theorem 4.11]. However, there are some problems with the proof presented by the authors – in particular, the main obstacle is a reliance on the proof of the analogous ungraded result in [3, Proposition 1.9]. The author thanks Professor Andrew Mathas profusely for communicating a problem with the original proof of [3, Proposition 1.9], the details of a fix in this classical case (which may be found in his "errata" for [33, Proposition 6.1]), and an explanation of how to put this proof into the graded context using results from [22]. Mathas's proof (for restriction to $\mathscr{H}_{n-1}$) is due to appear in a short note of his.

In [22], the authors are, in part, concerned with the following disparity: the Ariki–Koike algebra, as defined in Definition 1.6, the KLR algebra, and its cyclotomic quotient may all be defined as algebras over an integral domain $\mathcal{O}$, rather than a field. However, Brundan and Kleshchev's isomorphism theorem Theorem 1.25 only holds over a field.

In [22], the authors construct deformations of $\mathscr{H}_n^\kappa$ (viewed as an $\mathcal{O}$-algebra, where $\mathcal{O}$ must be what they call an *idempotent subring* of some field $\mathscr{K}$), which we denote by $\mathscr{H}_n(\mathcal{O})$. They show ([22, Theorem A]) that $\mathscr{H}_n(\mathcal{O}) \cong \mathcal{H}_{\mathcal{O},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$, and that over a field $\mathbb{F} = \mathcal{O}/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, the presentation of $\mathscr{H}_n(\mathcal{O})$ coincides with that of $\mathscr{H}_n^\kappa$. Crucially, for $\mathscr{K} = \mathrm{Frac}(\mathcal{O})$, $\mathcal{H}_{\mathscr{K},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$ is semisimple, and hence so is $\mathscr{H}_n(\mathscr{K}) \cong \mathscr{H}_n(\mathcal{O}) \otimes_{\mathcal{O}} \mathscr{K}$. Note that $\mathscr{H}_n(\mathcal{O})$ is *not* in general a graded algebra!

Importantly for us, we see in [22, Example 4.2b)] that given a field $\mathbb{F}$, the ring $\mathcal{O} = \mathbb{F}[x]_{(x)}$ satisfies the desired properties, where $\mathfrak{m}$ is the ideal generated by the indeterminate x and $\mathscr{K} = \mathrm{Frac}(\mathcal{O})$.

The authors define Specht modules $\mathrm{S}_\lambda(\mathcal{O})$ over $\mathscr{H}_n(\mathcal{O})$ and show that $\underline{\mathrm{S}_\lambda}(\mathbb{F}) \cong \underline{\mathrm{S}_\lambda}(\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F}$ and likewise $\underline{\mathrm{S}_\lambda}(\mathscr{K}) \cong \underline{\mathrm{S}_\lambda}(\mathcal{O}) \otimes_{\mathcal{O}} \mathscr{K}$. The Specht module $\mathrm{S}_\lambda(\mathcal{O})$ has a basis $\{\psi_{\mathtt{T}}^{\mathcal{O}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$, arising from a cellular basis of $\mathscr{H}_n(\mathcal{O})$ and $\psi_{\mathtt{T}}^{\mathcal{O}} \otimes_{\mathcal{O}} 1_{\mathbb{F}} = v_{\mathtt{T}}$.

Finally, in [22, Lemma 5.12], the authors show that $\mathrm{S}_\lambda(\mathscr{K})$ has a *seminormal basis* $\{f_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$. This completes the background we require to prove our branching rule; the proof will follow Mathas's proof (in his "errata") of [33, Proposition 6.1] very closely.

Definition 5.1. Let $\lambda \in \mathscr{P}_n^l$ and define

$$e_i^k \operatorname{Std}(\lambda) := \left\{ \mathtt{T} \in \operatorname{Std}(\lambda) \mid i_{\mathtt{T}} = (\dots \underbrace{i\, i \dots i}_{k \text{ times}}) \right\}$$

and $E = E(i, k, \lambda) := e_i^k \operatorname{Std}(\lambda)/\sim$, where $\sim$ denotes the equivalence relation defined by $\mathtt{T} \sim \mathtt{S}$ if $\mathtt{T}^{-1}(j) = \mathtt{S}^{-1}(j)$ for all $j > n - k$.

Note that $|E| = \binom{m}{k} k!$, where m is the number of removable i-nodes in λ.

Example. Let $e = 2$, $\kappa = (0, 1)$ and $\lambda = ((4, 3, 1^2), (3, 1)) \in \mathscr{P}_{13}^2$. The residue diagram for λ is

0	1	0	1
1	0	1	
0			
1			

1	0	1
0		

and thus we see that $e_1^k \operatorname{Std}(\lambda)$ is the set of standard λ-tableaux which contain the entries $13, 12, \dots, 13 - k + 1$ in removable 1-nodes, and $e_1^k \operatorname{Std}(\lambda) = \emptyset$ if $k > 4$. For $k = 1$, the four elements of E are given by the following four equivalence classes of tableaux: $\{\mathtt{T} \in \operatorname{Std}(\lambda) \mid \mathtt{T}^{-1}(n) = (1, 4, 1)\}$, $\{\mathtt{T} \in \operatorname{Std}(\lambda) \mid \mathtt{T}^{-1}(n) = (2, 3, 1)\}$, $\{\mathtt{T} \in \operatorname{Std}(\lambda) \mid \mathtt{T}^{-1}(n) = (4, 1, 1)\}$ and $\{\mathtt{T} \in \operatorname{Std}(\lambda) \mid \mathtt{T}^{-1}(n) = (1, 3, 2)\}$.

Recall Robinson's i-restriction functor; in [9, §4.4], the authors define a graded analogue of this functor, which we here write as

$$e_{i,\alpha} := \sum_{\substack{j \in I \\ j_n = i}} e(j) \mathscr{H}_{\alpha + \alpha_i} \otimes_{\mathscr{H}_{\alpha+\alpha_i}} - : \mathscr{H}_{\alpha+\alpha_i}\text{-mod} \longrightarrow \mathscr{H}_\alpha\text{- mod} .$$

The functor $e_{i,\alpha}$ is simply left multiplication by the idempotent $\sum_{\substack{j \in I \\ j_n = i}} e(j)$ followed by restriction to $\mathscr{H}_\alpha$ via the map $\operatorname{shift}_0 : \mathscr{H}_\alpha \to \mathscr{H}_{\alpha+\alpha_i}$. Define $e_i := \bigoplus_\alpha e_{i,\alpha}$ where the sum is over all $\alpha \in Q^+$ of height $n - 1$. Note that e_i is an exact functor. Composing such functors k times, we obtain the functor $e_i^k : \mathscr{H}_n\text{-mod} \to \mathscr{H}_{n-k}\text{-mod}$.

Theorem 5.2 (The graded branching rule). *Let $\lambda \in \mathscr{P}^l_n$ and suppose λ has m removable i-nodes. For any $k \leqslant m$, there is a filtration*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{\binom{m}{k}k!} = e_i^k \, \mathrm{S}_\lambda$$

such that for each $1 \leqslant j \leqslant \binom{m}{k}k!$, $V_j / V_{j-1} \cong \mathrm{S}_{\nu_j}\langle c\rangle$ for some ν_j obtained from λ by removing k removable i-nodes, and some $c \in \mathbb{Z}$ determined by j.

Remark. Note that when $k = 1$, this is just the branching rule in [10, Theorem 4.11], projected onto a block according to the residue i.

Proof. We define a total order $\succ$ on E as follows: for $[\mathtt{T}], [\mathtt{S}] \in E$, $[\mathtt{T}] \succ [\mathtt{S}]$ if and only if for some $j > n-k$ we have that $\mathtt{T}^{-1}(j)$ is lower than $\mathtt{S}^{-1}(j)$ but $\mathtt{T}^{-1}(d) = \mathtt{S}^{-1}(d)$ for all $d > j$. We label the elements of E by $E_1, \dots, E_{\binom{m}{k}k!}$ so that $E_1 \prec E_2 \prec \cdots \prec E_{\binom{m}{k}k!}$.

Next, we construct the desired filtration as an $\mathcal{O}$-filtration of $e_i^k \, \mathrm{S}_\lambda(\mathcal{O})$, where as explained at the start of Section 5.1 we may take $\mathcal{O} = \mathbb{F}[x]_{(x)}$. Note that over $\mathcal{O}$ this will not be a graded filtration, but when we tensor with $\mathbb{F}$ (to yield a filtration of $e_i^k \, \mathrm{S}_\lambda$ over $\mathbb{F}$) we will see that we do have a graded filtration.

Now, we define $V_j^{\mathcal{O}} := \langle \psi_{\mathtt{T}}^{\mathcal{O}} \mid \mathtt{T} \in E_l \text{ for some } l \leqslant j\rangle_{\mathcal{O}}$. To see that $V_j^{\mathcal{O}}$ is in general an $\mathscr{H}_{n-k}(\mathcal{O})$-module, it suffices to note that for any generator $h \in \mathscr{H}_{n-k}(\mathcal{O})$ and any $\mathtt{T} \in E_l$, $h\psi_{\mathtt{T}}^{\mathcal{O}} = \sum a_{\mathtt{S}}\psi_{\mathtt{S}}^{\mathcal{O}}$ and for each $\mathtt{S}$ with $a_{\mathtt{S}} \neq 0$ either $\mathtt{S} \sim \mathtt{T}$ or $\mathtt{S} \lhd \mathtt{T}$. This is a consequence of the set-up of seminormal forms in [22].

If $\mathtt{S} \sim \mathtt{T}$, $\mathtt{S} \in E_l$ so $\psi_{\mathtt{S}}^{\mathcal{O}} \in V_j^{\mathcal{O}}$. If $\mathtt{S} \lhd \mathtt{T}$, we have $\mathrm{Shape}(\mathtt{S}_{\downarrow n-t}) \trianglelefteq \mathrm{Shape}(\mathtt{T}_{\downarrow n-t})$ for all $0 \leqslant t \leqslant k$, with equality when $t = k$. If there is equality for each $0 \leqslant t \leqslant k$, then $\mathtt{T} = \mathtt{S}$, which is not possible. So let t be minimal such that $\mathrm{Shape}(\mathtt{S}_{\downarrow n-t}) \lhd \mathrm{Shape}(\mathtt{T}_{\downarrow n-t})$. So $\mathtt{T}^{-1}(n-t+1) \neq \mathtt{S}^{-1}(n-t+1)$ but $\mathtt{T}^{-1}(n-t') = \mathtt{S}^{-1}(n-t')$ for all $t' < t-1$. Since $\mathrm{Shape}(\mathtt{S}_{\downarrow n-t}) \lhd \mathrm{Shape}(\mathtt{T}_{\downarrow n-t})$, $\mathtt{S}^{-1}(n-t+1)$ must be in a higher row than $\mathtt{T}^{-1}(n-t+1)$. So $\mathtt{S} \in E_{l'}$ for some $l' < l$, and in particular $\psi_{\mathtt{S}}^{\mathcal{O}} \in V_j^{\mathcal{O}}$.

For each j, suppose the equivalence class E_j consists of tableaux with $n, n-1, \dots, n-k+1$ in nodes $A_1, A_2, \dots, A_k$ respectively. Define $\nu_j := \lambda \setminus \{A_1, \dots, A_k\}$.

We will show that $V_j^{\mathcal{O}}/V_{j-1}^{\mathcal{O}} \cong \mathrm{S}_{\nu_j}(\mathcal{O})$, and subsequently $V_j/V_{j-1} \cong \mathrm{S}_{\nu_j}\langle c\rangle$, where $V_j := V_j^{\mathcal{O}} \otimes_{\mathcal{O}} \mathbb{F}$ and $\mathbb{F} = \mathcal{O}/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. In particular, we show that the map $\Theta_j^{\mathcal{O}} : V_j^{\mathcal{O}}/V_{j-1}^{\mathcal{O}} \to \mathrm{S}_{\nu_j}(\mathcal{O})$ defined by $\psi_{\mathtt{T}}^{\mathcal{O}} + V_{j-1} \mapsto \psi_{\mathtt{T}\downarrow_{n-k}}^{\mathcal{O}}$ for each $\mathtt{T} \in E_j$ is an isomorphism. It is clear that $\dim V_j^{\mathcal{O}}/V_{j-1}^{\mathcal{O}} = \dim \mathrm{S}_{\nu_j}(\mathcal{O})$ and thus that $\Theta_j^{\mathcal{O}}$ is an isomorphism of vector spaces. It remains to show that $\Theta_j^{\mathcal{O}}$ is an $\mathscr{H}_n(\mathcal{O})$-homomorphism.

Recalling that $\underline{\mathrm{S}_\lambda}(\mathscr{K}) \cong \underline{\mathrm{S}_\lambda}(\mathcal{O}) \otimes_{\mathcal{O}} \mathscr{K}$, and considering $\mathrm{S}_\lambda(\mathcal{O})$ as an $\mathcal{O}$-submodule of $\underline{\mathrm{S}_\lambda}(\mathscr{K})$, we see that $\Theta_j^{\mathcal{O}}$ is an $\mathscr{H}_n(\mathcal{O})$-homomorphism if and only if $\Theta_j^{\mathcal{O}} \otimes 1_{\mathscr{K}}$ is an $\mathscr{H}_n(\mathscr{K})$-homomorphism, so we may work over $\mathscr{K}$, where $\mathscr{H}_n(\mathscr{K})$ is semisimple and $\mathrm{S}_\lambda(\mathscr{K})$ has a seminormal basis $\{f_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$; $\mathrm{S}_{\nu_j}(\mathscr{K})$ has an analogous seminormal basis.

We define a new homomorphism

$$\Theta^{\mathscr{K}} : e_i^k\, \mathrm{S}_\lambda(\mathscr{K}) \longrightarrow \bigoplus_{j=1}^{\binom{m}{k}k!} \mathrm{S}_{\nu_j}(\mathscr{K})$$

by $f_{\mathtt{T}} \mapsto f_{\mathtt{T}\downarrow_{n-k}}$. It is clear that $\Theta^{\mathscr{K}}$ is a vector space isomorphism. From the proof of [22, Theorem 5.7], it can be seen that there is a unitriangular transition matrix between the bases $\{f_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ and $\{\psi_{\mathtt{T}}^{\mathcal{O}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ and thus $V_j^{\mathscr{K}} := V_j^{\mathcal{O}} \otimes_{\mathcal{O}} \mathscr{K}$ has a basis $\{f_{\mathtt{T}} \mid \mathtt{T} \in E_l \text{ for some } l \leqslant j\}$. We see that $\Theta^{\mathscr{K}}(V_j^{\mathscr{K}}) = \bigoplus_{i=1}^{j} \mathrm{S}_{\nu_i}(\mathscr{K})$, so $\Theta^{\mathscr{K}}$ induces a map $V_j^{\mathscr{K}}/V_{j-1}^{\mathscr{K}} \to \mathrm{S}_{\nu_j}(\mathscr{K})$ for each $1 \leqslant j \leqslant \binom{m}{k}k!$. Furthermore, analogously to Mathas's (revised) proof of [33, Proposition 6.1], $\Theta^{\mathscr{K}}$ is an $\mathscr{H}_{n-k}(\mathscr{K})$-homomorphism, and it follows similarly that $\Theta_j^{\mathscr{K}}(m + V_{j-1}^{\mathscr{K}}) = \Theta^{\mathscr{K}}(m)$ for all $m \in V_j^{\mathscr{K}}$ and $1 \leqslant j \leqslant \binom{m}{k}k!$. This suffices to prove that each $\Theta_j^{\mathscr{K}}$ is an $\mathscr{H}_{n-k}(\mathscr{K})$-homomorphism; it follows that we have an $\mathcal{O}$-filtration, and tensoring with $\mathbb{F} = \mathcal{O}/\mathfrak{m}$ yields the desired filtration, as ungraded modules.

That $\Theta_j^{\mathbb{F}} : V_j/V_{j-1} \to \mathrm{S}_{\nu_j}\langle c\rangle$ is an isomorphism of *graded* modules follows from the combinatorics; the fact that $\mathrm{grdim}\, V_j/V_{j-1} = \mathrm{grdim}\, \mathrm{S}_\nu\langle c\rangle$ is clear, as is the degree shift by $c = d^{A_1}(\lambda) + d^{A_2}(\lambda \setminus \{A_1\}) + \cdots + d^{A_k}(\lambda \setminus \{A_1, A_2, \ldots, A_{k-1}\})$. □

Remark. We have been slightly sloppy in the statement of Theorem 5.2 and not explicitly stated over which algebra we are considering the Specht module S_λ. In fact, the proof works entirely over $\mathscr{H}_n^\kappa$, but $S_{\lambda|\kappa}$ is a module over both $\mathscr{H}_n^\kappa$ and $\mathscr{H}_n$. So in fact we have filtrations for both!

Example. Let $e = 2$, $l = 1$, $\kappa = (0)$ and $\lambda = ((3,2,1))$. Then we have the filtration

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5 \subset V_6 = e_0^2 S_\lambda$$

where for each j, $V_j := \langle v_{\mathtt{T}} \mid \mathtt{T} \in E_l \text{ for some } l \leqslant j \rangle_{\mathbb{F}}$ and

$$E_1 = \{\mathtt{T} \in \mathrm{Std}(\lambda) \mid \mathtt{T}^{-1}(n) = (1,3) \text{ and } \mathtt{T}^{-1}(n-1) = (2,2)\},$$
$$E_2 = \{\mathtt{T} \in \mathrm{Std}(\lambda) \mid \mathtt{T}^{-1}(n) = (1,3) \text{ and } \mathtt{T}^{-1}(n-1) = (3,1)\},$$
$$E_3 = \{\mathtt{T} \in \mathrm{Std}(\lambda) \mid \mathtt{T}^{-1}(n) = (2,2) \text{ and } \mathtt{T}^{-1}(n-1) = (1,3)\},$$
$$E_4 = \{\mathtt{T} \in \mathrm{Std}(\lambda) \mid \mathtt{T}^{-1}(n) = (2,2) \text{ and } \mathtt{T}^{-1}(n-1) = (3,1)\},$$
$$E_5 = \{\mathtt{T} \in \mathrm{Std}(\lambda) \mid \mathtt{T}^{-1}(n) = (3,1) \text{ and } \mathtt{T}^{-1}(n-1) = (1,3)\},$$
$$E_6 = \{\mathtt{T} \in \mathrm{Std}(\lambda) \mid \mathtt{T}^{-1}(n) = (3,1) \text{ and } \mathtt{T}^{-1}(n-1) = (2,2)\}.$$

Then we have the following isomorphisms:

$$V_1 \cong S_{((2,1^2))}\langle 1 \rangle, \qquad V_2/V_1 \cong S_{((2^2))}\langle 0 \rangle,$$
$$V_3/V_2 \cong S_{((2,1^2))}\langle -1 \rangle, \qquad V_4/V_3 \cong S_{((3,1))}\langle -1 \rangle,$$
$$V_5/V_4 \cong S_{((2^2))}\langle -2 \rangle, \qquad V_6/V_5 \cong S_{((3,1))}\langle -3 \rangle.$$

It can be checked that the graded dimensions of these six (shifted) Specht modules sum to $v^4 + 4v^2 + 6 + 4v^{-2} + v^{-4} = (v + v^{-1})^4 = \mathrm{grdim}\, S_{((3,2,1))}$.

5.2 Dominated homomorphisms for $e = 2$

In this section, we will mostly be concerned with the case $l = 1$ (cf. Section 3.1) though we end it with a conjecture for arbitrary l. Until further notice, fix $l = 1$.

For the most part, we will make steps towards proving that if $l = 1$, $e = 2$, $\lambda \vdash_2 n$ and $\mu \vdash n$ then $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) = \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$. Sadly, this work is currently incomplete, but we have several conjectures we are confident in.

Definition 5.3. If $\lambda \vdash n$ and $r \in \mathbb{N}$, we define the *rth ladder* of $[\lambda]$ to be

$$\mathbb{L}_r = \{(i, j) \in [\lambda] \mid i + (e-1)(j-1) = r\}.$$

If r is maximal such that $\mathbb{L}_r \neq \emptyset$, then we call $\mathbb{L}_r$ the *outer ladder* of $[\lambda]$.

Remark. Note that all nodes in a given ladder $\mathbb{L}_r$ have the same residues, so we may talk of the residue of a ladder.

For the remainder of the section, we fix $e = 2$. Thus the ladder $\mathbb{L}_r = \{(i, j) \in [\lambda] \mid i + j - 1 = r\}$. Furthermore, we will assume throughout that λ is 2-regular.

Definition 5.4. If $\lambda \vdash_2 n$ with an outer ladder of size k, and $\bar{\lambda}$ is the partition of $n - k$ obtained by removing the outer ladder of λ, then we define $\mathrm{Std}(\bar{\lambda})^+$ to be the set of all standard λ-tableaux with entries $n-k+1, n-k+2, \ldots, n$ in order going up the outer ladder of λ.

Henceforth we will assume that $\lambda \vdash_2 n$ such that $[\lambda]$ has an outer ladder of size k, residue i. By Theorem 5.2, we see that $e_i^k \mathrm{S}_\lambda$ has a bottom Specht factor (and therefore submodule) isomorphic to a shift of $\mathrm{S}_{\bar{\lambda}}$, given by $\langle v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\bar{\lambda})^+ \rangle_{\mathbb{F}}$. To emphasise that this is the degree shifted copy of $\mathrm{S}_{\bar{\lambda}}$ in S_λ, we denote it by $\mathrm{S}_{\bar{\lambda}^+}$.

Definition 5.5. Define $\mathtt{T}_{\bar{\lambda}}^+$ to be the $\trianglelefteq$-minimal tableau in $\mathrm{Std}(\bar{\lambda})^+$. That is, the tableau $\mathtt{T}_{\bar{\lambda}}$ with an outer ladder of size k adjoined to it, with entries $n-k+1, \ldots, n$ up the outer ladder.

Proposition 5.6. $\mathrm{S}_{\bar{\lambda}^+}$ *is generated by* $z_{\bar{\lambda}}^+ := v_{\mathtt{T}_{\bar{\lambda}}^+}$ *as an* $\mathscr{H}_{n-k}$*-module. Furthermore,*

$$\psi_{\mathtt{T}_{\bar{\lambda}}^+} = \psi_{n-k}\psi_{n-k-1}\cdots\psi_{n-t_{k-1}}\psi_{n-k+1}\cdots\psi_{n-t_{k-2}}\cdots\psi_{n-t_2}$$

where t_i is the ith triangle number. $w_{\mathtt{T}^+_{\bar{\lambda}}}$ is fully commutative, and therefore this does not depend on any choice of preferred reduced expression.

Proof. $w_{\mathtt{T}^+_{\bar{\lambda}}}$ is fully commutative by Lemma 1.9. It is clear that the above is a reduced expression for $\psi_{\mathtt{T}^+_{\bar{\lambda}}}$. □

Example. 1. Let $\lambda = (3,2,1)$. Then $k = 3$, $\mathtt{T}_{\bar{\lambda}} = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$ and $\psi_{\mathtt{T}^+_{\bar{\lambda}}} = \psi_3$.

2. Let $\lambda = (4,3,2,1)$. Then $k = 4$, $\mathtt{T}_{\bar{\lambda}} = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline 2 & 5 & 9 \\ \cline{1-3} 3 & 8 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$ and $\psi_{\mathtt{T}^+_{\bar{\lambda}}} = \psi_6\psi_5\psi_4\psi_7$.

Definition 5.7. If $\lambda \vdash n$, we call λ a 2-core if $\lambda = (m, m-1, m-2, \ldots, 1)$.

Remark. One may define an e-core for arbitrary e, but since the only place we will call on e-cores is in Lemma 5.8, we give the restricted (but equivalent) definition above.

It is a well known result that if λ is a 2-core, then $\mathrm{S}_\lambda = \mathrm{D}_\lambda$.

Lemma 5.8. *Let $\lambda \vdash_2 n$ and $\mu \vdash n$ and suppose $\varphi \in \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ is non-zero. Then $\varphi(z^+_{\bar{\lambda}}) \neq 0$. In particular, $\varphi(\mathrm{S}_{\bar{\lambda}^+}) \neq 0$.*

Proof. The lowest node in the outer ladder of $[\lambda]$ is (k, λ_k). Let $\nu = (\lambda_1 - \lambda_k + 1, \lambda_2 - \lambda_k + 1, \ldots, 1)$. This partition can be thought of as the first k rows of λ, from column λ_k to the right. $\nu \vdash t_k$ is a 2-core, and therefore S_ν is an irreducible $\mathscr{H}_{t_k}$-module, and thus any non-zero element of S_ν generates it. In particular, $z_\nu = hz^+_{\bar{\nu}} = h\psi_{\mathtt{T}^+_{\bar{\nu}}} z_\nu$ for some $h \in \mathscr{H}_{t_k}$. Since the first $\lambda_k - 1$ columns of $\mathtt{T}_\lambda$ and $\mathtt{T}^+_{\bar{\lambda}}$ agree, expressions for the basis of S_λ can be chosen in such a way that $\psi_{\mathtt{T}^+_{\bar{\lambda}}} = \mathrm{shift}_{n-t_k}(\psi_{\mathtt{T}^+_{\bar{\nu}}})$. It is now clear that $\mathrm{shift}_{n-t_k}(h)z^+_{\bar{\lambda}} = \mathrm{shift}_{n-t_k}(h\psi_{\mathtt{T}^+_{\bar{\nu}}})z_\lambda = z_\lambda$, and so $z^+_{\bar{\lambda}}$ generates S_λ. The result follows. □

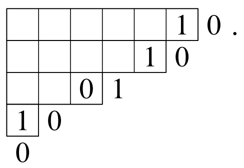
Example. If $k = 3$,

$$z_\lambda = \mathrm{shift}_{n-t_k}(-\psi_3\psi_4\psi_5)z^+_{\bar{\lambda}} = \mathrm{shift}_{n-t_k}(-\psi_3\psi_4\psi_5)\psi_{n-3}z_\lambda = -\psi_{n-3}\psi_{n-2}\psi_{n-1}\psi_{n-3}z_\lambda.$$

Definition 5.9. If $\lambda \vdash n$, we define the *i-signature* of λ to be the sequence of $+$ and $-$ signs obtained by examining the addable and removable i-nodes of $[\lambda]$ from top to bottom, writing a $+$ for each addable i-node and a $-$ for each removable i-node, and define the *reduced i-signature* of λ to be the subsequence of the i-signature obtained by successively deleting adjacent pairs $+-$. We call the removable i-nodes of $[\lambda]$ corresponding to the $-$ signs in the reduced i-signature the *normal i-nodes* of λ.

It is a well known fact (see [28, Theorem 7.4] for example) that if $\lambda \vdash_e n$ has x normal i-nodes then $e_i^x \, \mathrm{D}_\lambda \neq 0$.

Example. Let $e = 2$ and $\lambda = (6, 5, 3, 1)$. Consider the addable and removable nodes of $[\lambda]$:



So λ has 0-signature $+ + - + +$, reduced 0-signature $+ + +$ and therefore no normal 0-nodes. λ has 1-signature $- - + -$ and reduced 1-signature $- -$, corresponding to the nodes at the ends of the first two rows. Thus, these two nodes are normal 1-nodes. In particular, $e_1^2 \, \mathrm{D}_\lambda \neq 0$.

Lemma 5.10. *If $\lambda \vdash_e n$ has x normal i-nodes and $\mu \vdash n$ with $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) \neq 0$ then μ has at least x removable i-nodes.*

Proof. Since $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) \neq 0$, $[\mathrm{S}_\mu : \mathrm{D}_\lambda] \neq 0$. Thus, by the above comment we have that $e_i^x \, \mathrm{D}_\lambda \neq 0$, so $e_i^x \, \mathrm{S}_\mu \neq 0$. Recalling standard facts about the functors e_i from the proof of Theorem 3.2, we see that μ must have at least x removable i-nodes. □

Remark. All i-nodes in the outer ladder of $[\lambda]$ are normal nodes, and therefore we may assume that μ has at least k removable i-nodes.

From this point on we have many conjectures and few proofs. We will assume throughout that $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) \neq 0$, as otherwise our intended result holds trivially. The following conjecture is almost a refinement of Lemma 5.10, except that we only consider the normal i-nodes in the outer ladder of $[\lambda]$. We provide a counterexample to a corresponding statement for all normal i-nodes.

Conjecture 5.11. *Let $\lambda \vdash_2 n$ and $\mu \vdash n$, and suppose the outer ladder of $[\lambda]$ has residue i. Suppose $\varphi \in \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ with $\varphi(z_\lambda) = \sum_{\mathtt{T}\in\mathrm{Std}(\mu)} a_{\mathtt{T}} v_{\mathtt{T}}$. Whenever $a_{\mathtt{T}} \neq 0$ and j is in the outer ladder of $\mathtt{T}_\lambda$, j is in a removable i-node of $\mathtt{T}$.*

Remarks.

1. It it certainly *not* the case that the following stronger statement holds: suppose $\varphi \in \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ with $\varphi(z_\lambda) = \sum_{\mathtt{T}\in\mathrm{Std}(\mu)} a_{\mathtt{T}} v_{\mathtt{T}}$. Whenever $a_{\mathtt{T}} \neq 0$ and j is in a normal i-node of $\mathtt{T}_\lambda$, j is in a removable node of $\mathtt{T}$. For an easy counterexample to this, let $\lambda = (4, 1)$ and $\mu = (2, 1^3)$. For any p, there is a homomorphism $\varphi : z_\lambda \mapsto v_{\mathtt{T}}$ where $\mathtt{T}_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 \\ \cline{1-1}\end{array}$ and $\mathtt{T} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$. 2 is in a normal 1-node of $\mathtt{T}_\lambda$, but not a removable node of $\mathtt{T}$. Note however that the outer ladder of $\mathtt{T}_\lambda$ consists only of the node containing the entry 5, which *is* in a removable node of $\mathtt{T}$.

2. If $k = 1$ or 2, the conjecture is easily seen to be true by examining residues of standard tableaux.

We will be interested in the filtrations of $e_i^k \mathrm{S}_\lambda$ and $e_i^k \mathrm{S}_\mu$. For the former, we are in fact interested in the bottom factor, which is a submodule. From now on, we shall thus let $E_1 \prec E_2 \prec \cdots$ denote the equivalence classes of standard μ-tableaux as in the proof of Theorem 5.2. Analogously to the equivalence classes E_j, we may define equivalence classes E'_j, where we replace each $n - m$ in the definition of E_j with $n - t_m$. These equivalence classes involve tableaux with $n - t_{k-1}, n - t_{k-2}, \ldots, n - 1, n$ in removable nodes. Although we have a natural correspondence $E_j \leftrightarrow E'_j$, E_j and E'_j do not necessarily contain the same number of tableaux. Note that this definition relies on Conjecture 5.11.

Conjecture 5.12. *Let $\varphi \in \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ with*

$$\varphi(z_\lambda) = \sum_{\mathtt{T} \in E'_r} a_\mathtt{T} v_\mathtt{T} + \sum_{\substack{\mathtt{S} \in E'_j \\ j<r}} a_\mathtt{S} v_\mathtt{S} \quad \text{for some } a_\mathtt{T}, a_\mathtt{S} \in \mathbb{F}$$

and some r. Then for each $\mathtt{T} \in E'_r$ with $a_\mathtt{T} \neq 0$ and each $m = 1, 2, \ldots, k-1$, $n - t_m \nearrow_\mathtt{T} d$ for all $d > n - t_m$.

In particular, this conjecture says that in some sense the order of entries in the outer ladder is preserved – this property is desirable for our approach.

Example. Let $p = 0$, $\lambda = (7, 6)$ and $\mu = (4, 4, 2, 2, 1)$. Then $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ is spanned by $\varphi : z_\lambda \mapsto 3v_{\mathtt{T}_1} + 6v_{\mathtt{T}_2} + 3v_{\mathtt{T}_3} + 3v_{\mathtt{T}_4} - 6v_{\mathtt{T}_5} - 3v_{\mathtt{T}_6} + v_{\mathtt{T}_7}$ where

$$\mathtt{T}_1 = \begin{array}{cccc} 1 & 3 & 9 & 11 \\ 2 & 5 & 10 & 13 \\ 4 & 7 & & \\ 6 & 8 & & \\ 12 & & & \end{array}, \quad \mathtt{T}_2 = \begin{array}{cccc} 1 & 3 & 9 & 11 \\ 2 & 5 & 10 & 13 \\ 4 & 6 & & \\ 7 & 12 & & \\ 8 & & & \end{array}, \quad \mathtt{T}_3 = \begin{array}{cccc} 1 & 3 & 9 & 11 \\ 2 & 5 & 10 & 12 \\ 4 & 7 & & \\ 6 & 13 & & \\ 8 & & & \end{array}, \quad \mathtt{T}_4 = \begin{array}{cccc} 1 & 3 & 9 & 11 \\ 2 & 4 & 10 & 13 \\ 5 & 7 & & \\ 6 & 12 & & \\ 8 & & & \end{array},$$

$$\mathtt{T}_5 = \begin{array}{cccc} 1 & 3 & 8 & 11 \\ 2 & 5 & 10 & 13 \\ 4 & 7 & & \\ 6 & 12 & & \\ 9 & & & \end{array}, \quad \mathtt{T}_6 = \begin{array}{cccc} 1 & 3 & 5 & 7 \\ 2 & 9 & 11 & 13 \\ 4 & 10 & & \\ 6 & 12 & & \\ 8 & & & \end{array}, \quad \mathtt{T}_7 = \begin{array}{cccc} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 9 \\ 8 & 11 & & \\ 10 & 13 & & \\ 12 & & & \end{array}.$$

It is easy to see that $\mathtt{T}_7$ is in the ($\preccurlyeq$-)highest equivalence class out of all 7 tableaux above, and is the only tableau from this class which occurs here. Since $k = 2$, the conjecture predicts only that $12 \nearrow_{\mathtt{T}_7} 13$, which is seen to hold.

Remarks.

1. The conclusion of Conjecture 5.12 is certainly false if we don't include the condition that $\mathtt{T} \in E'_r$ – in the above example, $13 \nearrow_{\mathtt{T}_3} 12$!

2. If $k = 1$, the conjecture is holds trivially.

Conjecture 5.13. *Assume $\varphi \in \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ is as in the statement of Conjecture 5.12. Then for each $\mathtt{S} \in E'_j$ with $j < r$ and $a_\mathtt{S} \neq 0$, there exists $\mathtt{T} \in E'_r$ such that $a_\mathtt{T} \neq 0$ and $\mathtt{T} \rhd \mathtt{S}$.*

Remark. Examples suggest that perhaps only one tableau from E'_r occurs, and with coefficient 1; if this were true, then in fact there is a unique maximal tableau $\mathtt{T}$ such that $a_{\mathtt{T}} \neq 0$ and $\mathtt{T}$ dominates all other tableaux $\mathtt{S}$ for which $a_{\mathtt{S}} \neq 0$. This can be seen to be the case in the previous example. Here, $\mathtt{T}_7$ dominates all 6 other tableaux, which are pairwise incomparable in the dominance order.

Example. Let $p = 0$, $\lambda = (7, 6, 5)$ and $\mu = (7, 4, 2^2, 1^3)$. Now $k = 3$ so perhaps we have a more interesting example. There is a homomorphism $\varphi : z_\lambda \mapsto 8v_{\mathtt{T}_1} - 4v_{\mathtt{T}_2} + 2v_{\mathtt{T}_3} + v_{\mathtt{T}_4}$ where

$$\mathtt{T}_1 = \begin{array}{lllllll} 1 & 4 & 9 & 12 & 13 & 16 & 18 \\ 2 & 5 & 14 & 17 \\ 3 & 10 \\ 6 & 15 \\ 7 \\ 8 \\ 11 \end{array}, \qquad \mathtt{T}_2 = \begin{array}{lllllll} 1 & 4 & 7 & 12 & 13 & 16 & 18 \\ 2 & 5 & 14 & 17 \\ 3 & 8 \\ 6 & 15 \\ 9 \\ 10 \\ 11 \end{array},$$

$$\mathtt{T}_3 = \begin{array}{lllllll} 1 & 4 & 7 & 10 & 13 & 16 & 18 \\ 2 & 5 & 14 & 17 \\ 3 & 8 \\ 6 & 11 \\ 9 \\ 12 \\ 15 \end{array}, \qquad \mathtt{T}_4 = \begin{array}{lllllll} 1 & 4 & 7 & 10 & 13 & 16 & 18 \\ 2 & 5 & 8 & 11 \\ 3 & 14 \\ 6 & 17 \\ 9 \\ 12 \\ 15 \end{array}.$$

This time, the tableaux are totally ordered ($\mathtt{T}_1 \lhd \mathtt{T}_2 \lhd \mathtt{T}_3 \lhd \mathtt{T}_4$) and we see that the predictions of Conjectures 5.12 and 5.13 hold. Again, we see that the most dominant tableau occurs with coefficient 1.

Take a filtration of $e_i^k \mathrm{S}_\mu$ as in Theorem 5.2. For $\varphi \in \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$, there exists some r such that $\varphi(z_{\bar{\lambda}}^+) \subseteq V_r$ but $\varphi(z_{\bar{\lambda}}^+) \nsubseteq V_{r-1}$. Composing with the natural surjection $V_r \twoheadrightarrow V_r / V_{r-1}$ yields a homomorphism $\mathrm{S}_{\bar{\lambda}^+} \to \mathrm{S}_{\bar{\mu}}\langle c \rangle$ where $\bar{\mu}$ and c are given in Theorem 5.2. Let $\mathrm{S}_{\bar{\mu}^+}$ denote the copy of $\mathrm{S}_{\bar{\mu}}\langle c \rangle$ seen in V_r / V_{r-1} – by this we mean that $\mathrm{S}_{\bar{\mu}^+}$ has a basis of μ-tableaux in the equivalence class E_r.

Conjecture 5.14. *Suppose $e = 2$, $\lambda \vdash_2 n$ and $\mu \vdash n$. Then we have* $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) = \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$.

Proof assuming Conjectures 5.11–5.13. We prove by induction on n. If $n = 1$, the result is trivial. So suppose $n > 1$ and let $\varphi \in \mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$. By the paragraph above, there is a homogeneous $\mathscr{H}_{n-k}$-homomorphism $\bar{\varphi} : \mathrm{S}_{\bar{\lambda}} \to \mathrm{S}_{\bar{\mu}}$. By induction, $\bar{\varphi}$ is dominated. Suppose

$$\varphi(z_\lambda) = \sum_{\mathrm{T} \in E'_r} a_\mathrm{T} v_\mathrm{T} + \sum_{\substack{\mathrm{S} \in E'_j \\ j<r}} a_\mathrm{S} v_\mathrm{S} \quad \text{for some } a_\mathrm{T}, a_\mathrm{S} \in \mathbb{F}.$$

Then by Proposition 5.6 and Conjectures 5.11–5.13,

$$\begin{aligned}\varphi(z_{\bar{\lambda}}^+) = \varphi(\psi_{\mathrm{T}_{\bar{\lambda}}^+} z_\lambda) &= \sum_{\mathrm{T} \in E'_r} a_\mathrm{T} \psi_{\mathrm{T}_{\bar{\lambda}}^+} v_\mathrm{T} + \sum_{\substack{\mathrm{S} \in E'_j \\ j<r}} a_\mathrm{S} \psi_{\mathrm{T}_{\bar{\lambda}}^+} v_\mathrm{S} \\ &= \sum_{\mathrm{T} \in E'_r} a_\mathrm{T} v_{\bar{\mathrm{T}}^+} + \sum_{\substack{\mathrm{U} \lhd \mathrm{T} \\ \mathrm{T} \in E'_r \\ a_\mathrm{T} \neq 0}} b_\mathrm{U} v_\mathrm{U}\end{aligned}$$

where $\bar{\mathrm{T}}^+ = w_{\mathrm{T}_{\bar{\lambda}}^+} \mathrm{T}$ has the entries $n-k+1, n-k+2, \dots, n$ up the nodes of $\mu \setminus \bar{\mu}$ in order. It follows that

$$\bar{\varphi}(z_{\bar{\lambda}}) = \sum_{\mathrm{S}_\mathrm{T} \in E_r} a_\mathrm{T} v_{\bar{\mathrm{S}_\mathrm{T}}},$$

where $\mathrm{S}_\mathrm{T} = w_{\mathrm{T}_{\bar{\lambda}}^+} \mathrm{T}$ ($= \bar{\mathrm{T}}^+$ from above). Since $\bar{\varphi}$ is dominated, it follows that for each $\mathrm{S}_\mathrm{T} \in E_r$ such that $a_\mathrm{T} \neq 0$, $\bar{\mathrm{S}_\mathrm{T}}$ is dominated, so $\bar{\mathrm{T}}^+$ is dominated. Since $\mathrm{T} \trianglelefteq \bar{\mathrm{T}}^+$ the result follows by Corollary 2.2(1). □

Example. We build on our previous example, where $p = 0$, $\lambda = (7,6,5)$ and $\mu = (7,4,2^2,1^3)$. Here, $\varphi(z_\lambda) = v_{\mathrm{T}_4} + \sum_{\mathrm{S} \lhd \mathrm{T}_4} a_\mathrm{S} v_\mathrm{S}$.

$$\varphi(z_{\bar{\lambda}}^+) = \varphi(\psi_{15} z_\lambda) = v_{s_{15}\mathrm{T}_4} + \sum_{\mathrm{S} \lhd \mathrm{T}_4} a_\mathrm{S} v_{s_{15}\mathrm{S}}.$$

Noticing that 16, 17 and 18 are in nodes $(4,2)$, $(2,4)$ and $(1,7)$ of $s_{15}\mathrm{T}_4$ respectively, we have a homomorphism $\bar{\varphi} : \mathrm{S}_{\bar{\lambda}} \to \mathrm{S}_{\bar{\mu}}$ where $\bar{\lambda} = (6,5,4)$ and $\bar{\mu} = (6,4,2,1^3)$ given by taking a quotient of tableaux from less dominant classes. That is,

$$\bar{\varphi}(z_{\bar{\lambda}}) = v_{\mathrm{U}} \quad \text{where} \quad u = \begin{array}{cccccc} 1 & 4 & 7 & 10 & 13 & 15 \\ 2 & 5 & 8 & 11 & & \\ 3 & 14 & & & & \\ 6 & & & & & \\ 9 & & & & & \\ 12 & & & & & \end{array}$$

is the tableau obtained from $s_{15}\mathrm{T}_4$ by removing the nodes containing 16, 17 and 18.

We end with one final conjecture, which generalises the previous one to higher levels, as well as generalising Theorem 2.7 to include the case where the κ_i may not be distinct. First, we note that the level 1 notion of e-regular may be extended to the higher level notion of *regular* multipartitions; we will not discuss the definition here. It will suffice to note the following:

1. the definition depends on e and κ;
2. these multipartitions are often called *conjugate Kleshchev* in the literature;
3. the set of regular multipartitions indexes a family of Specht modules with non-isomorphic simple heads – these simple heads form a complete set of simple $\mathscr{H}_n^\kappa$-modules.

Conjecture 5.15. *Suppose $\lambda \in \mathscr{P}_n^l$ is regular. Then* $\mathrm{Hom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) = \mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$.

Index of notation

For the reader's convenience we conclude with an index of the notation we use in this thesis. We provide references to the relevant sections.

$\mathbb{F}$	a field	
$\mathbb{N}$	the set of positive integers	
$\mathfrak{S}_n$	the symmetric group of degree n	1.1
$s_1, \ldots, s_{n-1}$	the Coxeter generators of $\mathfrak{S}_n$	1.1
l	the Coxeter length function on $\mathfrak{S}_n$	1.1
$\preccurlyeq_{\mathrm{L}}$	the left order on $\mathfrak{S}_n$	1.1
$\preccurlyeq$	the Bruhat order on $\mathfrak{S}_n$	1.1
shift_k	the shift homomorphism $\mathfrak{S}_m \to \mathfrak{S}_n$	1.1
$\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$	the Iwahori–Hecke algebra of type A	1.2
$\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$	the Ariki–Koike algebra	1.3
I	the set $\mathbb{Z}/e\mathbb{Z}$ (or $\mathbb{Z}$, if $e = \infty$)	1.4
Γ	a quiver with vertex set I	1.4
$i \to j$	there is an arrow from i to j (but no arrow from j to i) in Γ	1.4
$i \rightleftarrows j$	there are arrows from i to j and from j to i in Γ	1.4
α_i	simple root labelled by $i \in I$	1.4
Λ_i	fundamental dominant weight labelled by $i \in I$	1.4
$(\mid)$	invariant bilinear form	1.4
Q^+	the positive root lattice	1.4

deg(T)	the degree of a tableau T	1.7
codeg(T)	the codegree of a tableau T	1.7
$\underline{M}$	the module obtained from M by forgetting the grading	1.8
$M\langle k\rangle$	the graded module M with the grading shifted by k	1.8
$\mathscr{H}_n$	the KLR algebra of degree n	1.9
shift_k	the shift homomorphism $\mathscr{H}_\beta \to \mathscr{H}_\alpha$	1.9
$\mathscr{H}_n^\kappa$	the cyclotomic KLR algebra determined by κ	1.9
$\mathbf{B}_A$	the column Garnir belt corresponding to a Garnir node A	1.10
G_A	the Garnir tableau corresponding to a Garnir node A	1.10
g_A	the (column) Garnir element corresponding to a Garnir node A	1.10
S_λ	the column Specht module corresponding to a multipartition λ	1.10
$\mathbf{B}^A$	the row Garnir belt corresponding to a Garnir node A	1.10
g^A	the (row) Garnir element corresponding to a Garnir node A	1.10
S^λ	the row Specht module corresponding to a multipartition λ	1.10
z_λ	the standard generator of S_λ	1.10
z^λ	the standard generator of S^λ	1.10
$\psi_\mathtt{T}$	$\psi_{t_1}\dots\psi_{t_b}$, where $s_{t_1}\dots s_{t_b}$ is the preferred reduced expression for $w_\mathtt{T}$	1.10
$v_\mathtt{T}$	$\psi_\mathtt{T} z_\lambda$	1.10
D_λ	the head of S_λ	1.11
$d_{\lambda\mu}$	the composition multiplicity $[\mathrm{S}_\lambda : \mathrm{D}_\lambda]$	1.12
$D^{e,p}$	the decomposition matrix of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ with $q^e = 1$ and $\mathrm{char}(\mathbb{F}) = p$	1.12
A_p	the adjustment matrix satisfying $D^{e,p} = D^{e,0}A_p$	1.12

$d_{\lambda\mu}(v)$	the graded composition multiplicity $[\mathrm{S}_\lambda : \mathrm{D}_\mu]_v$	1.12
$D^{e,p}(v)$	the graded decomposition matrix of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ with $q^e = 1$ and $\mathrm{char}(\mathbb{F}) = p$	1.12
$A_p(v)$	the graded adjustment matrix satisfying the equality $D^{e,p}(v) = D^{e,0}(v)A_p(v)$	1.12
$\mathrm{Std}_\lambda(\mu)$	the set of λ-dominated standard μ-tableaux	2.1
$\mathrm{Std}^\lambda(\mu)$	the set of λ-row-dominated standard μ-tableaux	2.1
$\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$	the space of dominated homomorphisms from S_λ to S_μ	2.2
$\mathrm{DHom}_{\mathscr{H}_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$	the space of dominated homomorphisms from S^λ to S^μ	2.2
$M^\circledast$	the graded dual of a graded module M	2.3
$\mathrm{Std}_{\mathrm{lr}}(\lambda)$	the set of λ-tableaux in which the entries $1, \dots, n_\mathrm{l}$ appear strictly to the left of the entries $n_\mathrm{l} + 1, \dots, n$	2.4
$\lambda_\mathrm{l}\#\lambda_\mathrm{r}$	the multipartition obtained by joining the left and right parts $\lambda_\mathrm{l}, \lambda_\mathrm{r}$ together	2.6
$\mathtt{T}_\mathrm{l}\#\mathtt{T}_\mathrm{r}$	the tableau obtained by joining the left and right parts $\mathtt{T}_\mathrm{l}, \mathtt{T}_\mathrm{r}$ together	2.6
$\mathscr{D}$	$\{v_\mathtt{T} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\} \cap e(i_\lambda)\mathrm{S}_\lambda = \{v_\mathtt{T} \mid i_\mathtt{T} = i_\lambda\}$	3.2
$\mathrm{Dom}(\lambda)$	the set of domino tableaux	3.2
$\Psi\overset{y}{\underset{x}{\downarrow}}$	$\Psi_y\Psi_{y-2}\dots\Psi_x$	3.4
$\Psi\overset{y}{\underset{x}{\uparrow}}$	$\Psi_x\Psi_{x+2}\dots\Psi_y$	3.4
$\mathtt{T}_{i,j}$	the tableau with dominoes $\{[2,3],[4,5],\dots,[b,b+1],[j-1,j]\} \setminus \{[i-1,i]\}$ in the leg	3.5
$D_2^{e,p}$	the submatrix of the $D^{e,p}(v)$ for $\mathcal{H} = \mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ which corresponds to two-part partitions	4
$\mathscr{H}_n(\mathcal{O})$	Hu and Mathas's $\mathcal{O}$-deformed cyclotomic KLR algebra	5.1
$\mathbb{L}_k$	the kth ladder of $[\lambda]$	5.2

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