SPECHT MODULES FOR QUIVER HECKE ALGEBRAS OF TYPE C

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Abstract. We construct and investigate Specht modules $\mathcal{S}^\lambda$ for cyclotomic quiver Hecke algebras in type $C_\ell^{(1)}$ and C_∞, which are labelled by multipartitions λ. It is shown that in type C_∞, the Specht module $\mathcal{S}^\lambda$ has a homogeneous basis indexed by standard tableaux of shape λ, which yields a graded character formula and good properties with the exact functors E_i^Λ and F_i^Λ. For type $C_\ell^{(1)}$, we propose a conjecture.

Introduction

Representations of Hecke algebras and the symmetric group have been studied for over a century and Specht modules play important roles in the representation theory. Nowadays, we realise that on the one hand the Hecke algebras generalise to *cyclotomic quiver Hecke algebras* (or *Khovanov–Lauda–Rouquier algebras*) in the direction of categorification of quantum groups $U_q(\mathfrak{g})$ [17, 18, 25], and on the other hand that the Hecke algebras are cellular algebras and Specht modules are their cell modules.

In the affine type $A_\ell^{(1)}$ case, cellular algebras and Specht modules for cyclotomic quiver Hecke algebras were studied via the isomorphism to cyclotomic Hecke algebras given in [6]. It was shown that the cyclotomic quiver Hecke algebras of affine type $A_\ell^{(1)}$ have graded cellular structure [12]. Graded Specht modules in type $A_\ell^{(1)}$ were constructed and studied using the combinatorics of multipartitions [8, 21]. But so far little is known about Specht modules for *cyclotomic* quiver Hecke algebras of *other types*. We remark that it was proved in [20] that quiver Hecke algebras of finite type are *graded affine cellular algebras*.

The first and second authors [3] studied cyclotomic quiver Hecke algebras $R^{\Lambda_0}(n)$ for the fundamental weight Λ_0 in type $C_\ell^{(1)}$, in which a graded dimension formula for $R^{\Lambda_0}(n)$ is given by using the C-type Fock space $\mathcal{F}$ [15, 19, 24]. This Fock space $\mathcal{F}$ is constructed by folding

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the usual A-type Fock space, so the dimension formula is described in terms of combinatorics of Young diagrams. In affine type $A_\ell^{(1)}$, graded Specht modules are deeply related to the A-type Fock space. It was shown in [7] that the graded decomposition numbers of graded Specht modules can be described in terms of combinatorics of Young diagrams via the Fock space which is the q-version of the first author's result [1]. One can expect that cyclotomic quiver Hecke algebras of type C and the Fock space $\mathcal{F}$ of type C exhibit similar properties to those of type A – thus it is worth considering Specht modules for cyclotomic quiver Hecke algebras of type C.

In this paper, we construct *Specht modules* $\mathcal{S}^\lambda$ for cyclotomic quiver Hecke algebras of affine type $C_\ell^{(1)}$ and type C_∞ which are labelled by multipartitions λ. This is inspired by [21]. Let A be the Cartan matrix of type $C_\ell^{(1)}$ or C_∞, and $U_q(\mathsf{A})$ the quantum group associated with A. We set $R(\beta)$ to be the quiver Hecke algebra associated with A and denote by E_i^Λ and F_i^Λ the functors categorifying Chevalley generators e_i and f_i of $U_q(\mathsf{A})$ on the highest weight irreducible module $V_q(\Lambda)$. Let $\mathscr{P}_n^l$ be the set of l-multipartitions of n with a multicharge $\kappa = (\kappa_1, \dots, \kappa_l) \in \mathbb{Z}^l$. For $\lambda \in \mathscr{P}_n^l$, we first construct a *permutation module* $\mathcal{M}_\kappa^\lambda$ which has a basis indexed by row-strict tableaux of λ. These permutation modules are built from more fundamental building blocks, namely they are convolution products of the one-dimensional $R(\beta)$-modules $\mathcal{L}(k;\ell)$ defined by (3.1). The modules $\mathcal{L}(k;\ell)$ take a role as the *segment modules*, which are given in [21], corresponding to *segments* in type A_∞ and $A_n^{(1)}$. We also define a module $\mathcal{M}_{\kappa,A}^\lambda$ for each Garnir node $A \in [\lambda]$ and construct homomorphisms between $\mathcal{M}_{\kappa,A}^\lambda$ and $\mathcal{M}_\kappa^\lambda$, which give an interpretation of Garnir elements in terms of quiver Hecke algebras. We then define a Specht module $\mathcal{S}^\lambda$ using the cokernel of homomorphisms between $\mathcal{M}_{\kappa,A}^\lambda$ and $\mathcal{M}_\kappa^\lambda$, see Definition 3.8. The Specht module $\mathcal{S}^\lambda$ is spanned by homogeneous elements indexed by standard tableaux of shape λ (Corollary 3.13). We prove in Corollary 3.21 that, in type C_∞, this spanning set of $\mathcal{S}^\lambda$ is in fact a basis. Thus we have a graded character formula for $\mathcal{S}^\lambda$ in terms of standard tableaux and a description of $[E_i^\Lambda \mathcal{S}^\lambda]$ in terms of $[\mathcal{S}^{\lambda \nearrow b}]$. Here, $\lambda \nearrow b$ is the Young diagram obtained from λ by deleting a *removable* node b. We remark that $\mathcal{S}^\lambda$ is not necessarily *simple*, even in the case of level 1 and type C_∞. We also investigate a connection between Specht modules $\mathcal{S}^\lambda$ and the Fock space $\mathcal{F}(\kappa)$ of type C, which provides a description of $[F_i^\Lambda \mathcal{S}^\lambda]$ in terms of $[\mathcal{S}^{\lambda \swarrow b}]$, where $\lambda \swarrow b$ is the Young diagram obtained from λ by adding an *addable* node b, see Corollary 3.23. Recently, the third author provided semisimplicity criteria for the cyclotomic quiver Hecke algebras of type C_∞ and $C_n^{(1)}$ using the Specht modules [26].

The paper is organised as follows. In Section 1, we review the combinatorics of tableaux and the Fock space of type C, and prove lemmas on Garnir nodes. In Section 2, we recall the notion of quiver Hecke algebras, and prove several lemmas on computations of products of ψ_i and convolution products of modules for proving our main theorem. In Section 3, we construct and investigate Specht modules $\mathcal{S}^\lambda$ and provide the main theorems with examples. Section 4 is devoted to proving Theorem 3.19. We may carry out the computation in a manner knot theorists do, but we have found an algebraic proof, which is easier to access for representation theorists. In Section 5, we propose a conjecture for type $C_\ell^{(1)}$.

1. Combinatorics of Tableaux

1.1. **Lie theory notation.**

Let $\ell \in \{2, 3, \dots\} \cup \{\infty\}$ and $I = \mathbb{Z}_{\geqslant 0}$ if $\ell = \infty$ or $I = \{0, 1, 2, \dots, \ell\}$ otherwise.

For $\ell = \infty$, the corresponding Cartan matrix $\mathsf{A} = (a_{ij})_{i,j \in I}$ of type C_∞ is given by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -2 & \text{if } (i,j) = (1,0), \\ -1 & \text{if } i = j \pm 1 \text{ and } (i,j) \neq (1,0), \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise, the affine Cartan matrix of type $C_\ell^{(1)}$ is given by

$$\mathsf{A} = (a_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

We adopt standard notation from [13] for the root datum; in particular we have simple roots $\{\alpha_i \mid i \in I\}$ and fundamental weights $\{\Lambda_i \mid i \in I\}$ in the *weight lattice* P, and simple coroots $\{\alpha_i^\vee \mid i \in I\}$ in the *dual weight lattice* $\mathsf{P}^\vee$. There is an invariant symmetric bilinear form $(-,-)$ on P satisfying $(\Lambda_i, \alpha_j) = d_j \delta_{ij}$ and $(\alpha_i, \alpha_j) = d_i a_{ij}$ where $d = (2, 1, 1, \dots)$ if $\ell = \infty$ or $d = (2, 1 \dots, 1, 2)$ if $\ell < \infty$. Let

$$\mathsf{P}^+ = \{\Lambda \in \mathsf{P} \mid \langle \alpha_i^\vee, \Lambda \rangle \geqslant 0 \text{ for all } i \in I\}$$

be the set of *dominant integral weights*, where $\langle \ , \ \rangle$ is the natural pairing. We denote by $\mathsf{Q} := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ the *root lattice* and $\mathsf{Q}^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geqslant 0}\alpha_i$ is the *positive cone* of the root lattice. Note that the *null root* in type $C_\ell^{(1)}$ is given by $\delta = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$.

1.2. **The symmetric group and multipartitions.**

Denote by $\mathfrak{S}_n$ the symmetric group on n letters, with Coxeter generators $s_1, \ldots, s_{n-1}$. For a permutation $w \in \mathfrak{S}_n$, a *reduced expression* for w is an expression $w = s_{i_1} \ldots s_{i_r}$ of minimal length; $r = \ell(w)$ is the *length* of w.

We denote by $\mathfrak{S}_{m+n}/\mathfrak{S}_m \times \mathfrak{S}_n$ the set of distinguished left coset representatives of $\mathfrak{S}_m \times \mathfrak{S}_n$ in $\mathfrak{S}_{m+n}$, i.e. $\ell(ws_i) = \ell(w) + 1$ for $w \in \mathfrak{S}_{m+n}/\mathfrak{S}_m \times \mathfrak{S}_n$ and $i \neq m$.

For $a, b \in \mathbb{Z}_{\geqslant 0}$ with $a + b \leqslant n$, we define $w[a, b] \in \mathfrak{S}_n$ by

$$w[a,b](x) = \begin{cases} x+b & \text{if } 1 \leqslant x \leqslant a, \\ x-a & \text{if } a < x \leqslant a+b, \\ x & \text{if } a+b < x \leqslant n. \end{cases}$$

In two-line notation, $w[a, b]$ is

$$\left(\begin{smallmatrix} 1 & 2 & \cdots & a & a+1 & a+2 & \cdots & a+b \\ b+1 & b+2 & \cdots & b+a & 1 & 2 & \cdots & b \end{smallmatrix}\right).$$

Throughout the paper, $w \in \mathfrak{S}_n$ permutes letters of a tableau, but permutes places of $\nu = (\nu_1, \ldots, \nu_n) \in I^n$ as $w\nu = (\nu_{w^{-1}(1)}, \ldots, \nu_{w^{-1}(n)})$. In particular,

$$w[a,b]\nu = (\nu_{a+1}, \ldots, \nu_{a+b}, \nu_1, \ldots, \nu_a, \nu_{a+b+1}, \ldots, \nu_n).$$

Lemma 1.1. *Let $a, b \geqslant 1$.*

(1) $w[a,b] = (s_b \ldots s_{a+b-1}) \ldots (s_1 \ldots s_a) = (s_b \ldots s_1) \ldots (s_{a+b-1} \ldots s_a)$.
(2) $s_{b+1} w[2,b] = w[2,b] s_1$.

Proof. It is easy to see (1). Using the braid relations, we have

$$\begin{aligned} s_{b+1} w[2,b] &= \underline{s_{b+1}}(\underline{s_b} s_{b-1} \ldots s_1)(\underline{s_{b+1}} s_b \ldots s_2) \\ &= (s_b s_{b+1}) \underline{s_b} (\underline{s_{b-1}} s_{b-2} \ldots s_1)(\underline{s_b} s_{b-1} \ldots s_2) \\ &= (s_b s_{b+1})(s_{b-1} s_b) \underline{s_{b-1}} (\underline{s_{b-2}} \ldots s_1)(\underline{s_{b-1}} \ldots s_2) \\ &\qquad \cdots\cdots\cdots\cdots \\ &= (s_b s_{b+1})(s_{b-1} s_b) \ldots (s_2 s_3) \underline{s_2} (\underline{s_1})(\underline{s_2}) \\ &= (s_b s_{b+1})(s_{b-1} s_b) \ldots (s_2 s_3)(s_1 s_2) s_1 = w[2,b] s_1, \end{aligned}$$

which complete the proof of (2). Here, the underlines indicate generators to which we can apply the braid relation. □

The following easy lemma will be useful to us later. Note that the equality $s_{b+1} w[2,b] = w[2,b] s_1$ in Lemma 1.1(2) is a special case of this, but the importance of Lemma 1.1(2) lies in the 'long-hand derivation' of this equality, which we will utilise later in Lemma 2.14.

Lemma 1.2. *Let $w \in \mathfrak{S}_n$ and $1 \leqslant i \leqslant n-1$. If $w(i+1) = w(i) + 1$, then $s_{w(i)} w = w s_i$.*

For a reduced expression $w = s_{i_1} \dots s_{i_r}$ and $k \in \mathbb{Z}_{\geqslant 0}$ with $i_j < n - k$ for $1 \leqslant j \leqslant r$, we set

$$\mathsf{sh}_k(w) = s_{i_1+k} \dots s_{i_r+k}. \tag{1.1}$$

Note that $\mathsf{sh}_k(w)$ does not depend on the choice of reduced expressions. For $a, b, c \in \mathbb{Z}_{\geqslant 0}$, we define the block permutation $\mathrm{S}_2(c, a, b)$ to be $\mathsf{sh}_c(w[a, b])$.

Definition 1.3. For $v, w \in \mathfrak{S}_n$, we write $v \succcurlyeq w$ if there is a reduced expression for v which has an expression for w as a subsequence. We write $v \succ w$ if $v \succcurlyeq w$ and $v \neq w$. This partial order is called the *Bruhat order*.

The *left order* (sometimes called the weak Bruhat order) is given by $v \geqslant_L w$ if there is a reduced expression for v which has a reduced expression for w as a suffix – that is, $v = s_{i_1} \dots s_{i_r} w$ for some $i_1, \dots, i_r$ with $r = \ell(v) - \ell(w)$.

We fix an integer $l \geqslant 1$ throughout, which we refer to as the *level*.

Definition 1.4. For $n \geqslant 0$, a partition of n is a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that the sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ is equal to n. If λ is a partition of n we write $\lambda \vdash n$. We write $\varnothing$ for the unique partition of 0. Note that we will in general omit trailing zeros for partitions.

An *l-multipartition* of n is an l-tuple of partitions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$ such that the total size $\sum_{i=1}^{l} |\lambda^{(i)}| = n$. We denote the set of l-multipartitions of n by $\mathscr{P}_n^l$ and set $\mathscr{P}^l := \cup_{n \geqslant 0} \mathscr{P}_n^l$.

Similarly, a *composition* is a sequence $\mu = (\mu_1, \mu_1, \dots)$ of non-negative integers, and an *l-multicomposition* is an l-tuple of compositions.

If λ and μ are l-multicompositions of n, we say that λ *dominates* μ, and write $\lambda \trianglerighteq \mu$ if

$$|\lambda^{(1)}| + \cdots + |\lambda^{(t-1)}| + \sum_{j=1}^{k} \lambda_j^{(t)} \geqslant |\mu^{(1)}| + \cdots + |\mu^{(t-1)}| + \sum_{j=1}^{k} \mu_j^{(t)}$$

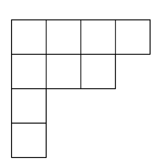
for all $1 \leqslant t \leqslant l$ and $k \geqslant 0$.

For any $\lambda \in \mathscr{P}_n^l$, we define its *Young diagram* $[\lambda]$ to be the set

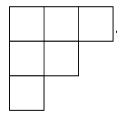
$$\{(r, c, t) \in \mathbb{Z}_{\geqslant 1} \times \mathbb{Z}_{\geqslant 1} \times \{1, \dots, l\} \mid c \leqslant \lambda_r^{(t)}\}.$$

We will depict a Young diagram for a partition using the English convention, and for a multipartition λ as a column vector of Young diagrams for the components $\lambda^{(1)}, \dots, \lambda^{(l)}$. If $l = 1$, then we write simply (r, c) for (r, c, t).

Example 1.5. Let $\lambda = ((4, 3, 1, 1), \varnothing, (3, 2, 1)) \in \mathscr{P}_{15}^3$. Then we write

$[\lambda] =$ 

$\varnothing$



With this convention, we say that for nodes $A = (r, c, t), A' = (r', c', t') \in [\lambda]$, A is *below* A' if $t > t'$ or if $t = t'$ with $r > r'$, and A is *above* A' if A' is below A.

We define $f_\ell : \mathbb{Z} \to I$ by $k \mapsto |k|$ if $\ell = \infty$ and, if $\ell \neq \infty$, $f_\ell : \mathbb{Z}/2\ell\mathbb{Z} \to I$ by

$$f_\ell(0 + 2\ell\mathbb{Z}) = 0, \quad f_\ell(\ell + 2\ell\mathbb{Z}) = \ell,$$
$$f_\ell(k + 2\ell\mathbb{Z}) = f_\ell(2\ell - k + 2\ell\mathbb{Z}) = k \quad \text{for } 1 \leqslant k \leqslant \ell - 1.$$

Let p be the natural projection $\mathbb{Z} \to \mathbb{Z}/2\ell\mathbb{Z}$ if $\ell \neq \infty$ and $p = \mathrm{id}$ if $\ell = \infty$. Then we define $\pi_\ell = f_\ell \circ p : \mathbb{Z} \to I$. We denote $\pi_\ell(k)$ by $\overline{k}$, for $k \in \mathbb{Z}$, if there is no confusion.

Now we fix a *multicharge* $\kappa = (\kappa_1, \dots, \kappa_l) \in \mathbb{Z}^l$ and define $\Lambda \in \mathsf{P}^+$ by $\Lambda = \sum_{i=1}^{l} \Lambda_{\overline{\kappa_i}}$. Let λ be an l-multipartition. Then, to any node $A = (r, c, t) \in [\lambda]$ we may associate its *residue* by

$$\mathrm{res}(A) = \overline{\kappa_t + c - r}.$$

If $\mathrm{res}(A) = i$, we call A an i-node. Thus, l-multipartitions may be coloured by I. We define the *content* of λ to be

$$\mathrm{cont}(\lambda) = \sum_{A \in [\lambda]} \alpha_{\mathrm{res}(A)} \in \mathsf{Q}^+.$$

Example 1.6. For $\lambda = ((4, 3, 1, 1), \varnothing, (3, 2, 1))$ as above, and $\kappa = (2, 0, -1)$, the residues of $[\lambda]$ are given as follows.

2	3	4	5
1	2	3	
0			
1			

$\varnothing$

1	0	1
2	1	
3		

We also have $\mathrm{cont}(\lambda) = 2\alpha_0 + 5\alpha_1 + 3\alpha_2 + 3\alpha_3 + \alpha_4 + \alpha_5$.

We say that a node A is *removable* (resp. *addable*) if $[\lambda] \setminus A$ (resp. $[\lambda] \cup A$) is a valid Young diagram for a multipartition of $n - 1$ (resp. $n + 1$). We write $\lambda \nearrow A$ (resp. $\lambda \swarrow A$) as shorthand for the multipartition whose Young diagram is $[\lambda] \setminus A$ (resp. $[\lambda] \cup A$). For an i-node $A \in [\lambda]$, we set

$$d_i(\lambda) = \#\{\text{addable } i\text{-nodes of } [\lambda]\} - \#\{\text{removable } i\text{-nodes of } [\lambda]\},$$

$$d_A(\lambda) = d_i(\#\{\text{addable } i\text{-nodes of } [\lambda] \text{ below } A\} - \#\{\text{removable } i\text{-nodes of } [\lambda] \text{ below } A\}),$$
$$d^A(\lambda) = d_i(\#\{\text{addable } i\text{-nodes of } [\lambda] \text{ above } A\} - \#\{\text{removable } i\text{-nodes of } [\lambda] \text{ above } A\}).$$

We define $\mathcal{F}(\kappa)$ to be a $\mathbb{Q}(q)$-vector space with basis consisting of the coloured l-multipartitions. Then $\mathcal{F}(\kappa)$ has a $U_q(\mathfrak{g}(\mathsf{A}))$-module structure defined by

$$q^{h_i}\lambda = q^{d_i(\lambda)}\lambda, \qquad e_i\lambda = \sum_A q^{d_A(\lambda)}\lambda \nearrow A, \qquad f_i\lambda = \sum_A q^{-d^A(\lambda)}\lambda \swarrow A, \tag{1.2}$$

where A runs over all removable i-nodes and all addable i-nodes respectively. The above description of $\mathcal{F}(\kappa)$ matches with that of the type A Fock space given in [7, Section 3.6], which is slightly different from [19, 24]. We call $\mathcal{F}(\kappa)$ the *level l Fock space with multicharge κ*. Note that the weight of a coloured l-multipartition λ is $\Lambda - \mathrm{cont}(\lambda)$, and there is a $U_q(\mathfrak{g}(\mathsf{A}))$-module isomorphism

$$\mathcal{F}(\kappa) \simeq \mathcal{F}(\kappa_1) \otimes \cdots \otimes \mathcal{F}(\kappa_l).$$

Here, the $U_q(\mathfrak{g}(\mathsf{A}))$-module structure of the tensor product comes from the comultiplication of $U_q(\mathfrak{g}(\mathsf{A}))$ given by, for $i \in I$,

$$\Delta : K_i \longmapsto K_i \otimes K_i, \quad e_i \longmapsto e_i \otimes K_i + 1 \otimes e_i, \quad f_i \longmapsto f_i \otimes 1 + K_i^{-1} \otimes f_i,$$

where $K_i = q^{\frac{(\alpha_i,\alpha_i)}{2}h_i}$. (cf. [7, Section 3.1])

Let $\Lambda = \sum_{t=1}^{l} \Lambda_{\overline{\kappa_t}}$ and $V_q(\Lambda)$ the irreducible highest weight $U_q(\mathfrak{g}(\mathsf{A}))$-module with highest weight Λ. As $\varnothing$ is a highest weight vector of $\mathcal{F}(\kappa)$ with highest weight Λ, we have a canonical $U_q(\mathfrak{g}(\mathsf{A}))$-module epimorphism

$$p_\kappa : \mathcal{F}(\kappa) \twoheadrightarrow V_q(\Lambda), \qquad \varnothing \longmapsto v_\Lambda, \tag{1.3}$$

where v_Λ is a highest weight vector of $V_q(\Lambda)$.

1.3. **Tableaux.**

We will mostly adopt the notation of [8, 21] for tableaux.

Let $\lambda \in \mathscr{P}^l_n$. A λ-tableau is a bijection $\mathtt{T} : [\lambda] \to \{1, \ldots, n\}$. We depict $\mathtt{T}$ by filling each node $(r, c, t) \in [\lambda]$ with $\mathtt{T}(r, c, t)$. We say that a tableau $\mathtt{T}$ is *row-strict* if the entries increase along the rows of each component of $\mathtt{T}$, and *column-strict* if the entries increase down the columns of each component of $\mathtt{T}$. If $\mathtt{T}$ is both row- and column-strict, we call it *standard*. We denote the set of standard tableaux by $\mathsf{Std}(\lambda)$, the set of row-strict tableaux by $\mathsf{RowStd}(\lambda)$ and the set of row-strict tableaux which are not standard by $\mathsf{Row}(\lambda) = \mathsf{RowStd}(\lambda) \setminus \mathsf{Std}(\lambda)$. Note that the symmetric group $\mathfrak{S}_n$ acts naturally on the left on the set of tableaux.

For each λ-tableau $\mathtt{T}$, we have the associated residue sequence

$$\mathrm{res}(\mathtt{T}) = (\mathrm{res}(\mathtt{T}^{-1}(1)),\ \mathrm{res}(\mathtt{T}^{-1}(2)), \ldots, \mathrm{res}(\mathtt{T}^{-1}(n))).$$

Let $\mathtt{T}^\lambda$ be the *initial tableau*, which is the distinguished tableau where we fill the nodes with $1, \dots, n$ first along successive rows in $\lambda^{(1)}$, then $\lambda^{(2)}$, and so on. Then for each λ-tableau, $\mathtt{T}$, we may define the permutation $w^{\mathtt{T}} \in \mathfrak{S}_n$ by $w^{\mathtt{T}}\mathtt{T}^\lambda = \mathtt{T}$ and the length $\ell(\mathtt{T}) \in \mathbb{Z}_{\geqslant 0}$ by $\ell(\mathtt{T}) = \ell(w^{\mathtt{T}})$.

Example 1.7. Continuing our previous example,

$$\mathtt{T}^\lambda = \begin{array}{l} \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6&7\\\cline{1-3} 8\\\cline{1-1} 9\\\cline{1-1}\end{array} \\[1ex] \varnothing \\[1ex] \begin{array}{|c|c|c|}\hline 10&11&12\\\hline 13&14\\\cline{1-2} 15\\\cline{1-1}\end{array}\end{array}.$$

We define the *dominance order* on λ-tableaux by setting $\mathtt{T} \trianglerighteq \mathtt{S}$ if and only if $w^{\mathtt{T}} \preccurlyeq w^{\mathtt{S}}$. The matchup of terminology and notation with the dominance order on partitions is justified by Lemma 1.8. Note in particular that $\mathtt{T}^\lambda \trianglerighteq \mathtt{T}$ for all λ-tableaux $\mathtt{T}$.

First, we introduce one more concept. Let $\mathtt{T}$ be a λ-tableau and $0 \leqslant m \leqslant n$. We denote by $\mathtt{T}_{\downarrow m}$ the set of nodes of $[\lambda]$ whose entries are less than or equal to m. If $\mathtt{T} \in \mathsf{Std}(\lambda)$, then $\mathtt{T}_{\downarrow m}$ is a tableau for some multipartition, which we call $\mathrm{Shp}(\mathtt{T}_{\downarrow m})$. If $\mathtt{T} \in \mathsf{Row}(\lambda)$, then $\mathtt{T}_{\downarrow m}$ is a tableau for some multicomposition, which we also call $\mathrm{Shp}(\mathtt{T}_{\downarrow m})$.

Lemma 1.8. [23, Theorem 3.8] *Suppose* $\lambda \in \mathscr{P}^l_n$ *and* $\mathtt{T}, \mathtt{S} \in \mathsf{RowStd}(\lambda)$. *Then* $\mathtt{T} \trianglerighteq \mathtt{S}$ *if and only if* $\mathrm{Shp}(\mathtt{T}_{\downarrow m}) \trianglerighteq \mathrm{Shp}(\mathtt{S}_{\downarrow m})$ *for all* $1 \leqslant m \leqslant n$.

For any $\lambda \in \mathscr{P}^l_n$ and $\mathtt{T} \in \mathsf{Std}(\lambda)$ we define the *degree* $\deg \mathtt{T}$ of $\mathtt{T}$ as follows. If $n = 0$ then $\mathtt{T}$ is the unique $\varnothing$-tableau and we set $\deg \mathtt{T} := 0$. Otherwise, let $A = \mathtt{T}^{-1}(n) \in [\lambda]$ and suppose A is an i-node. We set inductively

$$\deg \mathtt{T} := \deg \mathtt{T}_{\downarrow n-1} + d_A(\lambda). \tag{1.4}$$

Example 1.9. Let $\ell = \infty$, $\kappa = (2, -1)$ and $\lambda = ((2, 2, 1), (3, 2))$. Then the residue pattern of λ is

$$\begin{array}{l} \begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline 0\\\cline{1-1}\end{array} \\[1ex] \begin{array}{|c|c|c|}\hline 1&0&1\\\hline 2&1\\\cline{1-2}\end{array}\end{array}$$

and if $\mathtt{T}$ is the tableau

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 6 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 3 & 7 & 10 \\ \hline 8 & 9 \\ \cline{1-2}\end{array}$$

we have

$$\deg \mathtt{T} = 0+1+0+0+1+2+0+0+0-1 = 3.$$

The nodes contributing to the degree are those containing the entries 2 (a 1-node), 5 (a 2-node), 6 (a 0-node) and 10 (a 1-node).

1.4. **Garnir tableaux.**

Definition 1.10. Let $\lambda \in \mathscr{P}_n^l$ and $A = (r, c, t) \in [\lambda]$. We call A a *Garnir node* if $(r+1, c, t) \in [\lambda]$. For a Garnir node $A \in [\lambda]$, the *Garnir belt* $\mathbf{B}^A$ is the set of nodes

$$\{(r, a, t) \in [\lambda] \mid c \leqslant a \leqslant \lambda_r^{(t)}\} \cup \{(r+1, a, t) \in [\lambda] \mid 1 \leqslant a \leqslant c\}.$$

Finally, for a Garnir node $A \in [\lambda]$, the *Garnir tableau* $\mathtt{G}^A$ is the λ-tableau which agrees with the initial tableau $\mathtt{T}^\lambda$ outside of $\mathbf{B}^A$ and has the entries $u, u+1, \ldots, v$ from the bottom left to the top right of $\mathbf{B}^A$, where $u = \mathtt{T}^\lambda(r, c, t)$ and $v = \mathtt{T}^\lambda(r+1, c, t)$. Then

$$w^{\mathtt{G}^A} = S_2(a, \lambda_r^{(t)} - c + 1, c), \tag{1.5}$$

where $a = \sum_{i=1}^{t-1} |\lambda^{(i)}| + \sum_{j=1}^{r-1} \lambda_j^{(t)} + c - 1$. Note that $S_2(a, \lambda_r^{(t)} - c + 1, c)$ is 321-avoiding so that $w^{\mathtt{G}^A}$ is fully commutative. See [4, Lemma 2.1] for example.

Example 1.11. Let $\lambda = ((4,3,1,1), \varnothing, (3,2,1))$ and $A = (1,3,1)$. Then the Garnir tableau $\mathtt{G}^A$, with the Garnir belt $\mathbf{B}^A$ shaded, is as follows.

$$\mathtt{G}^A = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 7 \\ \hline 3 & 4 & 5 \\ \cline{1-3} 8 \\ \cline{1-1} 9 \\ \cline{1-1}\end{array}$$

$$\varnothing$$

$$\begin{array}{|c|c|c|}\hline 10 & 11 & 12 \\ \hline 13 & 14 \\ \cline{1-2} 15 \\ \cline{1-1}\end{array}$$

The following lemma is an easy generalisation of [23, Lemma 3.14] but we include a proof for the reader's convenience. This lemma and Lemma 1.13 will be used in the proof of Theorem 3.19 in Section 4.

Lemma 1.12. *Let $\lambda \in \mathscr{P}_n^l$ and $\mathtt{T} \in \mathsf{Row}(\lambda)$. If $A = (r,c,t) \in [\lambda]$ with $\mathtt{T}(r,c,t) > \mathtt{T}(r+1,c,t)$, then there is an element $w \in \mathfrak{S}_n$ such that $\mathtt{T} = w\mathtt{G}^A$ and $\ell(\mathtt{T}) = \ell(w) + \ell(\mathtt{G}^A)$. That is, $w^{\mathtt{T}} \geqslant_L w^{\mathtt{G}^A}$. Conversely, if $\mathtt{T} = w\mathtt{G}^A$ with $\ell(\mathtt{T}) = \ell(w) + \ell(\mathtt{G}^A)$ then $\mathtt{T} \in \mathsf{Row}(\lambda)$.*

Proof. Let $u := \mathtt{T}^\lambda(r,c,t)$, $v := \mathtt{T}^\lambda(r+1,c,t)$, $a := \mathtt{T}(r,c,t)$ and $b := \mathtt{T}(r+1,c,t)$. If $\mathtt{T} = \mathtt{G}^A$, the result is clear. So we suppose that $\mathtt{T} \neq \mathtt{G}^A$, and we will choose a basic transposition s_i such that $s_i\mathtt{T} \in \mathsf{Row}(\lambda)$ and $s_i\mathtt{T} \rhd \mathtt{T}$, from which the result follows by (reverse) induction on the dominance order $\rhd$.

If $\mathtt{T}$ coincides with $\mathtt{T}^\lambda$ outside of $\mathbf{B}^A$, there is a gap in the reading word of $\mathtt{T}$ in either the first or the second row of $\mathbf{B}^A$ – otherwise $u, u+1, \ldots, v$ are split into two sets of consecutive numbers and as $\mathtt{T}(r,c,t) > \mathtt{T}(r+1,c,t)$ the only way to fill in the numbers is $\mathtt{T} = \mathtt{G}^A$. Thus, we may choose $i+1$ in the first row and i in the second row for some i with $(i,i+1) \neq (b,a)$, so that $s_i\mathtt{T}(r,c,t) > s_i\mathtt{T}(r+1,c,t)$.

Otherwise, we may choose s_i so that $s_i\mathtt{T}(r,c,t) > s_i\mathtt{T}(r+1,c,t)$ as follows. First suppose that the reading word of $\mathtt{T}$ begins $1,2,\ldots,m,m'$ for some $m' > m+1$ and $m < u$. Then setting $i = m'-1$ suffices. Next suppose that the reading word of $\mathtt{T}$ ends $m', m, m+1, \ldots, n$ for some $v < m' < m-1$. Then, setting $i = m'$ suffices.

For the converse statement, we argue by induction on $\ell(\mathtt{T})$. Suppose $\mathtt{T} = w\mathtt{G}^A$ with $\ell(\mathtt{T}) = \ell(w) + \ell(\mathtt{G}^A)$ and $s_i\mathtt{T} \lhd \mathtt{T}$. Then if $s_i\mathtt{T}$ is standard, so is $\mathtt{T}$, by [9, Lemma 1.5]. But this contradicts the induction hypothesis. □

Lemma 1.13. *Let $\lambda \in \mathscr{P}_n^l$ and $\mathtt{T} \in \mathsf{Row}(\lambda)$.*

(1) If $\mathtt{T}(r,c,t) = \mathtt{T}(r+1,c,t)+1$, then there is an element $w \in \mathfrak{S}_n$ such that, for $A := (r,c,t) \in [\lambda]$,
   (i) $\mathtt{T} = w\mathtt{G}^A$,
   (ii) $s_p w = w s_q$, where $p = \mathtt{T}(r+1,c,t)$ and $q = \mathtt{G}^A(r+1,c,t)$.

(2) If $\mathtt{T}(r,c+1,t) = \mathtt{T}(r,c,t)+1$, then there is a Garnir node $A \in [\lambda]$ and $w \in \mathfrak{S}_n$ such that
   (i) $\mathtt{G}^A(r,c+1,t) = \mathtt{G}^A(r,c,t)+1$,
   (ii) $\mathtt{T} = w\mathtt{G}^A$,
   (iii) $s_p w = w s_q$, where $p = \mathtt{T}(r,c,t)$ and $q = \mathtt{G}^A(r,c,t)$.

Proof. (1) Part (i) follows from Lemma 1.12. For part (ii), note that $\mathtt{G}^A(r,c,t) = q+1$ by definition, so that

$$1 = \mathtt{T}(r,c,t) - \mathtt{T}(r+1,c,t) = w(\mathtt{G}^A(r,c,t)) - w(\mathtt{G}^A(r+1,c,t))$$

$= w(q+1) - w(q).$

It follows from Lemma 1.2 that $ws_q = s_{w(q)}w = s_p w$.

(2) We begin by choosing a node $A = (r', c', t') \in [\lambda]$ such that $\mathtt{T}(r', c', t') > \mathtt{T}(r'+1, c', t')$ and $\mathtt{G}^A(r, c+1, t) = \mathtt{G}^A(r, c, t) + 1$ as follows.

If $\mathtt{T}(r, c+1, t) > \mathtt{T}(r+1, c+1, t)$ then we know that $\mathtt{T}(r, c, t) > \mathtt{T}(r+1, c, t)$, so we may choose $A = (r, c, t)$.

Next, suppose $\mathtt{T}(r-1, c, t) > \mathtt{T}(r, c, t)$. Then we have $\mathtt{T}(r-1, c+1, t) > \mathtt{T}(r, c+1, t)$, and we may choose $A = (r-1, c+1, t)$.

Otherwise, if $\mathtt{T}(r, c+1, t) < \mathtt{T}(r+1, c+1, t)$ and $\mathtt{T}(r-1, c, t) < \mathtt{T}(r, c, t)$, then, as $\mathtt{T} \in \mathsf{Row}(\lambda)$, there is some node $A = (r', c', t')$ such that $\mathtt{T}(r', c', t') > \mathtt{T}(r'+1, c', t')$ and $(r', c', t') \neq (r-1, c, t), (r, c+1, t)$. Since $\mathtt{G}^A(x, y+1, z) = \mathtt{G}^A(x, y, z) + 1$ holds unless $(x, y, z) = (r', c'-1, t'), (r'+1, c', t')$, $(r, c, t) \neq (r', c'-1, t'), (r'+1, c', t')$ implies $\mathtt{G}^A(r, c+1, t) = \mathtt{G}^A(r, c, t) + 1$. Hence (i) is proved.

Now, by Lemma 1.12 and the fact that $\mathtt{T}(r', c', t') > \mathtt{T}(r'+1, c', t')$, there is some $w \in \mathfrak{S}_n$ such that $\mathtt{T} = w\mathtt{G}^A$ and $\ell(\mathtt{T}) = \ell(w) + \ell(\mathtt{G}^A)$. We have proved (ii). Moreover,

$$\begin{aligned} 1 = \mathtt{T}(r, c+1, t) - \mathtt{T}(r, c, t) &= w(\mathtt{G}^A(r, c+1, t)) - w(\mathtt{G}^A(r, c, t)) \\ &= w(q+1) - w(q). \end{aligned}$$

Thus, (iii) follows from Lemma 1.2. □

Lemma 1.14. *Let A and B be distinct Garnir nodes of $\lambda \in \mathscr{P}_n^l$. Then there is a unique tableau $\mathtt{G}^{A,B} \in \mathsf{Row}(\lambda)$ such that*

(1) $\mathtt{G}^{A,B} \geqslant_L \mathtt{G}^A$ and $\mathtt{G}^{A,B} \geqslant_L \mathtt{G}^B$,

(2) $\mathtt{T} \geqslant_L \mathtt{G}^{A,B}$ for any $\mathtt{T} \in \mathsf{RowStd}(\lambda)$ with $\mathtt{T} \geqslant_L \mathtt{G}^A$ and $\mathtt{T} \geqslant_L \mathtt{G}^B$.

Proof. It is known that $\mathsf{RowStd}(\lambda)$ is a lattice with respect to the left order. See for example [5, Theorem 7.1] (with some slight modification to generalise to $\mathsf{RowStd}(\lambda)$). Thus $\mathtt{G}^{A,B} = \mathtt{G}^A \vee \mathtt{G}^B$. □

We redefine $\mathtt{G}^{A,B}$ in Definition 1.15 below in a more concrete manner and show in Lemma 1.17 that it coincides with $\mathtt{G}^{A,B}$ in Lemma 1.14.

Definition 1.15. Suppose $A, B \in [\lambda]$ are distinct Garnir nodes. We define the sets $\mathbf{B}^A(2)$ and $\mathbf{B}^B(1)$ to be the second row of $\mathbf{B}^A$ and the first row of $\mathbf{B}^B$, respectively.

We define the *generalised Garnir belt* $\mathbf{B}^{A,B}$ of $[\lambda]$ to be the following set of nodes.

(1) If $\mathbf{B}^A \cap \mathbf{B}^B = \emptyset$, then $\mathbf{B}^{A,B} := \mathbf{B}^A \cup \mathbf{B}^B$.

(2) If $A = (r, c, t)$ and $B = (r, c', t)$ for some $c' > c$, then

$$\mathbf{B}^{A,B} := \{(r, a, t) \mid a \geqslant c\} \cup \{(r+1, a, t) \mid a \leqslant c'\}$$

$$= \mathbf{B}^A \cup \mathbf{B}^B$$
$$= (\mathbf{B}^A \setminus \mathbf{B}^B(1)) \cup (\mathbf{B}^B \setminus \mathbf{B}^A(2)).$$

In this case, we set $\mathbf{B}^{A,B}(1) = \mathbf{B}^A \setminus \mathbf{B}^B(1)$ and $\mathbf{B}^{A,B}(2) = \mathbf{B}^B \setminus \mathbf{B}^A(2)$.

(3) If $A = (r, c, t)$ and $B = (r-1, c', t)$ for some $c' \geqslant c$, then

$$\mathbf{B}^{A,B} := \{(r-1, a, t) \mid a \geqslant c'\} \cup \{(r, a, t) \mid c \leqslant a \leqslant c'\} \cup \{(r+1, a, t) \mid a \leqslant c\}.$$

Finally, we define the *generalised Garnir tableau* in the first two cases above to be the λ-tableau $\mathtt{G}^{A,B}$ which agrees with $\mathtt{T}^\lambda$ outside of $\mathbf{B}^{A,B}$ and has the entries of $\mathbf{B}^{A,B}$ as follows:

(1) If $\mathbf{B}^A \cap \mathbf{B}^B = \emptyset$, then we fill each of $\mathbf{B}^A$ and $\mathbf{B}^B$ as in $\mathtt{G}^A$ and $\mathtt{G}^B$, respectively.

(2) If $A = (r, c, t)$ and $B = (r, c', t)$ for some $c' \geqslant c$, then we first fill the entries of $\mathbf{B}^{A,B}(1)$, from bottom left to top right, and then we fill the entries of $\mathbf{B}^{A,B}(2)$, from bottom left to top right.

In the third case above, $\mathtt{G}^{A,B}$ is defined as follows.

(3) If $A = (r, c, t)$ and $B = (r-1, c', t)$ for some $c' \geqslant c$, $\mathtt{G}^{A,B}$ is defined to be the λ-tableau which agrees with $\mathtt{T}^\lambda$ outside of the three rows of $[\lambda]$ which contain elements of $\mathbf{B}^{A,B}$, and we fill the entries of these three rows first in order along rows above $\mathbf{B}^{A,B}$, then from bottom left to top right in $\mathbf{B}^{A,B}$, and finally in order along rows below $\mathbf{B}^{A,B}$.

Example 1.16. Let $\lambda = ((1), (10, 9, 6, 2))$ and $A = (2, 3, 2) \in [\lambda]$. Then we have the following tableaux $\mathtt{G}^{A,B}$ in cases corresponding to Definition 1.15, where we have shaded the generalised Garnir belts $\mathbf{B}^{A,B}$ in each case.

(1) Let $B = (1, 1, 2)$. Then

$\mathtt{G}^{A,B} =$

1

3	4	5	6	7	8	9	10	11	12
2	13	17	18	19	20	21	22	23	
14	15	16	24	25	26				
27	28								

(2) Let $B = (2, 6, 2)$. Then

$\mathtt{G}^{A,B} =$

1

2	3	4	5	6	7	8	9	10	11
12	13	17	18	19	23	24	25	26	
14	15	16	20	21	22				
27	28								

(3) Let $B = (1, 6, 2)$. Then

$$\mathtt{G}^{A,B} = \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 2 & 3 & 4 & 5 & 6 & 16 & 17 & 18 & 19 & 20 \\ \hline 7 & 8 & 12 & 13 & 14 & 15 & 21 & 22 & 23 \\ \cline{1-9} 9 & 10 & 11 & 24 & 25 & 26 \\ \cline{1-6} 27 & 28 \\ \cline{1-2}\end{array}$$

Lemma 1.17. *The construction of* $\mathtt{G}^{A,B}$ *in Definition 1.15 satisfies* $\mathtt{G}^{A,B} = \mathtt{G}^A \vee \mathtt{G}^B$ *and thus coincides with the tableau* $\mathtt{G}^{A,B}$ *defined in Lemma 1.14.*

Proof. It is easy to see that $\mathtt{G}^{A,B} \geqslant_L \mathtt{G}^A$ and $\mathtt{G}^{A,B} \geqslant_L \mathtt{G}^B$, so that we have $\mathtt{G}^{A,B} \geqslant_L \mathtt{G}^A \vee \mathtt{G}^B$. If the inequality were strict, then there exists a basic transposition s such that

$$\mathtt{G}^{A,B} >_L s\mathtt{G}^{A,B} \geqslant_L \mathtt{G}^A \vee \mathtt{G}^B.$$

However, the explicit construction of $\mathtt{G}^{A,B}$ shows that either $s\mathtt{G}^{A,B} \not\geqslant_L \mathtt{G}^A$ or $s\mathtt{G}^{A,B} \not\geqslant_L \mathtt{G}^B$ occurs for any basic transposition s with $\mathtt{G}^{A,B} >_L s\mathtt{G}^{A,B}$. Hence, we must have equality. □

Lemma 1.18. *Let* $A = (r, c, t)$ *and* $B = (r', c', t')$ *be Garnir nodes of* $\lambda \in \mathscr{P}^l_n$.

(1) If $\mathbf{B}^A \cap \mathbf{B}^B = \emptyset$ *then* $w^{\mathtt{G}^{A,B}}$ *is fully commutative.*
(2) If $r = r'$ *and* $t = t'$ *then* $w^{\mathtt{G}^{A,B}}$ *is fully commutative.*
(3) If $\mathtt{G}^{A,B} = w^A\mathtt{G}^A = w^B\mathtt{G}^B$, *then* w^A *and* w^B *are fully commutative.*

Proof. Take $w^A, w^B \in \mathsf{W}$ such that $\mathtt{G}^{A,B} = w^A\mathtt{G}^A = w^B\mathtt{G}^B$. We consider the three cases (1), (2) and (3) in Definition 1.15.

- In the first case, $\mathtt{G}^{A,B} = w^{\mathtt{G}^A}\mathtt{G}^B = w^{\mathtt{G}^B}\mathtt{G}^A$ and it is clear that each of $w^{\mathtt{G}^A}$ and $w^{\mathtt{G}^B}$ are of the form $S_2(c, a, b) = \mathsf{sh}_c w[a, b]$ for some a, b, c. Further, $w^{\mathtt{G}^{A,B}}$ has a unique descent pattern of 2143. Thus, $w^A = w^{\mathtt{G}^B}$, $w^B = w^{\mathtt{G}^A}$ and $w^{\mathtt{G}^{A,B}}$ are 321-avoiding. This implies that (1) holds, and (3) holds when $\mathbf{B}^A \cap \mathbf{B}^B = \emptyset$.
- In the second case, w^A is a shift of $w[\lambda^{(t)}_r - c' + 1, c' - c]$ and w^B is a shift of $w[c' - c, c]$. Thus w^A and w^B are 321-avoiding. Further, the two-line notation for $w^{\mathtt{G}^{A,B}}$ is

$$\begin{pmatrix} 1 & 2 & \cdots & c'-c & c'-c+1 & \cdots & \lambda^{(t)}_r & \lambda^{(t)}_r+1 & \cdots & \lambda^{(t)}_r+c & \lambda^{(t)}_r+c+1 & \cdots & \lambda^{(t)}_r+c' \\ c+1 & c+2 & \cdots & c' & 2c'-c+1 & \cdots & \lambda^{(t)}_r+c'-c+1 & 1 & \cdots & c & c'+1 & \cdots & 2c'-c \end{pmatrix}$$

  up to shift. Hence, $w^{\mathtt{G}^{A,B}}$ is 321-avoiding, which yields that (2) holds, and (3) holds when $r = r'$ and $t = t'$.
- In the third case, w^A and w^B are $w[\lambda^{(t)}_{r-1} - c' + 1, c + c']$ and $w[\lambda^{(t)}_{r-1} + \lambda^{(t)}_r - c - c' + 2, c]$ up to shift, respectively. Thus, they are also 321-avoiding, which completes the proof of (3).

□

2. Quiver Hecke algebras

2.1. **Affine and cyclotomic quiver Hecke algebras.**

In this subsection, A is an arbitrary symmetrisable Cartan matrix.

Let $\mathcal{O}$ be a unital commutative ring and we fix a system of polynomials $Q_{i,j}(u,v) \in \mathcal{O}[u,v]$ for $i,j \in I$ of the form

$$Q_{i,j}(u,v) = \begin{cases} \sum_{p(\alpha_i,\alpha_i)+q(\alpha_j,\alpha_j)+2(\alpha_i,\alpha_j)=0} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

where $t_{i,j;p,q} \in \mathcal{O}$ are such that $t_{i,j;-a_{ij},0} \in \mathcal{O}^\times$ and $Q_{i,j}(u,v) = Q_{j,i}(v,u)$.

For $\nu \in I^n$ and $\nu' \in I^{n'}$, we denote the concatenation of ν and ν' by $\nu * \nu' \in I^{n+n'}$. Here, we understand that $I^0 := \{\emptyset\}$ and $\emptyset * \nu = \nu * \emptyset = \nu$.

Definition 2.1. The *cyclotomic quiver Hecke algebra* $R^\Lambda(n)$ associated with polynomials $(Q_{i,j}(u,v))_{i,j\in I}$ and $\Lambda \in \mathsf{P}^+$ is the $\mathbb{Z}$-graded unital $\mathcal{O}$-algebra generated by

$$\{e(\nu) \mid \nu = (\nu_1, \dots, \nu_n) \in I^n\} \cup \{x_1, \dots, x_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}$$

subject to the following relations.

$$\begin{aligned}
e(\nu)e(\nu') &= \delta_{\nu,\nu'} e(\nu); \\
\sum_{\nu \in I^n} e(\nu) &= 1; \\
x_r e(\nu) &= e(\nu) x_r; \\
\psi_r e(\nu) &= e(s_r \nu) \psi_r; \\
x_r x_s &= x_s x_r; \\
\psi_r x_s &= x_s \psi_r && \text{if } s \neq r, r+1; \\
\psi_r \psi_s &= \psi_s \psi_r && \text{if } |r-s| > 1; \\
x_r \psi_r e(\nu) &= (\psi_r x_{r+1} - \delta_{\nu_r,\nu_{r+1}}) e(\nu); \\
x_{r+1} \psi_r e(\nu) &= (\psi_r x_r + \delta_{\nu_r,\nu_{r+1}}) e(\nu); \\
\psi_r^2 e(\nu) &= Q_{\nu_r,\nu_{r+1}}(x_r, x_{r+1}) e(\nu); \\
(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r) e(\nu) &= \begin{cases} \frac{Q_{\nu_r,\nu_{r+1}}(x_r,x_{r+1}) - Q_{\nu_r,\nu_{r+1}}(x_{r+2},x_{r+1})}{x_r - x_{r+2}} e(\nu) & \text{if } \nu_r = \nu_{r+2}, \\ 0 & \text{otherwise;} \end{cases}
\end{aligned}$$

for all admissible r, s, ν, ν', and $x_1^{\langle \alpha_{\nu_1}^\vee, \Lambda \rangle} e(\nu) = 0$ for $\nu \in I^n$.

The algebra $R^\Lambda(n)$ is given a $\mathbb{Z}$-grading by setting

$$\deg(e(\nu)) = 0, \qquad \deg(x_r e(\nu)) = (\alpha_{\nu_r}, \alpha_{\nu_r}), \qquad \deg(\psi_s e(\nu)) = -(\alpha_{\nu_s}, \alpha_{\nu_{s+1}})$$

for all admissible r, s and ν.

For $\beta \in \mathsf{Q}^+$ with $\mathrm{ht}(\beta) = n$, we set

$$I^\beta = \{\nu \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}.$$

Then $e(\beta) := \sum_{\nu \in I^\beta} e(\nu)$ is a central idempotent. We define $R^\Lambda(\beta) := R^\Lambda(n)e(\beta)$, which is also an $\mathcal{O}$-algebra. It is clear that $R^\Lambda(\beta)$ may be defined by the same set of relations if we replace I^n with I^β. We have the following decomposition of $R^\Lambda(n)$ into a direct sum of $\mathcal{O}$-algebras.

$$R^\Lambda(n) = \bigoplus_{\substack{\beta \in \mathsf{Q}^+ \\ \mathrm{ht}(\beta)=n}} R^\Lambda(\beta).$$

When we drop the relation $x_1^{\langle \alpha_{\nu_1}^\vee, \Lambda \rangle} e(\nu) = 0$ for $\nu \in I^\beta$, we obtain the *quiver Hecke algebra* $R(\beta)$.

For each element $w \in \mathfrak{S}_n$, we fix a preferred reduced expression $w = s_{i_1} \dots s_{i_t}$ and define

$$\psi_w := \psi_{i_1} \dots \psi_{i_t} \in R(\beta).$$

Note that ψ_w depends on the choice of reduced expressions of w unless w is fully commutative. The following comes from the defining relations.

Proposition 2.2 ([8, Proof of Proposition 2.5])**.** *For two reduced expressions $s_{i_1} \dots s_{i_t} = s_{j_1} \dots s_{j_t}$ for an element $w \in \mathfrak{S}_n$, $(\psi_{i_1} \dots \psi_{i_t} - \psi_{j_1} \dots \psi_{j_t})e(\nu)$ can be written as a linear combination of elements of the form $\psi_u f(x) e(\nu)$, where $u \prec w$ with $\ell(u) \leqslant \ell(w) - 3$, and $f(x)$ is a polynomial in the generators $x_1, \dots, x_n$.*

Theorem 2.3 ([17, 18, 25])**.** *Let $\beta \in \mathsf{Q}^+$ with $\mathrm{ht}(\beta) = n$. Then the set*

$$\{\psi_w x_1^{t_1} \dots x_n^{t_n} e(\nu) \mid w \in \mathfrak{S}_n, t_1, \dots, t_n \in \mathbb{Z}_{\geqslant 0}, \nu \in I^\beta\}$$

is an $\mathcal{O}$-basis of $R(\beta)$.

Proposition 2.4. *Suppose that $Q_{ij}(u,v)$ have integral coefficients. We denote the cyclotomic quiver Hecke algebra defined over $\mathbb{Z}$ by $R^\Lambda_\mathbb{Z}(n)$. Then $R^\Lambda_\mathbb{Z}(n)$ is free of finite rank over $\mathbb{Z}$. Further, $R^\Lambda(n) \simeq R^\Lambda_\mathbb{Z}(n) \otimes_\mathbb{Z} \mathcal{O}$ as $\mathcal{O}$-algebras.*

Proof. We prove by induction on n that $R^\Lambda(n)$ is a projective $\mathcal{O}$-module. It is clear that $R^\Lambda(0) = \mathcal{O}$ is a projective $\mathcal{O}$-module. Suppose that $R^\Lambda(n-1)$ is a projective $\mathcal{O}$-module. By [14, Thm 4.5] , $R^\Lambda(n)$ is a projective $R^\Lambda(n-1)$-module. Thus, the induction hypothesis implies that $R^\Lambda(n)$ is a projective $\mathcal{O}$-module. Applying the argument to $\mathcal{O} = \mathbb{Z}$ and noting that $\mathbb{Z}$ is a principal ideal domain, we deduce that $R^\Lambda_\mathbb{Z}(n)$ is a free $\mathbb{Z}$-module of finite rank.

As the defining relations of $R^\Lambda_\mathbb{Z}(n)$ hold in $R^\Lambda(n)$, the $\mathbb{Z}$-algebra homomorphism

$$R^\Lambda_\mathbb{Z}(n) \longrightarrow R^\Lambda(n)$$

given by mapping the generators $\psi_i, x_j, e(\nu)$ to the corresponding generators is well-defined. Hence we have a surjective $\mathcal{O}$-algebra homomorphism

$$R^\Lambda_\mathbb{Z}(n) \otimes_\mathbb{Z} \mathcal{O} \longrightarrow R^\Lambda(n).$$

On the other hand, as the defining relations of $R^\Lambda(n)$ hold in $R^\Lambda_\mathbb{Z}(n) \otimes_\mathbb{Z} \mathcal{O}$, we have a surjective $\mathcal{O}$-algebra homomorphism

$$R^\Lambda(n) \longrightarrow R^\Lambda_\mathbb{Z}(n) \otimes_\mathbb{Z} \mathcal{O}.$$

Thus, $R^\Lambda(n) \simeq R^\Lambda_\mathbb{Z}(n) \otimes_\mathbb{Z} \mathcal{O}$. □

Note that our choices (2.1) and (2.2) of $Q_{ij}(u,v)$ being integral coefficients allow us to define the cyclotomic Hecke algebra over $\mathbb{Z}$.

2.2. **The C_∞ case.** In this subsection, we carry out some computations in type C_∞. We choose the following system of polynomials $Q_{i,j}(u,v)$ as our preferred choice: if the Cartan matrix A is of type $C^{(1)}_\ell$ then, for $i < j$,

$$Q_{i,j}(u,v) = \begin{cases} u + v^2 & \text{if } (i,j) = (0,1), \\ u + v & \text{if } i \neq 0, j = i+1, j \neq \ell, \\ u^2 + v & \text{if } (i,j) = (\ell-1, \ell), \\ 1 & \text{otherwise,} \end{cases} \tag{2.1}$$

and if the Cartan matrix A is of type C_∞ then, for $i < j$,

$$Q_{i,j}(u,v) = \begin{cases} u + v^2 & \text{if } (i,j) = (0,1), \\ u + v & \text{if } i \neq 0, j = i+1, \\ 1 & \text{otherwise.} \end{cases} \tag{2.2}$$

Note that if we assume that $\mathcal{O}$ is a field and that any element of $\mathcal{O}$ has a square root, then other choices of the polynomials $Q_{i,j}(u,v)$ yield isomorphic algebras [2, Lemma 3.2]. Further we have the following graded dimension formulas. For $\nu \in I^n$, let

$$K_q(\lambda,\nu) := \sum_{\substack{\mathtt{T} \in \mathsf{Std}(\lambda) \\ \mathrm{res}(\mathtt{T}) = \nu}} q^{\deg(\mathtt{T})}, \qquad K_q(\lambda) := \sum_{\mathtt{T} \in \mathsf{Std}(\lambda)} q^{\deg(\mathtt{T})}.$$

Theorem 2.5. *For $\nu, \nu' \in I^\beta$, we have*

$$\operatorname{rank}_q e(\nu) R^\Lambda(\beta) e(\nu') = \sum_{\substack{\lambda \in \mathscr{P}^l_n \\ \mathrm{wt}(\lambda) = \Lambda - \beta}} K_q(\lambda, \nu) K_q(\lambda, \nu'),$$

$$\operatorname{rank}_q R^\Lambda(\beta) = \sum_{\substack{\lambda \in \mathscr{P}^l_n \\ \mathrm{wt}(\lambda) = \Lambda - \beta}} K_q(\lambda)^2,$$

$$\operatorname{rank}_q R^\Lambda(n) = \sum_{\lambda \in \mathscr{P}^l_n} K_q(\lambda)^2,$$

where $\operatorname{rank}_q M := \sum_{k \in \mathbb{Z}} \operatorname{rank}_{\mathcal{O}}(M_k) q^k$ *for a free graded $\mathcal{O}$-module* $M = \bigoplus_{k \in \mathbb{Z}} M_k$.

Proof. By virtue of Proposition 2.4, it suffices to prove the result when $\mathcal{O}$ is a field. The irreducible highest weight $U_q(\mathfrak{g}(\mathsf{A}))$-module with highest weight $\sum_{i=1}^l \Lambda_{\overline{\kappa_i}} \in \mathsf{P}^+$ is realised as the submodule $U_q(\mathfrak{g}(\mathsf{A}))\varnothing \subseteq \mathcal{F}(\kappa)$. Thus, the proof is entirely similar to [3]. The only difference is that we use the tensor product Fock space $\mathcal{F}(\kappa)$. □

Now we assume that the Cartan matrix A is of type C_∞ and prepare some technical results. We consider fully commutative elements $S_2(c, a, b) = \mathsf{sh}_c w[a, b]$. Then ψ_w for $w = S_2(c, a, b)$ does not depend on the choice of a preferred reduced expression. We denote it by $\Psi_2(c, a, b)$. If $c = 0$ we denote it by $\Psi[a, b]$ instead. We have

$$\deg(\Psi[a, b] e(\nu)) = -\left(\sum_{k=1}^{a} \alpha_{\nu_k}, \sum_{k=a+1}^{a+b} \alpha_{\nu_k} \right).$$

For $1 \leqslant x \leqslant y$, we define

$$\psi \uparrow_x^y = \psi_x \psi_{x+1} \dots \psi_y \quad \text{and} \quad \psi \downarrow_x^y = \psi_y \psi_{y-1} \dots \psi_x.$$

Then Lemma 1.1(1) implies

$$\Psi_2(c, a, b) = \psi \downarrow_{c+1}^{c+b} \dots \psi \downarrow_{c+a}^{c+a+b-1} = \psi \uparrow_{c+b}^{c+a+b-1} \dots \psi \uparrow_{c+1}^{c+a}.$$

In particular, ψ-generators that appear in $\Psi_2(c, a, b)$ are $\psi_{c+1}, \dots, \psi_{c+a+b-1}$. We also have the following formulae.

$$\begin{aligned} \Psi_2(c, a, b) &= \Psi_2(c, 1, b) \Psi_2(c+1, 1, b) \dots \Psi_2(c+a-1, 1, b) \\ &= \Psi_2(c+b-1, a, 1) \dots \Psi_2(c+1, a, 1) \Psi_2(c, a, 1). \end{aligned}$$

$$\begin{aligned} \Psi_2(c, a, b) &= \Psi_2(c, x, b) \Psi_2(c+x, a-x, b) \ \text{ for } 0 \leqslant x \leqslant a, \\ &= \Psi_2(c+y, a, b-y) \Psi_2(c, a, y) \ \text{ for } 0 \leqslant y \leqslant b. \end{aligned}$$

Remark 2.6. The algebra $R(\beta)$ admits an anti-involution which fixes the generators. Then it sends $\Psi_2(c,a,b)$ to $\Psi_2(c,b,a)$ because

$$\Psi_2(c,a,b) = \psi\downarrow_{c+1}^{c+b} \dots \psi\downarrow_{c+a}^{c+a+b-1} \mapsto \psi\uparrow_{c+a}^{c+a+b-1} \dots \psi\uparrow_{c+1}^{c+b} = \Psi_2(c,b,a).$$

Definition 2.7. For $a_1,\dots,a_t \in \mathbb{Z}_{\geqslant 0}$, we define a block transposition $S_i(a_1,\dots,a_t)$ by

$$S_i(a_1,\dots,a_t) = S_2(\sum_{k=1}^{i-1} a_k, a_i, a_{i+1}).$$

Then it is fully commutative and we may define $\Psi_i(a_1,\dots,a_t)$ by

$$\Psi_i(a_1,\dots,a_t) = \Psi_2(\sum_{k=1}^{i-1} a_k, a_i, a_{i+1}).$$

More generally, we define block permutations $S_{i_1}\dots S_{i_p}(a_1,\dots,a_t)$ by

$$S_{i_1}\dots S_{i_p}(a_1,\dots,a_t) = S_{i_1}\dots S_{i_{p-1}}(a_{s_{i_p}(1)},\dots,a_{s_{i_p}(t)})S_{i_p}(a_1,\dots,a_t),$$

and the corresponding $\Psi_{i_1}\dots\Psi_{i_p}(a_1,\dots,a_t)$ by

$$\Psi_{i_1}\dots\Psi_{i_p}(a_1,\dots,a_t) = \Psi_{i_1}\dots\Psi_{i_{p-1}}(a_{s_{i_p}(1)},\dots,a_{s_{i_p}(t)})\Psi_{i_p}(a_1,\dots,a_t).$$

Observing that $s_{i_1},\dots,s_{i_p}$ permute places, the following is clear.

Lemma 2.8. *Let $w = s_{i_1}\dots s_{i_p} \in \mathfrak{S}_t$ and $a_1,\dots,a_t \in \mathbb{Z}_{\geqslant 0}$. If we define*

$$\begin{aligned} A_{w^{-1}(1)} &= \{1,2,\dots,a_{w^{-1}(1)}\}, \\ A_{w^{-1}(2)} &= \{a_{w^{-1}(1)}+1,\dots,a_{w^{-1}(1)}+a_{w^{-1}(2)}\}, \\ &\vdots \\ A_{w^{-1}(t)} &= \{a_{w^{-1}(1)}+\dots+a_{w^{-1}(t-1)}+1,\dots,a_{w^{-1}(1)}+\dots+a_{w^{-1}(t)}\}, \end{aligned}$$

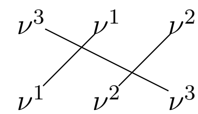
then the two-line notation of $S_{i_1}\dots S_{i_p}(a_1,\dots,a_t)$ is given as follows.

$$\begin{pmatrix} A_{w^{-1}(1)} & \cdots & A_{w^{-1}(t)} \\ A_1 & \cdots & A_t \end{pmatrix}$$

Corollary 2.9. *Suppose that each S_i is given by the reduced expressions in Lemma 1.1(1). Then $S_{i_1}\dots S_{i_p}(a_1,\dots,a_t)$ is a reduced expression if and only if $S_{i_1}\dots S_{i_p}(1,\dots,1)$ is.*

The two-line notation may be used to represent $\Psi_{i_1}\dots\Psi_{i_p}(a_1,\dots,a_t)e(\nu)$ by diagrams.

Example 2.10. Let $\nu^1 \in I^a$, $\nu^2 \in I^b$, $\nu^3 \in I^c$, for $a, b, c \geqslant 1$, and $\nu = \nu^1 * \nu^2 * \nu^3$. Then $\Psi[a+b, c]e(\nu)$ is represented by



and it follows that $\Psi[a+b, c]e(\nu) = \Psi_1\Psi_2(a, b, c)e(\nu)$.

Corollary 2.11. *Let $\underline{a} = (a_1, \dots, a_t)$. If $j \neq i \pm 1$ then $\Psi_i\Psi_j(\underline{a}) = \Psi_j\Psi_i(\underline{a})$.*

Lemma 2.12. *Suppose that the Cartan matrix A is of type C_∞. Let $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in I^n$ and $a, b \in \mathbb{Z}_{>0}$ with $a < n$ and $a + b \leqslant n$.*

(1) If $|\nu_i - \nu_{a+1}| \geqslant 2$, for $1 \leqslant i \leqslant a$, then $\Psi[1, a]\Psi[a, 1]e(\nu) = e(\nu)$.
(2) If $|\nu_1 - \nu_i| \geqslant 2$, for $2 \leqslant i \leqslant a+1$, then $\Psi[a, 1]\Psi[1, a]e(\nu) = e(\nu)$.
(3) If $\nu_1 = \nu_{a+1} \neq \nu_2, \dots, \nu_a$, then $x_{a+1}\Psi[1, a]e(\nu) = (\Psi[1, a]x_1 + \Psi[1, a-1])e(\nu)$.
(4) If $\nu_1 = \nu_{a+1} \neq \nu_2, \dots, \nu_a$, then $x_1\Psi[a, 1]e(\nu) = (\Psi[a, 1]x_{a+1} - \Psi_2(1, a-1, 1))e(\nu)$.
(5) If $|\nu_i - \nu_j| \geqslant 2$, for $1 \leqslant i \leqslant a$, $a+1 \leqslant j \leqslant a+b$, then $\Psi[b, a]\Psi[a, b]e(\nu) = e(\nu)$.

Proof. (1) As $\Psi[1, a] = \psi_a \dots \psi_1$ and $\Psi[a, 1] = \psi_1 \dots \psi_a$, we have

$$\begin{aligned}\Psi[1, a]\Psi[a, 1]e(\nu) &= (\psi_a \dots \psi_2)\psi_1^2 e(\mu^1)(\psi_2 \dots \psi_a) \\ &= (\psi_a \dots \psi_2)(\psi_2 \dots \psi_a)e(\nu),\end{aligned}$$

where $\mu^1 = (\nu_1, \nu_{a+1}, \nu_2, \dots, \nu_a, \nu_{a+2}, \dots, \nu_n)$, then

$$\begin{aligned}\Psi[1, a]\Psi[a, 1]e(\nu) &= (\psi_a \dots \psi_3)\psi_2^2 e(\mu^2)(\psi_3 \dots \psi_a) \\ &= (\psi_a \dots \psi_3)(\psi_3 \dots \psi_a)e(\nu),\end{aligned}$$

where $\mu^2 = (\nu_1, \nu_2, \nu_{a+1}, \nu_3, \dots, \nu_a, \nu_{a+2}, \dots, \nu_n)$, and so on. We continue the computation in this way and we reach $\Psi[1, a]\Psi[a, 1]e(\nu) = e(\nu)$.
(2) The proof is similar to (1) and left to the reader.
(3) By the assumption, we have

$$\begin{aligned}x_{a+1}\Psi[1, a]e(\nu) = x_{a+1}\psi_a \dots \psi_1 e(\nu) &= (\psi_a x_a \psi_{a-1} \dots \psi_2\psi_1 + \psi_{a-1} \dots \psi_1)e(\nu) \\ &= (\Psi[1, a]x_1 + \Psi[1, a-1])e(\nu).\end{aligned}$$

(4) By a similar computation to (3), we have

$$\begin{aligned}x_1\Psi[a, 1]e(\nu) = x_1\psi_1 \dots \psi_a e(\nu) &= (\psi_1 x_2 \psi_2 \dots \psi_a - \psi_2 \dots \psi_a)e(\nu) \\ &= (\Psi[a, 1]x_{a+1} - \Psi_2(1, a-1, 1))e(\nu).\end{aligned}$$

(5) We recall the formulas

$$\Psi[b, a] = \Psi_2(a-1, b, 1) \dots \Psi_2(1, b, 1)\Psi[b, 1],$$

$$\Psi[a,b] = \Psi[1,b]\Psi_2(1,1,b)\dots\Psi_2(a-1,1,b).$$

Then repeated use of (2) proves the result. □

Lemma 2.13. *Suppose that the Cartan matrix* A *is of type* C_∞. *Then*

$$\psi_{r+1}\psi_r\psi_{r+1}e(\nu) = \psi_r\psi_{r+1}\psi_r e(\nu) + \epsilon(r,\nu)e(\nu)$$

where $\epsilon(r,\nu)$ *is given as follows.*

$$\epsilon(r,\nu) = \begin{cases} x_r + x_{r+2} & \text{if } (\nu_r,\nu_{r+1},\nu_{r+2}) = (1,0,1), \\ 1 & \text{if } (\nu_r,\nu_{r+1},\nu_{r+2}) = (i,i\pm 1,i) \text{ and } \nu_{r+1}\neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.14. *Let* $\nu = (\mu_1\mu_2\nu_1\dots\nu_b) \in I^{b+2}$ *for* $b \geqslant 1$. *Then*

$$\psi_{b+1}\Psi[2,b]e(\nu) = \Psi[2,b]\psi_1 e(\nu) + \sum_{k=1}^{b}\Psi_2(k,2,b-k)c_k\Psi[2,k-1]e(\nu),$$

where c_k *is given by*

$$c_k = \begin{cases} x_k + x_{k+2} & \text{if } (\mu_1,\mu_2,\nu_k) = (1,0,1), \\ 1 & \text{if } (\mu_1,\mu_2,\nu_k) = (i,i\pm 1,i) \text{ and } \mu_2\neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We follow Lemma 1.1(2). Then,

$$\begin{aligned} \psi_{b+1}\Psi[2,b]e(\nu) &= \underline{\psi_{b+1}}(\underline{\psi_b}\psi_{b-1}\dots\psi_1)(\underline{\psi_{b+1}}\psi_b\dots\psi_2)e(\nu) \\ &= \psi_{b+1}\psi_b\psi_{b+1}e(\nu_1\dots\nu_{b-1}\mu_1\mu_2\nu_b)(\psi_{b-1}\dots\psi_1)(\psi_b\dots\psi_2) \\ &= (\psi_b\psi_{b+1})\underline{\psi_b}(\underline{\psi_{b-1}}\psi_{b-2}\dots\psi_1)(\underline{\psi_b}\psi_{b-1}\dots\psi_2)e(\nu) \\ &\qquad + c_b(\psi_{b-1}\psi_{b-2}\dots\psi_1)(\psi_b\psi_{b-1}\dots\psi_2)e(\nu) \\ &= \quad \dots\dots\dots\dots \\ &= \Psi[2,b]\psi_1 e(\nu) + \sum_{k=1}^{b}\Psi_2(k,2,b-k)c_k\Psi[2,k-1]e(\nu), \end{aligned}$$

where the error terms are computed by using Lemma 2.13. Here, the underlines indicate generators to which we can apply the braid-type relation. □

Lemma 2.15. *Let* $a,b \geqslant 1$ *and* $1 \leqslant i \leqslant a-1$. *Then*

$$\psi_{i+b}\Psi[a,b]e(\nu) - \Psi[a,b]\psi_i e(\nu)$$

$$= \sum_{k=1}^{b} \Psi[i-1,b]\Psi_2(i-1+k,2,b-k)c_k\Psi_2(i-1,2,k-1)\Psi_2(i+1,a-i-1,b)e(\nu)$$

$$= \sum_{k=1}^{b} \Psi_2\Psi_1\Psi_3(i-1,k,2,b-k)c_k\Psi_2\Psi_4\Psi_3(i-1,2,a-i-1,k-1,b-k+1)e(\nu),$$

where $c_k = x_{k+i-1}+x_{k+i+1}$ *if* $(\nu_i,\nu_{i+1},\nu_{a+k}) = (1,0,1)$, $c_k = 1$ *if* $(\nu_i,\nu_{i+1},\nu_{a+k}) = (j,j\pm 1,j)$ *for some* $j \geqslant 0$ *such that* $\nu_{i+1} \neq 0$, *and* $c_k = 0$ *otherwise.*

Proof. As the ψ-generators that appear in $\Psi_2(c,a,b)$ are $\psi_{c+1},\dots,\psi_{c+a+b-1}$, we have

$$\begin{aligned}\psi_{i+b}\Psi[a,b]e(\nu) &= \Psi[i-1,b]\psi_{i+b}\Psi_2(i-1,2,b)\Psi_2(i+1,a-i-1,b)e(\nu)\\ &= \Psi[i-1,b]\psi_{i+b}\Psi_2(i-1,2,b)e(\mu)\Psi_2(i+1,a-i-1,b)\end{aligned}$$

where $\mu = (\nu_1\dots\nu_{i+1}\nu_{a+1}\dots\nu_{a+b}\nu_{i+2}\dots\nu_a\nu_{a+b+1}\dots\nu_n)$. We apply Lemma 2.14 to substitute

$$\begin{aligned}\psi_{i+b}\Psi_2(i-1,2,b)e(\mu) &= \Psi_2(i-1,2,b)\psi_i e(\mu)\\ &\quad + \sum_{k=1}^{b} \Psi_2(i-1+k,2,b-k)c_k\Psi_2(i-1,2,k-1)e(\mu).\end{aligned}$$

Then we have the desired formula. ∎

We may also compute $(\psi_{b+i_1}\dots\psi_{b+i_r}\Psi[a,b]-\Psi[a,b]\psi_{i_1}\dots\psi_{i_r})e(\nu)$ by applying Lemma 2.15 to $(\psi_{b+i_k}\Psi[a,b]-\Psi[a,b]\psi_{i_k})e(s_{i_{k+1}}\dots s_{i_r}\nu)$ in the expression

$$\sum_{k=1}^{r} \psi_{b+i_1}\dots\psi_{b+i_{k-1}}(\psi_{b+i_k}\Psi[a,b]-\Psi[a,b]\psi_{i_k})e(s_{i_{k+1}}\dots s_{i_r}\nu)\psi_{i_{k+1}}\dots\psi_{i_r}.$$

In particular, we obtain the following.

Lemma 2.16. *Let* $a,b,c \geqslant 1$ *and* $1 \leqslant m \leqslant b$. *Then,*

$$\begin{aligned}&\Psi_2(c+m-1,a,1)\Psi[a+b,c]e(\nu) - \Psi[a+b,c]\Psi_2(m-1,a,1)e(\nu)\\ &= \sum_{s=1}^{a}\sum_{t=1}^{c} \Psi_2(c+m-1,s-1,1)\Psi[m+s-2,c]\Psi_2(m+s+t-2,2,c-t)\\ &\quad \times c_{st}\Psi_2(m+s-2,2,t-1)\Psi_2(m+s,a+b-m-s,c)\Psi_2(m+s-1,a-s,1)e(\nu),\end{aligned}$$

where $c_{st} = x_{m+s+t-2}+x_{m+s+t}$ *if* $(\nu_{m+s-1},\nu_{m+a},\nu_{a+b+t}) = (1,0,1)$, $c_{st} = 1$ *if* $(\nu_{m+s-1},\nu_{m+a},\nu_{a+b+t}) = (j,j\pm 1,j)$ *for some* j *such that* $\nu_{m+a} \neq 0$, $c_{st} = 0$ *otherwise.*

Proof. As $\Psi_2(c+m-1,a,1) = \psi_{c+m}\dots\psi_{c+m+a-1}$, the left-hand side is

$$\sum_{s=1}^{a} \psi \uparrow_{c+m}^{c+m+s-2} \big(\psi_{c+m+s-1}\Psi[a+b,c] - \Psi[a+b,c]\psi_{m+s-1}\big) e(\mu^s)\psi \uparrow_{m+s}^{m+a-1},$$

where $\mu^s = s_{m+s}\dots s_{m+a-1}\nu = (\nu_1\dots\nu_{m+s-1}\nu_{m+a}\nu_{m+s}\dots\widehat{\nu_{m+a}}\dots\nu_n)$. Thus, we apply Lemma 2.15 with $i = m+s-1$. □

Remark 2.17. We have prepared Lemma 2.16 for computing

$$\begin{aligned}
&\Psi_2\Psi_1\Psi_2(a,b,c)e(\nu) - \Psi_1\Psi_2\Psi_1(a,b,c)e(\nu) \\
&\qquad = \Psi_2(c,a,b)\Psi_1(a,c,b)\Psi_2(a,b,c)e(\nu) - \Psi_1(b,c,a)\Psi_2(b,a,c)\Psi_1(a,b,c)e(\nu)
\end{aligned}$$

in later sections. If $a = 0$ or $b = 0$ or $c = 0$ then it is zero. Thus we assume $a, b, c \geqslant 1$. First we observe that

$$\begin{aligned}
\Psi_1(a,c,b)\Psi_2(a,b,c) &= \Psi_2(0,a,c)\Psi_2(0+a,(a+b)-a,c) = \Psi[a+b,c], \\
\Psi_1(b,c,a)\Psi_2(b,a,c) &= \Psi_2(0,b,c)\Psi_2(0+b,(a+b)-b,c) = \Psi[a+b,c].
\end{aligned}$$

Hence we compute $\Psi_2(c,a,b)\Psi[a+b,c]e(\nu) - \Psi[a+b,c]\Psi_1(a,b,c)e(\nu)$, which is equal to

$$\sum_{m=1}^{b} \Psi_2(c+m,a,b-m)\big(\Psi_2(c+m-1,a,1)\Psi[a+b,c] - \Psi[a+b,c]\Psi_2(m-1,a,1)\big)\Psi[a,m-1]e(\nu)$$

since $\Psi_2(c,a,b) = \Psi_2(c+b-1,a,1)\dots\Psi_2(c,a,1)$. Then

$$\Psi[a,m-1]e(\nu) = e(w[a,m-1]\nu)\Psi[a,m-1],$$

where $w[a,m-1]\nu = (\nu_{a+1}\dots\nu_{a+m-1}\nu_1\dots\nu_a\nu_{a+m}\dots\nu_n)$. To compute

$$(\Psi_2(c+m-1,a,1)\Psi[a+b,c] - \Psi[a+b,c]\Psi_2(m-1,a,1))e(w[a,m-1]\nu)$$

using Lemma 2.16, we check whether $(\nu_s, \nu_{a+m}, \nu_{a+b+t})$ is $(1,0,1)$ or $(j,j+1,j)$ for $j \geqslant 0$, or $(j,j-1,j)$ for $j \geqslant 2$, for any $1 \leqslant s \leqslant a$ and $1 \leqslant t \leqslant c$.

2.3. **Module categories.**

In the subsequent Subsections 2.3 to 2.5, we keep the assumption that A is an arbitrary symmetrisable Cartan matrix but assume that $\mathcal{O}$ is a field.

We denote by $R(\beta)$-proj and $R(\beta)$-gmod the full subcategories in the category $R(\beta)$-Mod of graded $R(\beta)$-modules which consist of finitely generated projective graded $R(\beta)$-modules or finite dimensional graded $R(\beta)$-modules, respectively. We set

$$R\text{-proj} := \bigoplus_{\beta\in\mathsf{Q}^+} R(\beta)\text{-proj} \quad \text{and} \quad R\text{-gmod} := \bigoplus_{\beta\in\mathsf{Q}^+} R(\beta)\text{-gmod}.$$

Similarly, $R^\Lambda(\beta)$-proj and $R^\Lambda(\beta)$-gmod are the full subcategories in the category $R^\Lambda(\beta)$-Mod of graded $R^\Lambda(\beta)$-modules which consist of finitely generated projective graded $R^\Lambda(\beta)$-modules or finite dimensional graded $R^\Lambda(\beta)$-modules, respectively. We set

$$R^\Lambda\text{-proj} := \bigoplus_{\beta \in \mathsf{Q}^+} R^\Lambda(\beta)\text{-proj} \quad \text{and} \quad R^\Lambda\text{-gmod} := \bigoplus_{\beta \in \mathsf{Q}^+} R^\Lambda(\beta)\text{-gmod}.$$

Let us denote by q the *grading shift functor*, i.e. $(qM)_k = M_{k-1}$ for a graded module $M = \bigoplus_{k \in \mathbb{Z}} M_k$. For $M \in R(\beta)$-gmod, the *q-character* of M is defined by

$$\mathrm{ch}_q(M) := \sum_{\nu \in I^\beta} \dim_q(e(\nu)M)\nu,$$

where $\dim_q V := \sum_{k \in \mathbb{Z}} \dim(V_k) q^k$ for a graded vector space $V = \bigoplus_{k \in \mathbb{Z}} V_k$.

For graded $R(\beta)$-modules M and N, we denote by $\mathrm{Hom}_{R(\beta)}(M, N)$ the space of degree preserving module homomorphisms. If $f \in \mathrm{Hom}_{R(\beta)}(q^k M, N)$, we set $\deg(f) := k$. Then we define the following graded vector space:

$$\mathrm{HOM}_{R(\beta)}(M, N) := \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_{R(\beta)}(q^k M, N).$$

We write $\mathrm{Hom}(M, N)$ and $\mathrm{HOM}(M, N)$ if there is no confusion. For $\beta, \beta' \in \mathsf{Q}^+$, we set

$$e(\beta, \beta') := \sum_{\nu \in I^\beta, \nu' \in I^{\beta'}} e(\nu * \nu').$$

Definition 2.18. Let M be a graded $R(\beta)$-module, N a graded $R(\beta')$-module. Then the *convolution product* $M \circ N$ is the graded $R(\beta + \beta')$-module defined by

$$M \circ N := R(\beta + \beta')e(\beta, \beta') \otimes_{R(\beta) \otimes R(\beta')} (M \otimes N).$$

Let $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$. We denote by $[R\text{-proj}]$ and $[R\text{-gmod}]$ the Grothendieck groups of R-proj and R-gmod respectively. The convolution product makes $[R\text{-proj}]$ and $[R\text{-gmod}]$ into $\mathbb{A}$-algebras, and we have the following theorem.

Theorem 2.19 ([17, 18, 25])**.** *There exist isomorphisms of $\mathbb{A}$-algebras*

$$[R\text{-proj}] \simeq U_{\mathbb{A}}^-(\mathfrak{g}) \quad \textit{and} \quad [R\text{-gmod}] \simeq U_{\mathbb{A}}^-(\mathfrak{g})^\vee.$$

Now we explain the cyclotomic categorification theorem proved by Kang and Kashiwara. For this, we introduce the induction and restriction functors F_i^Λ and E_i^Λ, for $i \in I$, as follows.

- The induction functors $F_i^\Lambda : R^\Lambda(\beta)\text{-Mod} \to R^\Lambda(\beta + \alpha_i)\text{-Mod}$ are defined by

$$F_i^\Lambda = R^\Lambda(\beta + \alpha_i)e(\beta, \alpha_i) \otimes_{R^\Lambda(\beta)} -.$$

- The restriction functors $E_i^\Lambda : R^\Lambda(\beta)\text{-Mod} \to R^\Lambda(\beta - \alpha_i)\text{-Mod}$ are defined by

$$E_i^\Lambda = e(\beta - \alpha_i, \alpha_i)R^\Lambda(\beta) \otimes_{R^\Lambda(\beta)} -.$$

The following theorem is proved by showing that F_i^Λ and E_i^Λ are biadjoint functors. The action of the Chevalley generators on the left-hand side of each of the isomorphisms in the theorem is given by the linear operators induced by the functors: for $\beta \in \mathsf{Q}^+$,

$$\begin{array}{c} R^\Lambda(\beta)\text{-proj} \underset{q^{(1-\langle \alpha_i^\vee, \Lambda-\beta\rangle)(\alpha_i,\alpha_i)/2} E_i^\Lambda}{\overset{F_i^\Lambda}{\rightleftarrows}} R^\Lambda(\beta+\alpha_i)\text{-proj} \\ R^\Lambda(\beta)\text{-gmod} \underset{E_i^\Lambda}{\overset{q^{(1-\langle \alpha_i^\vee, \Lambda-\beta\rangle)(\alpha_i,\alpha_i)/2} F_i^\Lambda}{\rightleftarrows}} R^\Lambda(\beta+\alpha_i)\text{-gmod} \end{array} \tag{2.3}$$

Theorem 2.20 ([14, Theorem 6.2])**.** *For $\Lambda \in \mathsf{P}^+$, there exist $U_{\mathbb{A}}(\mathfrak{g})$-module isomorphisms*

$$[R^\Lambda\text{-proj}] \simeq V_{\mathbb{A}}(\Lambda), \quad [R^\Lambda\text{-gmod}] \simeq V_{\mathbb{A}}(\Lambda)^\vee.$$

2.4. **Convolution product for cyclotomic quiver Hecke algebras.**

The aim of this subsection is to prove the following.

Proposition 2.21. *If $M \in R^\Lambda(\beta)$-gmod for $\Lambda \in \mathsf{P}^+$ and $N \in R^{\Lambda'}(\beta')$-gmod for $\Lambda' \in \mathsf{P}^+$, then $M \circ N \in R^{\Lambda+\Lambda'}(\beta+\beta')$-gmod.*

Proof. Let $m = \mathrm{ht}(\beta)$ and $n = \mathrm{ht}(\beta')$. We may assume that M and N are non-zero modules and $m, n > 0$, so that we may take non-zero elements $a \in e(\nu_1 \dots \nu_m)M$ and $b \in e(\nu_{m+1} \dots \nu_{m+n})N$, for some $\nu = (\nu_1, \nu_2, \dots, \nu_{m+n}) \in I^{m+n}$. As $e(\nu_1 \dots \nu_m) \neq 0$ and $e(\nu_{m+1} \dots \nu_{m+n}) \neq 0$, the defining relations

$$x_1^{\langle \alpha_{\nu_1}^\vee,\, \Lambda\rangle} e(\nu_1 \dots \nu_m) = 0 \quad \text{and} \quad x_1^{\langle \alpha_{\nu_{m+1}}^\vee,\, \Lambda'\rangle} e(\nu_{m+1} \dots \nu_{m+n}) = 0$$

imply $\langle \alpha_{\nu_1}^\vee, \Lambda\rangle > 0$ and $\langle \alpha_{\nu_{m+1}}^\vee, \Lambda'\rangle > 0$. We take $w \in \mathfrak{S}_{m+n}/\mathfrak{S}_m \times \mathfrak{S}_n$. Note that

$$w(1) < \cdots < w(m) \quad \text{and} \quad w(m+1) < \cdots < w(m+n),$$

so that w is fully commutative and

$$\begin{aligned} &w^{-1}(1) = 1 \text{ or } m+1, \\ &e(\nu_{w^{-1}(1)}, \nu_{w^{-1}(2)}, \dots, \nu_{w^{-1}(m+n)}) \psi_w (a \otimes b) = \psi_w (a \otimes b). \end{aligned}$$

To prove the assertion, it suffices to show that

$$x_1^{l+l'} \psi_w (a \otimes b) = 0,$$

where $l = \langle \alpha_{\nu_{w^{-1}(1)}}^\vee, \Lambda\rangle$ and $l' = \langle \alpha_{\nu_{w^{-1}(1)}}^\vee, \Lambda'\rangle$.

If $w^{-1}(1) = 1$, then $x_1 \psi_w = \psi_w x_1$. Thus, we have

$$x_1^{l+l'} \psi_w (a \otimes b) = \psi_w x_1^{l+l'} (a \otimes b) = \psi_w ((x_1^{l+l'} a) \otimes b) = 0.$$

Suppose that $w^{-1}(1) = m+1$. We set $u = ws_m s_{m-1} \dots s_1$, whose two-line notation is

$$\left(\begin{smallmatrix} 1 & 2 & \cdots & m+1 & m+2 & \cdots & m+n \\ 1 & w(1) & \cdots & w(m) & w(m+2) & \cdots & w(m+n) \end{smallmatrix}\right),$$

so that $u(1) = 1$ and $\ell(us_1s_2\dots s_m) = \ell(u) + \ell(s_1s_2\dots s_m) = \ell(u)+m$. As w is fully commutative, $\psi_w = \psi_u\psi_1\psi_2\dots\psi_m$. It follows from

$$x_1\psi_1\psi_2\dots\psi_m e(\nu) = \psi_1\psi_2\dots\psi_m x_{m+1}e(\nu) - \sum_{t=1}^{m} \delta_{\nu_{m+1},\nu_t}\psi_1\dots\psi_{t-1}\psi_{t+1}\dots\psi_m e(\nu)$$

and $s_1\dots s_{t-1}(\nu_1\dots\nu_m) = (\nu_t\nu_1\dots\widehat{\nu_t}\dots\nu_m)$ that

$$\begin{aligned} x_1^{l+l'}\psi_w(a\otimes b) &= x_1^{l+l'-1}\psi_u x_1\psi_1\psi_2\dots\psi_m(a\otimes b) \\ &= x_1^{l+l'-1}\psi_w x_{m+1}(a\otimes b) \\ &\quad - \sum_{t=1}^{m} x_1^{l+l'-1}\delta_{\nu_{m+1},\nu_t}\psi_u\psi_1\dots\psi_{t-1}\psi_{t+1}\dots\psi_m(a\otimes b) \\ &= x_1^{l+l'-1}\psi_w x_{m+1}(a\otimes b) \\ &\quad - \sum_{t=1}^{m} \delta_{\nu_{m+1},\nu_t}\psi_u\psi_{t+1}\dots\psi_m((x_1^{l+l'-1}\psi_1\dots\psi_{t-1}a)\otimes b) \\ &= x_1^{l+l'-1}\psi_w x_{m+1}(a\otimes b). \end{aligned}$$

Continuing this process, we have

$$x_1^{l+l'}\psi_w(a\otimes b) = x_1^{l}\psi_w x_{m+1}^{l'}(a\otimes b) = x_1^{l}\psi_w(a\otimes(x_1^{l'}b)) = 0,$$

which completes the proof. □

2.5. **Dual space for the convolution product.**

Let $\tau : R(\beta)\to R(\beta)$ be the graded anti-involution which is the identity on generators. For $M\in R(\beta)$-gmod, we define $M^{\circledast} := \mathrm{HOM}_{\mathcal{O}}(M,\mathcal{O})$ to be the dual of M whose $R(\beta)$-action is given by $(xf)(v) = f(\tau(x)v)$ for $x\in R(\beta)$, $f\in M^{\circledast}$ and $v\in M$.

We take self-dual simple modules $M\in R(\beta)$-gmod and $N\in R(\gamma)$-gmod with $m=\mathrm{ht}(\beta)$ and $n=\mathrm{ht}(\gamma)$. Let $\mathfrak{b}_M$ and $\mathfrak{b}_N$ be bases of M and N over $\mathcal{O}$ respectively. Then

$$\mathfrak{b}_{M\circ N} = \{\psi_w\otimes x\otimes y \mid w\in\mathfrak{S}_{m+n}/\mathfrak{S}_m\times\mathfrak{S}_n,\ x\in\mathfrak{b}_M, y\in\mathfrak{b}_N\}$$

is a basis of $M\circ N$. We define

$$\mathfrak{b}^{\circledast}_{M\circ N} = \{\xi_w^{x,y} \mid w\in\mathfrak{S}_{m+n}/\mathfrak{S}_m\times\mathfrak{S}_n,\ x\in\mathfrak{b}_M, y\in\mathfrak{b}_N\}\subseteq (M\circ N)^{\circledast}$$

to be the dual basis of $\mathfrak{b}_{M\circ N}$, i.e. $\xi_w^{x,y}(\psi_{w'}\otimes x'\otimes y') = \delta_{(w,x,y),(w',x',y')}$. It is known that there is an $R(\beta+\gamma)$-module isomorphism

$$N\circ M \xrightarrow{\sim} q^{-(\beta,\gamma)}(M\circ N)^{\circledast}$$

which sends $1 \otimes y \otimes x$ to $\xi^{x,y}_{w[m,n]}$ for $y \in N$ and $x \in M$. See [22, Theorem 2.2(2)].

Lemma 2.22. *The isomorphism* $N \circ M \simeq q^{-(\beta,\gamma)}(M \circ N)^{\circledast}$ *sends* $\psi_w \otimes y \otimes x \in N \circ M$ *to*

$$\xi^{x,y}_{w^{-1}w[m,n]} + \sum_{\substack{w' \succ w^{-1}w[m,n] \\ x' \in M, y' \in N}} a_{w',x',y'} \xi^{x',y'}_{w'} \in q^{-(\beta,\gamma)}(M \circ N)^{\circledast}$$

for some $a_{w',x',y'} \in \mathcal{O}$, *and* $\xi^{x,y}_{w^{-1}w[m,n]} \in q^{-(\beta,\gamma)}(M \circ N)^{\circledast}$ *to*

$$\psi_w \otimes y \otimes x + \sum_{\substack{w' \prec w, \\ x' \in M, y' \in N}} b_{w',x',y'} \psi_{w'} \otimes y' \otimes x' \in N \circ M \tag{2.4}$$

for some $b_{w',x',y'} \in \mathcal{O}$.

Proof. The first assertion is clear because $\psi_w \xi^{x,y}_{w[m,n]}$ has the desired form. The second assertion follows from the first. □

3. Specht modules in affine and infinite type C

In this section, we introduce Specht modules for cyclotomic quiver Hecke algebras in type $C^{(1)}_\ell$ or C_∞ and provide a basis theorem for Specht modules in type C_∞. From now until Definition 3.11, we assume that $\mathcal{O}$ is a field.

3.1. The modules $\mathcal{L}(k;\ell)$.

For $k \in \mathbb{Z}$ and $\ell \in \mathbb{Z}_{>0}$, let

$$\beta_{(k;\ell)} = \sum_{t=k}^{k+\ell-1} \alpha_{\overline{t}} \quad \text{and} \quad \nu_{(k;\ell)} = (\overline{k}, \overline{k+1}, \dots, \overline{k+\ell-1}) \in I^{\beta_{(k;\ell)}}.$$

Then $\mathcal{L}(k;\ell) = \mathcal{O}\mathfrak{l}_{(k;\ell)}$ is the one-dimensional graded $R(\beta_{(k;\ell)})$-module defined by $\deg(\mathfrak{l}_{(k;\ell)}) = 0$ and

$$x_i \mathfrak{l}_{(k;\ell)} = \psi_j \mathfrak{l}_{(k;\ell)} = 0, \quad e(\nu)\mathfrak{l}_{(k;\ell)} = \delta_{\nu,\nu_{(k;\ell)}} \mathfrak{l}_{(k;\ell)} \tag{3.1}$$

for $1 \leqslant i \leqslant \ell$, $1 \leqslant j \leqslant \ell - 1$, $\nu \in I^{\beta_{(k;\ell)}}$. If there is no confusion, we write $\mathfrak{l}$ for $\mathfrak{l}_{(k;\ell)}$ and we sometimes write $\mathcal{L}(\overline{k}, \overline{k+1}, \dots, \overline{k+\ell-1})$ instead of $\mathcal{L}(k;\ell)$.

Let $k \in \mathbb{Z}$ and $\ell_1, \ell_2 \in \mathbb{Z}_{\geqslant 0}$. As

$$\mathcal{L}(k;\ell_1) \otimes \mathcal{L}(k+\ell_1;\ell_2) \simeq e(\beta_{(k;\ell_1)}, \beta_{(k+\ell_1;\ell_2)})\mathcal{L}(k;\ell_1+\ell_2)$$

as an $R(\beta_{(k;\ell_1)}) \otimes R(\beta_{(k+\ell_1;\ell_2)})$-module by construction, we have

$$\mathrm{Hom}_{R(\beta_{(k;\ell_1+\ell_2)})}(\mathcal{L}(k;\ell_1) \circ \mathcal{L}(k+\ell_1;\ell_2), \mathcal{L}(k;\ell_1+\ell_2)) \neq 0$$

by Frobenius reciprocity so that there exists a surjective $R(\beta_{(k;\ell_1+\ell_2)})$-module homomorphism

$$p : \mathcal{L}(k;\ell_1) \circ \mathcal{L}(k+\ell_1;\ell_2) \longrightarrow \mathcal{L}(k;\ell_1+\ell_2) \tag{3.2}$$

sending $\mathfrak{l} \otimes \mathfrak{l}$ to $\mathfrak{l}$. Taking the dual of p, we have the graded monomorphism

$$\iota : \mathcal{L}(k;\ell_1+\ell_2) \hookrightarrow q^{(\beta_{(k;\ell_1)},\ \beta_{(k+\ell_1;\ell_2)})} \mathcal{L}(k+\ell_1;\ell_2) \circ \mathcal{L}(k;\ell_1).$$

Then, noting that $p(\psi_w \otimes \mathfrak{l} \otimes \mathfrak{l}) = \psi_w \mathfrak{l}$ implies $\iota(\mathfrak{l}) = \xi_1^{\mathfrak{l},\mathfrak{l}}$, (2.4) from Lemma 2.22 shows that

$$\iota(\mathfrak{l}_{(k;\ell_1+\ell_2)}) = \psi_{w[\ell_2,\ell_1]}(\mathfrak{l} \otimes \mathfrak{l}) + \sum_{\substack{w \in \mathfrak{S}_{\ell_1+\ell_2} \\ w \prec w[\ell_2,\ell_1]}} a_w \psi_w(\mathfrak{l} \otimes \mathfrak{l}) \quad \text{for some } a_w \in \mathcal{O},$$

with $\psi_w(\mathfrak{l} \otimes \mathfrak{l}) \in e(\nu_{(k;\ell_1+\ell_2)})\mathcal{L}(k+\ell_1;\ell_2) \circ \mathcal{L}(k;\ell_1)$ and $\deg(\psi_w(\mathfrak{l} \otimes \mathfrak{l})) = 0$ whenever $a_w \neq 0$. Here, $\deg(\mathfrak{l} \otimes \mathfrak{l}) = (\beta_{(k;\ell_1)}, \beta_{(k+\ell_1;\ell_2)})$ because of the shift. The following lemma is easy to see by construction.

Lemma 3.1. *Define*

$$r := \iota \circ p : \mathcal{L}(k;\ell_1) \circ \mathcal{L}(k+\ell_1;\ell_2) \to q^{(\beta_{(k;\ell_1)},\ \beta_{(k+\ell_1;\ell_2)})} \mathcal{L}(k+\ell_1;\ell_2) \circ \mathcal{L}(k;\ell_1).$$

(1) Let $\mathfrak{l}_1 = \mathfrak{l}_{(k;\ell_1)}$ *and* $\mathfrak{l}_2 = \mathfrak{l}_{(k+\ell_1;\ell_2)}$. *Then*

$$r(\mathfrak{l}_1 \otimes \mathfrak{l}_2) = \psi_{w[\ell_2,\ell_1]}(\mathfrak{l}_2 \otimes \mathfrak{l}_1) + \sum_{\substack{w \in \mathfrak{S}_{\ell_1+\ell_2} \\ w \prec w[\ell_2,\ell_1]}} a_w \psi_w(\mathfrak{l}_2 \otimes \mathfrak{l}_1) \quad \text{for some } a_w \in \mathcal{O},$$

with $\psi_w(\mathfrak{l}_2 \otimes \mathfrak{l}_1) \in e(\nu_{(k;\ell_1+\ell_2)})\mathcal{L}(k+\ell_1;\ell_2) \circ \mathcal{L}(k;\ell_1)$ *and* $\deg(\psi_w(\mathfrak{l}_2 \otimes \mathfrak{l}_1)) = 0$ *whenever* $a_w \neq 0$.

(2) $\mathrm{im}(r)$ *is isomorphic to* $\mathcal{L}(k;\ell_1+\ell_2)$.

Corollary 3.2. *If the Cartan matrix is of type* C_∞*, then*

$$r(\mathfrak{l}_1 \otimes \mathfrak{l}_2) = \psi_{w[\ell_2,\ell_1]}(\mathfrak{l}_2 \otimes \mathfrak{l}_1).$$

Proof. In type C_∞, we know by examining residues that $e(\nu)\mathcal{L}(k+\ell_1;\ell_2) \circ \mathcal{L}(k;\ell_1) \neq 0$ if and only if ν is a shuffle of $\nu_{(k+\ell_1;\ell_2)}$ and $\nu_{(k;\ell_1)}$. Thus it is straightforward to check that

$$e(\nu_{(k;\ell_1+\ell_2)})\mathcal{L}(k+\ell_1;\ell_2) \circ \mathcal{L}(k;\ell_1) = \mathrm{Span}_{\mathcal{O}}\{\psi_{w[\ell_2,\ell_1]}(\mathfrak{l}_2 \otimes \mathfrak{l}_1)\},$$

which completes the proof by Lemma 3.1. □

Remark 3.3. It is easy to show that $\mathcal{L}(k;\ell)$ admits an affinization for any k and ℓ. If A is of type C_∞, then $\mathcal{L}(k;\ell)$ is real and r in Lemma 3.1 is the R-matrix [16]. Note that, if A is affine, $\mathcal{L}(k;\ell)$ is not real in general.

Proposition 3.4. *Let $k \in \mathbb{Z}$ and $a, b, c \in \mathbb{Z}_{\geqslant 0}$ with $b \geqslant c > 0$. Then, there is a non-zero $R(\beta_{(k;a)} + \beta_{(k-1;a+b+1)} + \beta_{(k+a;c-1)})$-module homomorphism*

$$\mathsf{g} : \mathcal{L}(k;a) \circ \mathcal{L}(k-1;a+b+1) \circ \mathcal{L}(k+a;c-1) \longrightarrow q^{(\beta_{(k-1;a+1)},\beta_{(k+a;b)})} \mathcal{L}(k;a+b) \circ \mathcal{L}(k-1;a+c)$$

such that

$$\mathsf{g}(\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{l}) = \Psi_2(a,b,a+1)(\mathfrak{l} \otimes \mathfrak{l}) + \sum_{w \prec S_2(a,b,a+1)} a_w \psi_w (\mathfrak{l} \otimes \mathfrak{l}) \quad \text{for some } a_w \in \mathcal{O}.$$

If the Cartan matrix A is of type C_∞ then $a_w = 0$ for all $w \prec S_2(a,b,a+1)$.

Proof. Combining Lemma 3.1 with the surjectivity of p, we have a non-zero homomorphism

$$\begin{aligned} &\mathcal{L}(k;a) \circ \mathcal{L}(k-1;a+1) \circ \mathcal{L}(k+a;b) \circ \mathcal{L}(k+a;c-1) \\ &\xrightarrow{\mathrm{id} \circ r \circ \mathrm{id}} q^{(\beta_{(k-1;a+1)},\beta_{(k+a;b)})} \mathcal{L}(k;a) \circ \mathcal{L}(k+a;b) \circ \mathcal{L}(k-1;a+1) \circ \mathcal{L}(k+a;c-1) \\ &\xrightarrow{p \circ p} q^{(\beta_{(k-1;a+1)},\beta_{(k+a;b)})} \mathcal{L}(k;a+b) \circ \mathcal{L}(k-1;a+c). \end{aligned}$$

Lemma 3.1 (2) tells us that the image of the first homomorphism is isomorphic to

$$\mathcal{L}(k;a) \circ \mathcal{L}(k-1;a+b+1) \circ \mathcal{L}(k+a;c-1),$$

which is generated by

$$\begin{aligned} &\mathfrak{l} \otimes \left(\Psi[b,a+1](\mathfrak{l} \otimes \mathfrak{l}) + \sum_{w \prec w[b,a+1]} a_w \psi_w (\mathfrak{l} \otimes \mathfrak{l}) \right) \otimes \mathfrak{l} \\ &\qquad = \Psi_2(a,b,a+1)(\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{l}) + \sum_{w \prec S_2(a,b,a+1)} a_w \psi_w (\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{l}) \end{aligned}$$

by Lemma 3.1 (1). Thus it gives a non-zero homomorphism

$$\mathsf{g} : \mathcal{L}(k;a) \circ \mathcal{L}(k-1;a+b+1) \circ \mathcal{L}(k+a;c-1) \longrightarrow q^{(\beta_{(k-1;a+1)},\beta_{(k+a;b)})} \mathcal{L}(k;a+b) \circ \mathcal{L}(k-1;a+c)$$

such that $\mathsf{g}(\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{l})$ has the desired form. ∎

3.2. **The modules $\mathcal{S}^\lambda$.**

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \in \mathscr{P}^1_n$ with a charge $\kappa \in \mathbb{Z}$. Note that the level of λ is 1. Let $\beta := \mathrm{cont}(\lambda)$ and define

$$\mathcal{M}^\lambda_\kappa := \mathcal{L}(\kappa;\lambda_1) \circ \mathcal{L}(\kappa-1;\lambda_2) \circ \cdots \circ \mathcal{L}(\kappa-t+1;\lambda_t) \in R(\beta)\text{-gmod}.$$

For a Garnir node $A = (r, c) \in [\lambda]$, let

$$p^{\lambda}_{\kappa,A} := (\beta_{(\kappa-r;c)}, \beta_{(\kappa-r+c;\lambda_r-c+1)}),$$
$$\mathcal{M}^{\lambda}_{\kappa,A} := \mathcal{M}^{\lambda_{<r}}_{\kappa} \circ \mathcal{L}(\kappa-r+1; c-1) \circ \mathcal{L}(\kappa-r; \lambda_r+1) \circ \mathcal{L}(\kappa-r+c; \lambda_{r+1}-c) \circ \mathcal{M}^{\lambda_{>r+1}}_{\kappa-r-1},$$

where $\lambda_{<r} = (\lambda_1, \dots, \lambda_{r-1})$ and $\lambda_{>r+1} = (\lambda_{r+2}, \dots, \lambda_t)$. We denote by m^{λ}_{κ} (resp. $m^{\lambda}_{\kappa,A}$) the distinguished generator $\mathfrak{l} \otimes \cdots \otimes \mathfrak{l}$ of $\mathcal{M}^{\lambda}_{\kappa}$. (resp. $\mathcal{M}^{\lambda}_{\kappa,A}$.) By Proposition 3.4, we have the non-zero homomorphism

$$q^{-p^{\lambda}_{\kappa,A}} \mathcal{L}(\kappa-r+1; c-1) \circ \mathcal{L}(\kappa-r; \lambda_r+1) \circ \mathcal{L}(\kappa-r+c; \lambda_{r+1}-c) \longrightarrow \mathcal{L}(\kappa-r+1; \lambda_r) \circ \mathcal{L}(\kappa-r; \lambda_{r+1}).$$

which gives the induced homomorphism

$$\mathcal{H}^{\lambda}_{\kappa,A} : q^{-p^{\lambda}_{\kappa,A}} \mathcal{M}^{\lambda}_{\kappa,A} \longrightarrow \mathcal{M}^{\lambda}_{\kappa}.$$

Definition 3.5. Let $\lambda \in \mathscr{P}^1_n$ with a charge $\kappa \in \mathbb{Z}$. Then we define, for a Garnir node A,

$$g^{\lambda}_{\kappa,A} = \mathcal{H}^{\lambda}_{\kappa,A}(m^{\lambda}_{\kappa,A}).$$

By Proposition 3.4 and (1.5), we have

$$g^{\lambda}_{\kappa,A} - \psi_{w^{\mathtt{G}^A}} m^{\lambda}_{\kappa} = \sum_{u \prec w^{\mathtt{G}^A}} a_u \psi_u m^{\lambda}_{\kappa} \quad \text{for some } a_u \in \mathcal{O}. \tag{3.3}$$

Remark 3.6. If the Cartan matrix A is of type C_∞, then $g^{\lambda}_{\kappa,A} = \psi_{w^{\mathtt{G}^A}} m^{\lambda}_{\kappa}$. This is reminiscent of the Garnir element defined in [21] for type A_∞.

Lemma 3.7. *We have*

(i) $x_i g^{\lambda}_{\kappa,A} = 0$, *for* $1 \leqslant i \leqslant n$,
(ii) $\psi_j g^{\lambda}_{\kappa,A} = 0$ *unless* $s_j \mathtt{G}^A \in \mathsf{Row}(\lambda)$.

Proof. For $1 \leqslant i \leqslant n$, we have

$$x_i g^{\lambda}_{\kappa,A} = x_i \mathcal{H}^{\lambda}_{\kappa,A}(m^{\lambda}_{\kappa,A}) = \mathcal{H}^{\lambda}_{\kappa,A}(x_i m^{\lambda}_{\kappa,A}) = 0.$$

Let $A = (r, c) \in \lambda$ and $l_p = \sum_{k=1}^{p} \lambda_p$. Considering the definition of $\mathtt{G}^A$, we know that

$$s_j \mathtt{G}^A \in \mathsf{Row}(\lambda) \text{ if and only if } j = l_1, \dots, l_{r-1}, l_{r-1}+c-1, l_r+c, l_{r+1}, \dots, l_t.$$

Thus, by the construction of $m^{\lambda}_{\kappa,A}$,

$$\psi_j g^{\lambda}_{\kappa,A} = \psi_j \mathcal{H}^{\lambda}_{\kappa,A}(m^{\lambda}_{\kappa,A}) = \mathcal{H}^{\lambda}_{\kappa,A}(\psi_j m^{\lambda}_{\kappa,A}) = 0$$

unless $s_j \mathtt{G}^A \in \mathsf{Row}(\lambda)$. □

We define $\mathcal{H}^\lambda_\kappa : \oplus_A q^{-p^\lambda_{\kappa,A}} \mathcal{M}^\lambda_{\kappa,A} \to \mathcal{M}^\lambda_\kappa$ as the sum of $\mathcal{H}^\lambda_{\kappa,A}$ over Garnir nodes A of λ and set

$$\mathcal{G}^\lambda_\kappa = \mathrm{im}\ \mathcal{H}^\lambda_\kappa \subset \mathcal{M}^\lambda_\kappa, \quad \mathcal{S}^\lambda_\kappa = q^{\deg(\mathtt{T}^\lambda)} \mathrm{coker}\ \mathcal{H}^\lambda_\kappa.$$

If there is no possibility of confusion, we will drop the subscript κ from our notation, i.e. we will simply write $\mathcal{M}^\lambda$, m^λ, g^λ_A, $\mathcal{S}^\lambda$, etc.

Definition 3.8. For $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \mathscr{P}^l_n$ and $\kappa = (\kappa_1, \dots, \kappa_l) \in \mathbb{Z}^l$, we define

$$\mathcal{M}^\lambda = \mathcal{M}^\lambda_\kappa := \mathcal{M}^{\lambda^{(1)}}_{\kappa_1} \circ \cdots \circ \mathcal{M}^{\lambda^{(l)}}_{\kappa_l}.$$

$$\mathcal{S}^\lambda = \mathcal{S}^\lambda_\kappa := q^{\deg(\mathtt{T}^\lambda)} \mathrm{coker}\ \mathcal{H}^{\lambda^{(1)}}_{\kappa_1} \circ \cdots \circ \mathrm{coker}\ \mathcal{H}^{\lambda^{(l)}}_{\kappa_l}.$$

We write $\mathcal{S}^\lambda(\mathcal{O})$ when we need to emphasise the field.

Remark 3.9. By Theorem 2.3, the set

$$\{\psi_{w^{\mathtt{T}}} m^\lambda \mid \mathtt{T} \in \mathsf{RowStd}(\lambda)\} \tag{3.4}$$

is an $\mathcal{O}$-basis of $\mathcal{M}^\lambda$.

Note that $\deg \psi_{w^{\mathtt{T}}} m^\lambda = \deg \psi_{w^{\mathtt{T}}} e(\mathrm{res}(\mathtt{T}))$ by definition. Then we have the following result.

Proposition 3.10. *If* $\mathtt{T} \in \mathsf{Std}(\lambda)$, *then* $\deg \psi_{w^{\mathtt{T}}} m^\lambda = \deg \mathtt{T} - \deg \mathtt{T}^\lambda$.

Proof. We closely follow the proof of [8, Proposition 3.13]. If $\mathtt{T} = \mathtt{T}^\lambda$, then we have

$$\deg \psi_{w^{\mathtt{T}^\lambda}} m^\lambda = \deg m^\lambda = 0 = \deg \mathtt{T}^\lambda - \deg \mathtt{T}^\lambda.$$

Thus, it suffices to prove our statement in the case that $\mathtt{S}, \mathtt{T} \in \mathsf{Std}(\lambda)$ are such that $\ell(\mathtt{S}) = \ell(\mathtt{T}) + 1$ and $\mathtt{S} = s_r \mathtt{T}$. Let $\mathrm{res}(\mathtt{T}) = (\nu_1, \nu_2, \dots, \nu_n)$. We may assume that $r = n-1$. We want to show that $\deg \mathtt{T} - \deg \mathtt{S} = (\alpha_{\nu_{n-1}}, \alpha_{\nu_n})$. Let $A = \mathtt{T}^{-1}(n)$ and $B = \mathtt{T}^{-1}(n-1)$. By assumption, B is above A in $[\lambda]$. Now,

$$\deg \mathtt{T} = d_A(\lambda) + d_B(\lambda \nearrow A) + \deg(\mathtt{T}_{\downarrow n-2}),$$

$$\deg \mathtt{S} = d_B(\lambda) + d_A(\lambda \nearrow B) + \deg(\mathtt{S}_{\downarrow n-2}).$$

Note that $\mathtt{T}_{\downarrow n-2} = \mathtt{S}_{\downarrow n-2}$, and since B is above A, $d_A(\lambda) = d_A(\lambda \nearrow B)$. So we must show that $d_B(\lambda \nearrow A) - d_B(\lambda) = (\alpha_{\nu_{n-1}}, \alpha_{\nu_n})$.

If $\mathrm{res}(A) = \mathrm{res}(B) = i$, then removing A leads to the disappearance of a removable i-node and the appearance of a new addable i-node below B, so that $d_B(\lambda \nearrow A) - d_B(\lambda) = 4$ if $i = 0$, or 2 otherwise.

If $\mathrm{res}(A) = 0$ and $\mathrm{res}(B) = 1$, removing A leaves either one fewer addable 1-node and one extra removable 1-node, or two extra removable 1-nodes, or two fewer addable 1-nodes, so that $d_B(\lambda \nearrow A) - d_B(\lambda) = -2$.

If $\mathrm{res}(A) = 1$ and $\mathrm{res}(B) = 0$, removing A leaves either one extra removable 0-node or one fewer addable 0-node, so that $d_B(\lambda \nearrow A) - d_B(\lambda) = -2$.

If $\mathrm{res}(A) = \ell - 1$ and $\mathrm{res}(B) = \ell$ or $\mathrm{res}(A) = \ell$ and $\mathrm{res}(B) = \ell - 1$ in type $C_\ell^{(1)}$, similar arguments show that $d_B(\lambda \nearrow A) - d_B(\lambda) = -2$.

If $\mathrm{res}(A) = i \pm 1$ and $\mathrm{res}(B) = i$, with neither residue equal to 0 or ℓ, then removing A leaves either one extra removable i-node or one fewer addable i-node, so $d_B(\lambda \nearrow A) - d_B(\lambda) = -1$.

In all other cases, removing A does not change the degree, so $d_B(\lambda \nearrow A) - d_B(\lambda) = 0$. □

We denote by $\overline{m}^\lambda$ the image of m^λ under the projection $q^{\deg(\mathtt{T}^\lambda)}\mathcal{M}^\lambda \to \mathcal{S}^\lambda$.

Definition 3.11. Let $\mathcal{O}$ be an integral domain. Then for $\lambda \in \mathscr{P}_n^l$ and $\kappa \in \mathbb{Z}^l$, we define $\mathcal{S}_\kappa^\lambda(\mathcal{O})$ over $\mathcal{O}$ to be the lattice $R_\mathcal{O}(\mathrm{cont}(\lambda))\overline{m}^\lambda$ generated by $\overline{m}^\lambda$ in $\mathcal{S}_\kappa^\lambda(\mathbb{F})$, where $\mathbb{F} = \mathrm{Frac}(\mathcal{O})$ and $R_\mathcal{O}(\mathrm{cont}(\lambda))$ is the quiver Hecke algebra over $\mathcal{O}$.

From now on, let $\mathcal{O}$ denote an arbitrary integral domain.

Theorem 3.12. *Let $\lambda \in \mathscr{P}_n^1$ with a charge $\kappa \in \mathbb{Z}$, and let $\beta = \mathrm{cont}(\lambda)$.*
(1) $\mathcal{S}^\lambda$ is generated by $\{\psi_{w^\mathtt{T}}\overline{m}^\lambda \mid \mathtt{T} \in \mathsf{Std}(\lambda)\}$ as an $\mathcal{O}$-module.
(2) $\mathcal{S}^\lambda$ is a graded $R^{\Lambda_{\overline{\kappa}}}(\beta)$-module.

Proof. (1) For $\ell = 0, 1, \dots$, we define

$$A_\ell := \{\psi_{w^\mathtt{T}}\overline{m}^\lambda \mid \mathtt{T} \in \mathsf{RowStd}(\lambda),\ \ell(\mathtt{T}) \leqslant \ell\} \subseteq \mathcal{S}^\lambda,$$
$$B_\ell := \{\psi_{w^\mathtt{T}}\overline{m}^\lambda \mid \mathtt{T} \in \mathsf{Std}(\lambda),\ \ell(\mathtt{T}) \leqslant \ell\} \subseteq A_\ell.$$

Then (3.4) implies that $\mathcal{S}^\lambda$ is generated by $\bigcup_{\ell \geqslant 0} A_\ell$ as an $\mathcal{O}$-module, so it suffices to show that

$$\mathrm{Span}_\mathcal{O} A_\ell = \mathrm{Span}_\mathcal{O} B_\ell$$

for all $\ell \geqslant 0$ by induction on ℓ. If $\ell = 0$, there is nothing to prove. Suppose that $\ell > 0$ and take $\mathtt{T} \in \mathsf{RowStd}(\lambda)$ with $\ell = \ell(\mathtt{T})$. We will show that $\psi_{w^\mathtt{T}}\overline{m}^\lambda \in \mathrm{Span}_\mathcal{O} B_\ell$. Since it is trivial when $\mathtt{T} \in \mathsf{Std}(\lambda)$, we assume that $\mathtt{T} \in \mathsf{Row}(\lambda)$ and prove $\psi_{w^\mathtt{T}}\overline{m}^\lambda \in \mathrm{Span}_\mathcal{O} B_{\ell-1}$. We set

$$\mathcal{M}_{\ell-1}^\lambda := \mathrm{Span}_\mathcal{O}\{\psi_{w^\mathtt{T}} m^\lambda \mid \mathtt{T} \in \mathsf{RowStd}(\lambda),\ \ell(\mathtt{T}) \leqslant \ell - 1\} \subseteq \mathcal{M}^\lambda,$$
$$\mathcal{S}_{\ell-1}^\lambda := \mathrm{Span}_\mathcal{O} A_{\ell-1} \subseteq \mathcal{S}^\lambda.$$

By Lemma 1.12, there are a Garnir node $A \in [\lambda]$ and an element $w \in \mathfrak{S}_n$ such that $\mathtt{T} = w\mathtt{G}^A$ and $\ell(\mathtt{T}) = \ell(w) + \ell(\mathtt{G}^A)$. It follows from Proposition 2.2 and (3.4) that

$$\psi_{w^\mathtt{T}} m^\lambda - \psi_w \psi_{w^{\mathtt{G}^A}} m^\lambda \equiv 0 \pmod{\mathcal{M}_{\ell-1}^\lambda}.$$

By (3.3), we have

$$\psi_w \psi_{w^{\mathtt{G}^A}} m^\lambda - \psi_w g_A^\lambda \equiv 0 \pmod{\mathcal{M}_{\ell-1}^\lambda},$$

which implies that

$$\psi_{w^{\mathtt{T}}}\overline{m}^{\lambda} \equiv 0 \pmod{\mathcal{S}^{\lambda}_{\ell-1}},$$

proving $\psi_{w^{\mathtt{T}}}\overline{m}^{\lambda} \in \mathrm{Span}_{\mathcal{O}} B_{\ell-1}$ by the induction hypothesis $\mathrm{Span}_{\mathcal{O}} A_{\ell-1} = \mathrm{Span}_{\mathcal{O}} B_{\ell-1}$.

(2) It follows from (1) that it suffices to prove $x_1^{\langle \alpha^\vee_{\nu_1}, \Lambda_{\overline{\kappa}}\rangle} e(\nu)\psi_{w^{\mathtt{T}}} m^{\lambda} = 0$, for $\mathtt{T} \in \mathsf{Std}(\lambda)$. But if $\mathtt{T} \in \mathsf{Std}(\lambda)$ then $w^{\mathtt{T}}(1) = 1$, so that $\psi_{w^{\mathtt{T}}}$ is a product of $\psi_2, \dots, \psi_{n-1}$ and $x_1\psi_{w^{\mathtt{T}}} = \psi_{w^{\mathtt{T}}} x_1$ holds. Then, since $e(\nu)\psi_{w^{\mathtt{T}}} m^{\lambda} \neq 0$ implies $\nu_1 = \overline{\kappa}$,

$$x_1^{\langle \alpha^\vee_{\nu_1}, \Lambda_{\overline{\kappa}}\rangle} e(\nu)\psi_{w^{\mathtt{T}}} m^{\lambda} = x_1 e(\nu)\psi_{w^{\mathtt{T}}} m^{\lambda} = e(\nu)\psi_{w^{\mathtt{T}}} x_1 m^{\lambda} = 0. \qquad \square$$

Corollary 3.13. *Let $l \in \mathbb{Z}_{>0}$, $\lambda \in \mathscr{P}^l_n$, $\kappa = (\kappa_1, \dots, \kappa_l) \in \mathbb{Z}^l$, and let $\beta = \mathrm{cont}(\lambda)$.*

(1) $\mathcal{S}^{\lambda}$ is generated by $\{\psi_{w^{\mathtt{T}}}\overline{m}^{\lambda} \mid \mathtt{T} \in \mathsf{Std}(\lambda)\}$ as an $\mathcal{O}$-module.

(2) Let $\Lambda = \Lambda_{\overline{\kappa_1}} + \cdots + \Lambda_{\overline{\kappa_l}}$. Then $\mathcal{S}^{\lambda}$ is a graded $R^{\Lambda}(\beta)$-module.

Proof. This follows from Theorem 3.12 and Proposition 2.21. $\square$

Definition 3.14. Let $l \in \mathbb{Z}_{>0}$, $\kappa = (\kappa_1, \dots, \kappa_l) \in \mathbb{Z}^l$ and $\Lambda = \Lambda_{\overline{\kappa_1}} + \cdots + \Lambda_{\overline{\kappa_l}}$. Then we call the graded $R^{\Lambda}(\beta)$-modules $\mathcal{S}^{\lambda}$, for $\lambda \in \mathscr{P}^l_n$, *Specht modules*.

Remark 3.15. One can easily construct a 'column version' of the Specht modules by the same argument. (cf. [21, Section 7]).

Example 3.16. Let $\kappa \in \mathbb{Z}$ and $\lambda = (n), \lambda' = (1^n) \in \mathscr{P}^1_n$. It is straightforward to prove that

$$\mathcal{S}^{\lambda}_{\kappa} \simeq \mathcal{L}(\kappa; n) \simeq \mathcal{S}^{\lambda'}_{-\kappa}.$$

In particular, $\mathcal{S}^{\lambda}_0 \simeq \mathcal{S}^{\lambda'}_0$.

Example 3.17. Suppose that A is of type C_∞ or $C^{(1)}_\ell$ with $\ell > 2$. Let $\kappa = -1$ and $\lambda = (4), \mu = (3,1) \in \mathscr{P}^1_4$. As λ has no Garnir nodes, we have

$$\mathcal{S}^{\lambda} = \mathcal{L}(1012).$$

Since μ has only one Garnir node $(1,1)$, we have $p^{\mu}_{(1,1)} = (\alpha_2, \alpha_1 + \alpha_0 + \alpha_1) = -2$ and

$$\mathcal{H}^{\mu} : q^2\mathcal{L}(2101) \longrightarrow \mathcal{M}^{\mu} := \mathcal{L}(101) \circ \mathcal{L}(2), \qquad \mathfrak{l} \mapsto \psi_1\psi_2\psi_3 m^{\mu}.$$

Thus, we have $\mathcal{G}^{\lambda} \simeq q^2\mathcal{L}(2101)$ and

$$\mathrm{ch}_q \mathcal{S}^{\mu} = (1012) + q(1021) + q(1201).$$

The epimorphism $\mathcal{L}(101) \circ \mathcal{L}(2) \twoheadrightarrow \mathcal{L}(1012)$ gives the epimorphism

$$\mathcal{S}^{\mu} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{S}^{\lambda},$$

which tells us that $\mathcal{S}^{\mu}$ is not simple and the head of $\mathcal{S}^{\mu}$ is isomorphic to $\mathcal{S}^{\lambda}$.

Example 3.18. Suppose that A is of type C_∞ or $C_\ell^{(1)}$ with $\ell > 2$. Let $\kappa = 0$ and $\lambda = (3,2,1) \in \mathscr{P}_6^1$. Then $\deg(\mathtt{T}^\lambda) = 1$, $\mathcal{M}^\lambda = \mathcal{L}(012) \circ \mathcal{L}(10) \circ \mathcal{L}(2)$ and the Garnir nodes of λ are $A_1 := (1,1)$, $A_2 := (1,2)$ and $A_3 := (2,1)$. Since

$$\begin{aligned} p_{A_1}^\lambda &= (\alpha_1, \alpha_0 + \alpha_1 + \alpha_2) = -1, \\ p_{A_2}^\lambda &= (\alpha_1 + \alpha_0, \alpha_1 + \alpha_2) = -1, \\ p_{A_3}^\lambda &= (\alpha_2, \alpha_1 + \alpha_0) = -1, \end{aligned}$$

we have

$$\begin{aligned} \mathcal{H}_{A_1}^\lambda &: q\mathcal{L}(1012) \circ \mathcal{L}(0) \circ \mathcal{L}(2) \longrightarrow \mathcal{M}^\lambda, & m_{A_1}^\lambda &\longmapsto \psi_1\psi_2\psi_3 m^\lambda, \\ \mathcal{H}_{A_2}^\lambda &: q\mathcal{L}(0) \circ \mathcal{L}(1012) \circ \mathcal{L}(2) \longrightarrow \mathcal{M}^\lambda, & m_{A_2}^\lambda &\longmapsto \psi_3\psi_2\psi_4\psi_3 m^\lambda, \\ \mathcal{H}_{A_3}^\lambda &: q\mathcal{L}(012) \circ \mathcal{L}(210) \longrightarrow \mathcal{M}^\lambda, & m_{A_3}^\lambda &\longmapsto \psi_4\psi_5 m^\lambda. \end{aligned}$$

Thus, $\mathcal{G}^\lambda = \langle \psi_1\psi_2\psi_3 m^\lambda, \psi_3\psi_2\psi_4\psi_3 m^\lambda, \psi_4\psi_5 m^\lambda \rangle \subset \mathcal{M}^\lambda$ and

$$\mathcal{S}^\lambda = q\mathcal{M}^\lambda / \mathcal{G}^\lambda.$$

3.3. Basis theorem for type C_∞.

Suppose that the Cartan matrix is of type C_∞. Then we have the following basis theorem for Specht modules, whose proof is postponed to Section 4.

Theorem 3.19. *Let $\lambda \in \mathscr{P}_n^1$ with a charge $\kappa \in \mathbb{Z}$. Then the set $\{\psi_{w^{\mathtt{T}}} \overline{m}^\lambda \mid \mathtt{T} \in \mathsf{Std}(\lambda)\}$ is an $\mathcal{O}$-basis of $\mathcal{S}^\lambda$. Moreover, we have the following graded character formula.*

$$\mathrm{ch}_q \mathcal{S}^\lambda = \sum_{\mathtt{T} \in \mathsf{Std}(\lambda)} q^{\deg(\mathtt{T})} \mathrm{res}(\mathtt{T}).$$

Corollary 3.20. *In the Grothendieck group of $R^{\Lambda_{\overline{\kappa}}}(n-1)$-gmod, we have*

$$[E_i^{\Lambda_{\overline{\kappa}}} \mathcal{S}^\lambda] = \sum_b q^{d_b(\lambda)} [\mathcal{S}^{\lambda \nearrow b}],$$

where b runs over all removable i-nodes.

Proof. We rewrite the graded character formula from Theorem 3.19 as follows.

$$\mathrm{ch}_q \mathcal{S}^\lambda = \sum_b \sum_{\mathtt{T} \in \mathsf{Std}(\lambda \nearrow b)} q^{\deg(\mathtt{T}) + d_b(\lambda)} \mathrm{res}(\mathtt{T}) * \mathrm{res}(b),$$

where b runs over all removable nodes. Thus,

$$\mathrm{ch}_q(E_i^{\Lambda_{\overline{\kappa}}} \mathcal{S}^\lambda) = \sum_b q^{d_b(\lambda)} \mathrm{ch}_q(\mathcal{S}^{\lambda \nearrow b}),$$

where b runs over all removable i-nodes. □

Let $l \in \mathbb{Z}_{>0}$, $\kappa = (\kappa_1, \dots, \kappa_l) \in \mathbb{Z}^l$ and $\Lambda = \Lambda_{\overline{\kappa}_1} + \cdots + \Lambda_{\overline{\kappa}_l}$. One can easily prove Corollary 3.21 from Theorem 3.19 and Corollary 3.20.

Corollary 3.21. *Let $\lambda \in \mathscr{P}^l_n$.*

(1) The set $\{\psi_{w^{\mathtt{T}}} \overline{m}^\lambda \mid \mathtt{T} \in \mathsf{Std}(\lambda)\}$ is an $\mathcal{O}$-basis of $\mathcal{S}^\lambda$. Moreover,

$$\mathrm{ch}_q \mathcal{S}^\lambda = \sum_{\mathtt{T} \in \mathsf{Std}(\lambda)} q^{\deg(\mathtt{T})} \mathrm{res}(\mathtt{T}).$$

(2) In the Grothendieck group of $R^\Lambda(n-1)$-gmod, we have

$$[E_i^\Lambda \mathcal{S}^\lambda] = \sum_b q^{d_b(\lambda)} [\mathcal{S}^{\lambda \nearrow b}],$$

where b runs over all removable i-nodes.

We revisit the Fock space $\mathcal{F}(\kappa)$. As $V_q(\Lambda) \simeq V_q(\Lambda)^\vee$, by Theorem 2.20, we can identify $V_q(\Lambda) \simeq V_q(\Lambda)^\vee \simeq \mathbb{Q}(q) \otimes_{\mathbb{A}} [R^\Lambda\text{-gmod}]$. Thus, from (1.3), we have the $U_q(\mathfrak{g}(\mathsf{A}))$-module epimorphism

$$p_\kappa : \mathcal{F}(\kappa) \longrightarrow\!\!\!\!\rightarrow \mathbb{Q}(q) \otimes_{\mathbb{A}} [R^\Lambda\text{-gmod}].$$

Proposition 3.22. *For $\lambda \in \mathscr{P}^l_n$, we have*

$$p_\kappa(\lambda) = [\mathcal{S}^\lambda].$$

Proof. It is obvious that $p_\kappa(\varnothing) = [\mathcal{S}^\varnothing]$ and $\mathrm{wt}(\varnothing) = \mathrm{wt}([\mathcal{S}^\varnothing]) = \Lambda$. Since both of $\mathcal{F}(\kappa)$ and $\mathbb{Q}(q) \otimes_{\mathbb{A}} [R^\Lambda\text{-gmod}]$ are integrable $U_q(\mathfrak{g}(\mathsf{A}))$-modules and $\mathbb{Q}(q) \otimes_{\mathbb{A}} [R^\Lambda\text{-gmod}]$ is simple, it suffices to show that $e_i(p_\kappa(\lambda) - [\mathcal{S}^\lambda]) = 0$ for all λ. By (1.2), Corollary 3.21 and the induction hypothesis, we have

$$\begin{aligned} e_i p_\kappa(\lambda) &= p_\kappa(e_i \lambda) \\ &= p_\kappa\left(\sum_A q^{d_A(\lambda)} \lambda \nearrow A\right) \\ &= \sum_A q^{d_A(\lambda)} p_\kappa(\lambda \nearrow A) = \sum_A q^{d_A(\lambda)} [\mathcal{S}^{\lambda \nearrow A}] = [E_i^\Lambda \mathcal{S}^\lambda], \end{aligned}$$

which completes the proof. □

Corollary 3.23 follows from (1.2), (2.3) and Proposition 3.22.

Corollary 3.23. *Let $\beta = \mathrm{cont}(\lambda)$. In the Grothendieck group of $R^\Lambda(n+1)$-gmod, we have*

$$[F_i^\Lambda \mathcal{S}^\lambda] = \sum_b q^{-d^b(\lambda) + (\langle \alpha_i^\vee, \Lambda - \beta \rangle - 1)(\alpha_i, \alpha_i)/2} [\mathcal{S}^{\lambda \swarrow b}],$$

where b runs over all addable i-nodes.

Example 3.24. We use the same notation as in Example 3.18. Let $\mu = (2,2)$, $\mu_1 = (3,2)$ and $\mu_2 = (2,2,1)$. By Theorem 3.19, we have

$$\begin{aligned}
\mathrm{ch}_q \mathcal{S}^\lambda &= [2]_q(012102) + [2]_q(012120) + [2]_q^2(011202) + [2]_q^2(011220) + [2]_q^2(011022),\\
\mathrm{ch}_q \mathcal{S}^\mu &= [2]_q(0110),\\
\mathrm{ch}_q \mathcal{S}^{\mu_1} &= q(01210) + q[2]_q(01120) + q[2]_q(01102),\\
\mathrm{ch}_q \mathcal{S}^{\mu_2} &= (01210) + [2]_q(01120) + [2]_q(01102),
\end{aligned}$$

where $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$ for $n \in \mathbb{Z}_{\geqslant 0}$. Let $B_1 = (1,3)$ and $B_2 = (3,1)$. Then

$$d_{B_1}(\lambda) = -1, \quad d_{B_2}(\lambda) = 0, \quad d^{B_1}(\mu) = 0, \quad d^{B_2}(\mu) = 1, \quad \langle \alpha_2^\vee, \Lambda_0 - 2\alpha_0 - 2\alpha_1 \rangle = 2.$$

By Corollary 3.20 and Corollary 3.23, we have

$$\begin{aligned}
\mathrm{ch}_q E_2^{\Lambda_0} \mathcal{S}^\lambda &= \mathrm{ch}_q \mathcal{S}^{\mu_1} + q^{-1} \mathrm{ch}_q \mathcal{S}^{\mu_2} = [2]_q(01210) + [2]_q^2(01120) + [2]_q^2(01102),\\
\mathrm{ch}_q F_2^{\Lambda_0} \mathcal{S}^\mu &= q\,\mathrm{ch}_q \mathcal{S}^{\mu_1} + \mathrm{ch}_q \mathcal{S}^{\mu_2} = q[2]_q(01210) + q[2]_q^2(01120) + q[2]_q^2(01102).
\end{aligned}$$

Remark 3.25. It looks like we need a modified version of the upper global basis in the Fock space to describe the simple modules. It is an interesting problem to characterise the elements in the Fock space which correspond to the simple modules.

4. Proof of Theorem 3.19

We assume that the Cartan matrix A is of type C_∞ and take the parameters (2.2) for the quiver Hecke algebra $R(\beta)$. Let us fix $\lambda \in \mathscr{P}_n^1$, $\kappa \in \mathbb{Z}$ and $\beta = \mathrm{cont}(\lambda)$.

Definition 4.1. For $t \in \mathbb{Z}_{>0}$, we define

$$\begin{aligned}
\mathcal{G}^\lambda_{<t} := \mathcal{O}\text{-submodule of } \mathcal{G}^\lambda \text{ generated by } \psi_w g^\lambda_A \text{ for all Garnir nodes } A \in [\lambda] \text{ and}\\
\text{all } w \in \mathfrak{S}_n \text{ such that } w\mathtt{G}^A \in \mathsf{Row}(\lambda) \text{ and } \ell(w\mathtt{G}^A) = \ell(w) + \ell(\mathtt{G}^A) < t.
\end{aligned}$$

Remark 4.2. Note that we require $w\mathtt{G}^A \in \mathsf{Row}(\lambda)$ in the definition of $\mathcal{G}^\lambda_{<t}$. In Theorem 4.15 below, we will eliminate the possibility that $\mathcal{G}^\lambda$ is strictly larger than $\sum_{t \in \mathbb{Z}_{>0}} \mathcal{G}^\lambda_{<t}$.

Lemmas 4.3 and 4.4 are needed for proving Lemma 4.5.

Lemma 4.3. *Let* $\mathtt{T} \in \mathsf{RowStd}(\lambda)$.

(1) If
- *(i)* $\lambda = (\lambda_1, \lambda_2)$ *with* $\lambda_2 > 0$,
- *(ii)* $\mathrm{res}(\mathtt{T}^{-1}(n)) = \overline{\kappa + \lambda_1 - 1}$ *and* $\mathrm{res}(\mathtt{T}^{-1}(n-1)) = \overline{\kappa + \lambda_1 - 2}$,

then $\mathtt{T}(1, \lambda_1) = n$.

(2) If

(i) $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *with* $\lambda_3 > 0$,

(ii) $\mathrm{res}(\mathtt{T}^{-1}(n)) = \overline{\kappa + \lambda_1 - 1}$, $\mathrm{res}(\mathtt{T}^{-1}(n-1)) = \overline{\kappa + \lambda_1 - 2}$ *and* $\mathrm{res}(\mathtt{T}^{-1}(n-2)) = \overline{\kappa + \lambda_1 - 3}$,

(iii) $\mathrm{res}(3, \lambda_3) \neq \mathrm{res}(1, \lambda_2)$,

then $\mathtt{T}(1, \lambda_1) = n$.

Proof. (1) As $\mathtt{T} \in \mathsf{RowStd}(\lambda)$, $\mathtt{T}^{-1}(n) = (1, \lambda_1)$ or $(2, \lambda_2)$. If $\kappa + \lambda_1 - 1 \leqslant 0$ then $(1, \lambda_1)$ is the only node of residue $\overline{\kappa + \lambda_1 - 1}$, so that $\mathtt{T}(1, \lambda_1) = n$. Suppose that $\kappa + \lambda_1 - 1 > 0$ and $\mathtt{T}^{-1}(n) = (2, \lambda_2)$. Then the assumption (ii) implies that

$$\kappa + \lambda_2 - 2 = -(\kappa + \lambda_1 - 1) < 0$$

and $\mathtt{T}^{-1}(n-1) = (1, \lambda_1)$ or $(2, \lambda_2 - 1)$. Thus, $\mathrm{res}(\mathtt{T}^{-1}(n-1))$ is either $\kappa + \lambda_1 - 1$ or $\kappa + \lambda_1$, which are not equal to $\overline{\kappa + \lambda_1 - 2} = \kappa + \lambda_1 - 2$.

(2) As $\mathtt{T} \in \mathsf{RowStd}(\lambda)$, we know that $\mathtt{T}^{-1}(n) = (1, \lambda_1)$, $(2, \lambda_2)$ or $(3, \lambda_3)$. Further, by the same reasoning as in (1), $\kappa + \lambda_1 - 1 > 0$ and $\mathtt{T}^{-1}(n-1) \neq (1, \lambda_1)$ hold if $\mathtt{T}^{-1}(n) = (2, \lambda_2)$ or $(3, \lambda_3)$.

Suppose that $\mathtt{T}^{-1}(n) = (2, \lambda_2)$. Then

$$\kappa + \lambda_2 - 2 = -(\kappa + \lambda_1 - 1) < 0$$

and $\mathtt{T}^{-1}(n-1) = (2, \lambda_2 - 1)$ or $(3, \lambda_3)$. Thus, $\mathrm{res}(\mathtt{T}^{-1}(n-1)) \geqslant \kappa + \lambda_1$, which is not equal to $\overline{\kappa + \lambda_1 - 2} = \kappa + \lambda_1 - 2$.

Now suppose that $\mathtt{T}^{-1}(n) = (3, \lambda_3)$. Then

$$\kappa + \lambda_3 - 3 = -(\kappa + \lambda_1 - 1) < 0$$

and $\mathtt{T}^{-1}(n-1) = (2, \lambda_2)$ or $(3, \lambda_3 - 1)$. Then $\mathrm{res}(\mathtt{T}^{-1}(n-1)) = \kappa + \lambda_1 - 2 < \mathrm{res}(\mathtt{T}^{-1}(n))$ implies that $\mathtt{T}^{-1}(n-1) = (2, \lambda_2)$. In particular, $\mathtt{T}^{-1}(n-2) = (1, \lambda_1)$, $(2, \lambda_2 - 1)$ or $(3, \lambda_3 - 1)$.

If $\kappa + \lambda_1 - 1 = 1$, then $\mathrm{res}(\mathtt{T}^{-1}(n-1)) = 0$ by condition (ii). It follows that $\kappa + \lambda_2 - 2 = 0$ and $\kappa + \lambda_3 - 3 = -1$, so that $\lambda_1 = \lambda_2 = \lambda_3$. But this contradicts condition (iii).

If $\kappa + \lambda_1 - 1 \geqslant 2$, then, since $\mathrm{res}(\mathtt{T}^{-1}(n-1)) = \kappa + \lambda_1 - 2$, we have one of the following.

(a) $\kappa + \lambda_2 - 2 = \kappa + \lambda_1 - 2 > 0$ and $\lambda_1 = \lambda_2$.

(b) $\kappa + \lambda_2 - 2 = -(\kappa + \lambda_1 - 2) < 0$ and $\lambda_3 = \lambda_2$.

Suppose that we are in case (a). Then

$$\mathrm{res}(1, \lambda_2) = \mathrm{res}(2, \lambda_2) + 1 = \mathrm{res}(\mathtt{T}^{-1}(n-1)) + 1 = \kappa + \lambda_1 - 1 = \mathrm{res}(\mathtt{T}^{-1}(n)) = \mathrm{res}(3, \lambda_3),$$

which contradicts condition (iii). Suppose that we are in case (b). Then none of $(1, \lambda_1)$, $(2, \lambda_2 - 1)$ or $(3, \lambda_3 - 1)$ have residue $\kappa + \lambda_1 - 3$. □

Lemma 4.4. *Let $A = (r, c)$ be a Garnir node of $[\lambda]$. Then there is no tableau $\mathtt{T} \in \mathsf{RowStd}(\lambda)$ such that*

$$\mathtt{T} \rhd \mathtt{G}^A \quad \textit{and} \quad \mathrm{res}(\mathtt{T}) = \mathrm{res}(\mathtt{G}^A).$$

Proof. We take $\mathtt{T} \in \mathsf{RowStd}(\lambda)$ such that

$$\mathtt{T} \trianglerighteq \mathtt{G}^A \quad \text{and} \quad \mathrm{res}(\mathtt{T}) = \mathrm{res}(\mathtt{G}^A).$$

Note that Lemma 1.8 implies that

$$\mathtt{T}(1, k) = k \qquad \text{for } k = 1, 2, \ldots, c-1. \tag{4.1}$$

Thus, we may assume that $\lambda = (\lambda_1, c)$ and $r = 1$. As $\mathrm{res}(\mathtt{T}) = \mathrm{res}(\mathtt{G}^A)$, we have $\mathtt{T}(1, \lambda_1) = n$ by Lemma 4.3(1).

If $\lambda_1 = c$, then we have $\mathtt{T} = \mathtt{G}^A$ by (4.1).

If $\lambda_1 > c$, then we have

$$\mathtt{T}_{\downarrow n-1} \trianglerighteq \mathtt{G}^A_{\downarrow n-1} \quad \text{and} \quad \mathrm{res}(\mathtt{T}_{\downarrow n-1}) = \mathrm{res}(\mathtt{G}^A_{\downarrow n-1}),$$

which, by induction on $\lambda_1 - c$, implies that $\mathtt{T}_{\downarrow n-1} = \mathtt{G}^A_{\downarrow n-1}$. Therefore, we have $\mathtt{T} = \mathtt{G}^A$. $\square$

Lemma 4.5. *Let $A = (r, c)$ and $B = (r', c')$ be Garnir nodes of $[\lambda]$ with $c \leqslant c'$. Suppose that either $(r \neq r' + 1)$ or $(r = r' + 1$ and $\mathrm{res}(B) \neq \mathrm{res}(r + 1, c))$. Then there is no tableau $\mathtt{T} \in \mathsf{RowStd}(\lambda)$ such that*

$$\mathtt{T} \rhd \mathtt{G}^{A,B} \quad \textit{and} \quad \mathrm{res}(\mathtt{T}) = \mathrm{res}(\mathtt{G}^{A,B}).$$

Proof. Let $\mathtt{T} \in \mathsf{RowStd}(\lambda)$ such that

$$\mathtt{T} \trianglerighteq \mathtt{G}^{A,B} \quad \text{and} \quad \mathrm{res}(\mathtt{T}) = \mathrm{res}(\mathtt{G}^{A,B}),$$

and $\mathbf{B}^A$ and $\mathbf{B}^B$ the Garnir belts corresponding to A and B respectively. If $\mathbf{B}^A \cap \mathbf{B}^B = \emptyset$, then we may argue as in the proof of Lemma 4.4 to see that $\mathtt{T} = \mathtt{G}^{A,B}$. So we assume that $\mathbf{B}^A \cap \mathbf{B}^B \neq \emptyset$. Then, we have two cases – either $r = r'$, or $r = r' + 1$.

First suppose that $r = r'$. By Lemma 1.8, we may assume that $\lambda = (\lambda_1, c')$, $c \neq c'$ and $r = r' = 1$. Since $\mathrm{res}(\mathtt{T}) = \mathrm{res}(\mathtt{G}^{A,B})$, we have $\mathtt{T}(1, \lambda_1) = n$ by Lemma 4.3(1).

Suppose that $\lambda_1 = c'$ and consider the condition

$$\mathtt{T}_{\downarrow n-p} \trianglerighteq \mathtt{G}^{A,B}_{\downarrow n-p} \quad \text{and} \quad \mathrm{res}(\mathtt{T}_{\downarrow n-p}) = \mathrm{res}(\mathtt{G}^{A,B}_{\downarrow n-p}),$$

where $p = c' - c + 1$. As $\mathtt{G}^{A,B}_{\downarrow n-p}$ is a Garnir tableau with Garnir node A, we conclude that $\mathtt{T}_{\downarrow n-p} = \mathtt{G}^{A,B}_{\downarrow n-p}$ by Lemma 4.4. Since $\mathtt{T} \in \mathsf{RowStd}(\lambda)$ and $\mathtt{T}^{-1}(n) = (1, \lambda_1)$, it follows that the entries $n - p + 1, \ldots, n - 1$ must appear in the nodes $(2, c + 1), \ldots, (2, c')$ respectively, and thus $\mathtt{T} = \mathtt{G}^{A,B}$.

If $\lambda_1 > c'$, then

$$\mathtt{T}_{\downarrow n-1} \trianglerighteq \mathtt{G}^{A,B}_{\downarrow n-1} \quad \text{and} \quad \mathrm{res}(\mathtt{T}_{\downarrow n-1}) = \mathrm{res}(\mathtt{G}^{A,B}_{\downarrow n-1}),$$

which, by induction on $\lambda_1 - c'$, implies that $\mathtt{T}_{\downarrow n-1} = \mathtt{G}^{A,B}_{\downarrow n-1}$. Thus we have $\mathtt{T} = \mathtt{G}^{A,B}$.

Next, suppose that $r = r' + 1$. By Lemma 1.8, we may assume that $\lambda = (\lambda_1, \lambda_2, c)$ and $r' = 1$. Note that $\mathrm{res}(1, c') \neq \mathrm{res}(3, c)$ by our assumption and $\mathtt{T}(1, k) = k$ for $k = 1, 2, \ldots, c'-1$ by Lemma 1.8. We now proceed by induction on $\lambda_2 - c'$.

First, suppose that $\lambda_2 = c'$. We must have $\mathtt{T}(1, \lambda_1) = n$, by Lemma 4.3(2).

If $\lambda_1 = c'$, then we define row-strict tableaux $\mathtt{T}'$ and $\mathtt{G}'$ of shape (λ_2, c) by

$$\mathtt{T}'(i,j) = \mathtt{T}(i+1, j) - c' + 1, \qquad \mathtt{G}'(i,j) = \mathtt{G}^{A,B}(i+1, j) - c' + 1.$$

Then $\mathtt{G}'$ becomes a Garnir tableau with Garnir node A and

$$\mathtt{T}' \trianglerighteq \mathtt{G}' \quad \text{and} \quad \mathrm{res}(\mathtt{T}') = \mathrm{res}(\mathtt{G}'),$$

which implies that $\mathtt{T}' = \mathtt{G}'$ by Lemma 4.4. Thus we have $\mathtt{T} = \mathtt{G}^{A,B}$.

If $\lambda_1 > c'$, then

$$\mathtt{T}_{\downarrow n-1} \trianglerighteq \mathtt{G}^{A,B}_{\downarrow n-1} \quad \text{and} \quad \mathrm{res}(\mathtt{T}_{\downarrow n-1}) = \mathrm{res}(\mathtt{G}^{A,B}_{\downarrow n-1}),$$

which implies that $\mathtt{T}_{\downarrow n-1} = \mathtt{G}^{A,B}_{\downarrow n-1}$ by induction on $\lambda_1 - c'$. Thus we have $\mathtt{T} = \mathtt{G}^{A,B}$.

Now suppose that $\lambda_2 > c'$. Then n is located in the node $(2, \lambda_2)$ in $\mathtt{G}^{A,B}$. Thus $\mathtt{T}^{-1}(n) = (2, \lambda_2)$ or $(3, c)$ since $\mathtt{T} \trianglerighteq \mathtt{G}^{A,B}$. Suppose that $\mathtt{T}^{-1}(n) = (3, c)$. Since $\mathrm{res}(\mathtt{T}) = \mathrm{res}(\mathtt{G}^{A,B})$, we have $\mathrm{res}(3, c) = \mathrm{res}(2, \lambda_2)$ and therefore $\kappa + \lambda_2 - 2 > 0$ and $\kappa + c - 3 < 0$.

If $\lambda_2 = c' + 1$, then $\mathrm{res}(3, c) = \mathrm{res}(2, \lambda_2) = \mathrm{res}(1, c')$ which is a contradiction.

If $\lambda_2 > c' + 1$, then we have $\mathrm{res}(\mathtt{T}^{-1}(n-1)) = \kappa + \lambda_2 - 3 = \mathrm{res}(\mathtt{T}^{-1}(n)) - 1$. Then $\kappa + c - 3 < 0$ implies that $\mathtt{T}^{-1}(n-1)$ cannot be in the third row, so that $\mathtt{T}^{-1}(n-1) = (1, \lambda_1)$ or $(2, \lambda_2)$. But then $\mathrm{res}(\mathtt{T}^{-1}(n-1)) \neq \kappa + \lambda_2 - 3$, another contradiction. Therefore we must have $\mathtt{T}^{-1}(n) = (2, \lambda_2)$.

Thus we have

$$\mathtt{T}_{\downarrow n-1} \trianglerighteq \mathtt{G}^{A,B}_{\downarrow n-1} \quad \text{and} \quad \mathrm{res}(\mathtt{T}_{\downarrow n-1}) = \mathrm{res}(\mathtt{G}^{A,B}_{\downarrow n-1}),$$

which implies, by induction on $\lambda_2 - c'$, that $\mathtt{T}_{\downarrow n-1} = \mathtt{G}^{A,B}_{\downarrow n-1}$. We conclude that $\mathtt{T} = \mathtt{G}^{A,B}$. $\square$

4.1. **A lemma for block braid relations.**

Lemma 4.6. *Let $\nu = \nu^1 * \nu^2 * \nu^3$ where $a_1, a_3 \geqslant 1$ and*

$$\begin{aligned}
\nu^1 &= (i+1, i+2, \ldots, i+a_1),\\
\nu^2 &= (i, i-1, \ldots, 1, 0, 1, \ldots, i-1, i),\\
\nu^3 &= (i+a_3, i+a_3-1, \ldots, i+1),
\end{aligned}$$

for some $i \geqslant 0$. We set $a_2 = 2i+1$ and $\underline{a} = (1, a_1-1, a_2, a_3-1, 1)$. Then

$$\Psi_2\Psi_1\Psi_2(a_1, a_2, a_3)e(\nu) - \Psi_1\Psi_2\Psi_1(a_1, a_2, a_3)e(\nu)$$

is given as follows.

(1) Suppose $i \neq 0$. Then it is equal to

$$\Psi_1\Psi_4\Psi_2\Psi_3\Psi_2(\underline{a})(x_1 + x_{a_1+1} + x_{a_1+a_2} + x_{a_1+a_2+a_3})e(\nu).$$

(2) Suppose $i = 0$. Then it is equal to $\Psi_1\Psi_4\Psi_2\Psi_3\Psi_2(\underline{a})(x_1 + x_{a_1+a_3+1})e(\nu)$.

Proof. Following Remark 2.17, we compute

$$\begin{aligned}\Psi_2\Psi_1\Psi_2(a_1, a_2, a_3)e(\nu) &- \Psi_1\Psi_2\Psi_1(a_1, a_2, a_3)e(\nu)\\ &= \sum_{k=1}^{a_2} \Psi_2(a_3+k, a_1, a_2-k)X_k\Psi[a_1, k-1]e(\nu),\end{aligned}$$

where $X_k = \Psi_2(a_3+k-1, a_1, 1)\Psi[a_1+a_2, a_3] - \Psi[a_1+a_2, a_3]\Psi_2(k-1, a_1, 1)$.

Then, Lemma 2.16 tells that the term $\Psi_2(a_3+k, a_1, a_2-k)X_k\Psi[a_1, k-1]e(\nu)$ survives only if for some $1 \leqslant s \leqslant a_1$ and $1 \leqslant t \leqslant a_3$,

- the sth entry of $\nu^1 = (i+1, i+2, \ldots, i+a_1)$,
- the kth entry of $\nu^2 = (i, \ldots, 0 \ldots, i)$,
- the tth entry of $\nu^3 = (i+a_3, \ldots, i+1)$

form a triple of the form $(1,0,1)$, $(j, j+1, j)$ for $j \geqslant 0$, or $(j, j-1, j)$ for $j \geqslant 2$. Hence, either $(s,k,t) = (1,1,a_3)$ or $(1, a_2, a_3)$ are possible. Thus, if $i \neq 0$ we insert

$$\begin{aligned}X_1 &= \Psi[2, a_3-1]\Psi_2(2, a_1+a_2-2, a_3)\Psi_2(1, a_1-1, 1),\\ X_{a_2} &= \Psi[a_2-1, a_3]\Psi_2(a_2-1, 2, a_3-1)\Psi_2(a_2+1, a_1-1, a_3)\Psi_2(a_2, a_1-1, 1),\end{aligned}$$

and $X_k = 0$ for $k \neq 1, a_2$, to obtain

$$\begin{aligned}\Psi_2\Psi_1\Psi_2(a_1, a_2, a_3)e(\nu) &- \Psi_1\Psi_2\Psi_1(a_1, a_2, a_3)e(\nu)\\ &= \left(\Psi_2(a_3+1, a_1, a_2-1)X_1 + X_{a_2}\Psi[a_1, a_2-1]\right)e(\nu).\end{aligned}$$

On the other hand, if $i = 0$ we obtain

$$\begin{aligned}\Psi_2\Psi_1\Psi_2(a_1, a_2, a_3)e(\nu) &- \Psi_1\Psi_2\Psi_1(a_1, a_2, a_3)e(\nu)\\ &= (x_{a_3} + x_{a_3+2})\Psi[2, a_3-1]\Psi_2(2, a_1-1, a_3)\Psi_2(1, a_1-1, 1)e(\nu).\end{aligned}$$

(1) Suppose that $i \neq 0$. We write

$$\nu^1 = (i+1) * \nu^a,\ \nu^2 = (i) * \nu^b * (i),\ \nu^3 = \nu^c * (i+1)$$

and let $\nu = (i+1) * \nu^a * (i) * \nu^b * (i) * \nu^c * (i+1)$. Then the first term is

$$\begin{aligned}&\Psi_2(a_3+1, a_1, a_2-1)\Psi[2, a_3-1]\Psi_2(2, a_1+a_2-2, a_3)\Psi_2(1, a_1-1, 1)e(\nu)\\ &= (\Psi_5\Psi_4\Psi_6\Psi_5)(\Psi_1\Psi_2)(\Psi_4\Psi_3\Psi_5\Psi_4\Psi_6\Psi_5)(\Psi_2)(\underline{a})e(\nu),\end{aligned}$$

where $\underline{a} = (1, a_1-1, 1, a_2-2, 1, a_3-1, 1)$. Then, following the recipe in Remark 2.17, we know that there is no error term in $\Psi_5\Psi_4\Psi_5(\underline{b})e(\mu) - \Psi_4\Psi_5\Psi_4(\underline{b})e(\mu)$, so that

$$\begin{aligned}&=(\Psi_1\Psi_2)(\Psi_5\Psi_4\Psi_6)\Psi_5\Psi_4\Psi_5(\underline{b})e(\mu)\Psi_3\Psi_4\Psi_6\Psi_5\Psi_2(\underline{a})\\ &=\Psi_1\Psi_2\Psi_5\Psi_4\Psi_6\Psi_4\Psi_5\Psi_4\Psi_3\Psi_4\Psi_6\Psi_5\Psi_2(\underline{a})e(\nu),\end{aligned}$$

where $\mu = (i+1) * (i) * \nu^c * \nu^a * \nu^b * (i+1) * (i)$ and $\underline{b} = (1, 1, a_3-1, a_1-1, a_2-2, 1, 1)$. Then, Lemma 2.12(1) implies

$$\begin{aligned}&=(\Psi_1\Psi_2\Psi_5)\Psi_4^2(\underline{b}')e(\mu')\Psi_6\Psi_5\Psi_4\Psi_3\Psi_4\Psi_6\Psi_5\Psi_2(\underline{a})\\ &=\Psi_1\Psi_2\Psi_5\Psi_6\Psi_5\Psi_4\Psi_3\Psi_4\Psi_6\Psi_5\Psi_2(\underline{a})e(\nu)\end{aligned}$$

where $\mu' = (i+1) * (i) * \nu^c * \nu^b * (i+1) * (i) * \nu^a$ and $\underline{b}' = (1, 1, a_3-1, a_2-2, 1, 1, a_1-1)$. We continue with similar arguments:

$$\begin{aligned}&=(\Psi_1\Psi_2)\Psi_5\Psi_6\Psi_5(\underline{b}'')e(\mu'')\Psi_4\Psi_3\Psi_4\Psi_6\Psi_5\Psi_2(\underline{a})\\ &=\Psi_1\Psi_2\Psi_6\Psi_5\Psi_6\Psi_4\Psi_3\Psi_4\Psi_6\Psi_5\Psi_2(\underline{a})e(\nu)\end{aligned}$$

where $\mu'' = (i+1) * (i) * \nu^c * \nu^b * \nu^a * (i+1) * (i)$ and $\underline{b}'' = (1, 1, a_3-1, a_2-2, a_1-1, 1, 1)$,

$$\begin{aligned}&=(\Psi_1\Psi_2\Psi_6\Psi_5\Psi_4\Psi_3\Psi_4)\Psi_6^2(\underline{b}''')e(\mu''')\Psi_5\Psi_2(\underline{a})\\ &=\Psi_1\Psi_2\Psi_6\Psi_5\Psi_4\Psi_3\Psi_4(x_{a_1+a_2+a_3-1} + x_{a_1+a_2+a_3})\Psi_5\Psi_2(\underline{a})e(\nu)\\ &=\Psi_1\Psi_2\Psi_6\Psi_5\Psi_4\Psi_3\Psi_4\Psi_5\Psi_2(\underline{a})(x_{a_1+a_2} + x_{a_1+a_2+a_3})e(\nu)\\ &=(\Psi_1\Psi_2\Psi_6\Psi_5)\Psi_4\Psi_3\Psi_4(\underline{b}''')e(\mu''')\Psi_5\Psi_2(\underline{a})(x_{a_1+a_2} + x_{a_1+a_2+a_3})\\ &=\Psi_1\Psi_2\Psi_6\Psi_5\Psi_3\Psi_4\Psi_3\Psi_5\Psi_2(\underline{a})(x_{a_1+a_2} + x_{a_1+a_2+a_3})e(\nu)\\ &=\Psi_1\Psi_6\Psi_2\Psi_3\Psi_5\Psi_4\Psi_5\Psi_3\Psi_2(\underline{a})(x_{a_1+a_2} + x_{a_1+a_2+a_3})e(\nu),\end{aligned}$$

where $\mu''' = (i+1) * (i) * \nu^a * \nu^b * \nu^c * (i) * (i+1)$ and $\underline{b}''' = (1, 1, a_1-1, a_2-2, a_3-1, 1, 1)$, and after one more step, we obtain

$$\Psi_1\Psi_6\Psi_2\Psi_3\Psi_4\Psi_5\Psi_4\Psi_3\Psi_2(\underline{a})(x_{a_1+a_2} + x_{a_1+a_2+a_3})e(\nu).$$

Then, we can check that this is equal to $\Psi_1\Psi_4\Psi_2\Psi_3\Psi_2(\underline{a})(x_{a_1+a_2}+x_{a_1+a_2+a_3})e(\nu)$ if we change $\underline{a}$ to $\underline{a} = (1, a_1-1, a_2, a_3-1, 1)$.

Next, we consider the second term

$$\Psi[a_2-1, a_3]\Psi_2(a_2-1, 2, a_3-1)\Psi_2(a_2+1, a_1-1, a_3)\Psi_2(a_2, a_1-1, 1)\Psi[a_1, a_2-1]e(\nu)$$

$$= (\Psi_2\Psi_1\Psi_3\Psi_2)(\Psi_3\Psi_4)(\Psi_6\Psi_5)(\Psi_4)(\Psi_2\Psi_1\Psi_3\Psi_2)(\underline{a})e(\nu),$$

where $\underline{a} = (1, a_1 - 1, 1, a_2 - 2, 1, a_3 - 1, 1)$. Then

$$\begin{aligned}
&=(\Psi_2\Psi_1)\Psi_3\Psi_2\Psi_3(\underline{b})e(\mu)\Psi_4\Psi_6\Psi_5\Psi_4\Psi_2\Psi_1\Psi_3\Psi_2(\underline{a})\\
&=\Psi_2\Psi_1\Psi_2\Psi_3\Psi_2\Psi_4\Psi_6\Psi_5\Psi_4\Psi_2\Psi_1\Psi_3\Psi_2(\underline{a})e(\nu),
\end{aligned}$$

where $\mu = (i) * \nu^b * (i+1) * \nu^c * (i) * (i+1) * \nu^a$ and $\underline{b} = (1, a_2 - 2, 1, a_3 - 1, 1, 1, a_1 - 1)$,

$$\begin{aligned}
&=(\Psi_2\Psi_1\Psi_2\Psi_3\Psi_4\Psi_6\Psi_5\Psi_4)\Psi_2^2(\underline{b}')e(\mu')\Psi_1\Psi_3\Psi_2(\underline{a})\\
&=\Psi_2\Psi_1\Psi_2\Psi_3\Psi_4\Psi_6\Psi_5\Psi_4\Psi_1\Psi_3\Psi_2(\underline{a})e(\nu),
\end{aligned}$$

where $\mu' = (i) * (i+1) * \nu^b * \nu^a * (i) * \nu^c * (i+1)$ and $\underline{b}' = (1, 1, a_2 - 2, a_1 - 1, 1, a_3 - 1, 1)$,

$$\begin{aligned}
&=\Psi_2\Psi_1\Psi_2(\underline{b}'')e(\mu'')\Psi_3\Psi_4\Psi_6\Psi_5\Psi_4\Psi_1\Psi_3\Psi_2(\underline{a})\\
&=\Psi_1\Psi_2\Psi_1\Psi_3\Psi_4\Psi_6\Psi_5\Psi_4\Psi_1\Psi_3\Psi_2(\underline{a})e(\nu),
\end{aligned}$$

where $\mu'' = (i) * (i+1) * \nu^c * \nu^b * (i) * (i+1) * \nu^a$ and $\underline{b}'' = (1, 1, a_3 - 1, a_2 - 2, 1, 1, a_1 - 1)$,

$$\begin{aligned}
&=(\Psi_1\Psi_2\Psi_3\Psi_4\Psi_6\Psi_5\Psi_4)\Psi_1^2(\underline{b}''')e(\mu''')\Psi_3\Psi_2(\underline{a})\\
&=\Psi_1\Psi_2\Psi_3\Psi_4\Psi_6\Psi_5\Psi_4(x_1 + x_2)\Psi_3\Psi_2(\underline{a})e(\nu)\\
&=\Psi_6\Psi_1\Psi_2\Psi_3\Psi_4\Psi_5\Psi_4\Psi_3\Psi_2(\underline{a})(x_1 + x_{a_1+1})e(\nu),
\end{aligned}$$

where $\mu''' = (i+1)*(i)*\nu^b*\nu^a*(i)*\nu^c*(i+1)$ and $\underline{b}''' = (1, 1, a_2-2, a_1-1, 1, a_3-1, 1)$. Then this is equal to $\Psi_1\Psi_4\Psi_2\Psi_3\Psi_2(\underline{a})(x_1+x_{a_1+1})e(\nu)$ if we change $\underline{a}$ to $\underline{a} = (1, a_1-1, a_2, a_3-1, 1)$.

(2) Suppose that $i = 0$. We write $\nu^1 = (1) * \nu^a,\ \nu^2 = (0),\ \nu^3 = \nu^c * (1)$ as before, and let $\nu = (1) * \nu^a * (0) * \nu^c * (1)$ and $\underline{a} = (1, a_1 - 1, a_3 - 1, 1)$. Then

$$\begin{aligned}
&(x_{a_3} + x_{a_3+2})\Psi[2, a_3 - 1]\Psi_2(2, a_1 - 1, a_3)\Psi_2(1, a_1 - 1, 1)e(\nu)\\
&\qquad = (x_{a_3} + x_{a_3+2})\Psi_1\Psi_2\Psi_4\Psi_3\Psi_2(\underline{a})e(\nu) = \Psi_1\Psi_4\Psi_2\Psi_3\Psi_2(\underline{a})(x_1 + x_{a_1+a_3+1})e(\nu),
\end{aligned}$$

which is the desired result. □

4.2. **A special three row case.** To handle the case that the Garnir belt of $\mathtt{G}^{A,B}$ has three rows, we may assume that $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_3 > 0$ and that Garnir nodes are $A = (2, c)$, $B = (1, c')$ with $c \leqslant c'$. In this subsection, we consider a special case, that is, we assume

(i) The first row of the Garnir belt has residues $\nu^1_r = (i+1, i+2, \ldots, \lambda_1 - c' + i + 1)$ from left to right.

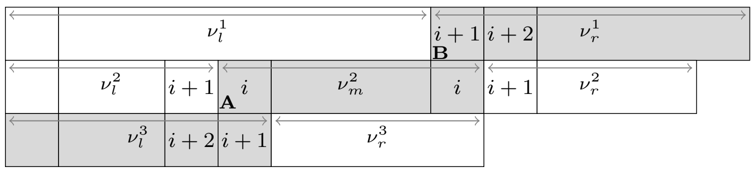
(ii) The second row of the Garnir belt has residues $\nu^2_m = (i, i-1, \ldots, 1, 0, 1, \ldots, i-1, i)$ from left to right.

(iii) The third row of the Garnir belt has residues $\nu^3_l = (c+i, c+i-1, \ldots, i+1)$ from left to right.

In particular, $\operatorname{res}(A) = i$ and $\operatorname{res}(B) = i+1$. We denote

(i) the residues of the first row of λ by $\nu_l^1 * \nu_r^1$,
(ii) the residues of the second row of λ by $\nu_l^2 * \nu_m^2 * \nu_r^2$,
(ii) the residues of the third row of λ by $\nu_l^3 * \nu_r^3$,

respectively. Pictorially, if $c \neq 1$ and $c' \neq \lambda_2$ then



Recall from Lemma 1.18 that there are fully commutative elements w^A and w^B such that

$$\mathtt{G}^{A,B} = w^A \mathtt{G}^A = w^B \mathtt{G}^B,$$
$$\ell(\mathtt{G}^{A,B}) = \ell(w^A) + \ell(\mathtt{G}^A) = \ell(w^B) + \ell(\mathtt{G}^B).$$

We will show that $\psi_{w^A} g_A^\lambda \equiv \psi_{w^B} g_B^\lambda \pmod{\mathcal{G}^\lambda_{<\ell(\mathtt{G}^{A,B})}}$ in Lemma 4.11. We build up to this with several smaller lemmas, as the calculation is quite lengthy.

We denote the length of $\nu_l^1, \nu_r^1, \nu_l^2, \nu_m^2, \nu_r^2, \nu_l^3, \nu_r^3$ by $a_l^1, a_r^1, a_l^2, a_m^2, a_r^2, a_l^3, a_r^3$, respectively, and define

$$\underline{a} = (a_l^1, a_r^1, a_l^2, a_m^2, a_r^2, a_l^3, a_r^3),$$
$$\underline{a}' = (a_l^1, a_r^1, a_l^2, a_l^3, a_m^2, a_r^2, a_r^3),$$
$$\underline{a}'' = (a_l^1, a_l^2, a_m^2, a_r^1, a_r^2, a_l^3, a_r^3),$$
$$\underline{a}''' = (a_l^1, a_l^2, a_r^1, a_m^2, a_l^3, a_r^2, a_r^3).$$

Lemma 4.7. *We have*

$$\psi_{w^A} g_A^\lambda - \psi_{w^B} g_B^\lambda = \Psi_3 \Psi_6 \Psi_4 \Psi_5 \Psi_4(\underline{b}')(x_{a_l^1 + a_l^2 + 1} + x_{a_l^1 + a_r^1 + a_l^2 + a_m^2 + a_l^3}) \Psi_7 \Psi_6 \Psi_2 \Psi_3(\underline{b}) m^\lambda$$

where

$$\underline{b}' = (a_l^1, a_l^2, 1, a_r^1 - 1, a_m^2, a_l^3 - 1, 1, a_r^2, a_r^3),$$
$$\underline{b} = (a_l^1, 1, a_r^1 - 1, a_l^2, a_m^2, a_r^2, a_l^3 - 1, 1, a_r^3).$$

Proof. Note that $a_r^1 \geqslant 1$, $a_m^2 \geqslant 1$, $a_l^3 \geqslant 1$ by definition. Then, by considering the two-line notation for $w^{\mathtt{G}^A}$ and $w^{\mathtt{G}^B}$ as in Lemma 2.8, we know that

$$w^{\mathtt{G}^A} = S_4 S_5(\underline{a}) \quad \text{and} \quad w^{\mathtt{G}^B} = S_3 S_2(\underline{a}).$$

Similarly, we have $w^A = S_4S_3S_2(\underline{a}')$ and $w^B = S_3S_4S_5(\underline{a}'')$. Thus,

$$\begin{aligned}\psi_{w^A} g_A^\lambda - \psi_{w^B} g_B^\lambda &= \Psi_4\Psi_3\Psi_2\Psi_4\Psi_5(\underline{a})m^\lambda - \Psi_3\Psi_4\Psi_5\Psi_3\Psi_2(\underline{a})m^\lambda \\ &= \left(\Psi_4\Psi_3\Psi_4(\underline{a}''') - \Psi_3\Psi_4\Psi_3(\underline{a}''')\right) e(\mu)\Psi_2\Psi_5(\underline{a})m^\lambda,\end{aligned}$$

where $\mu = \nu_l^1 * \nu_l^2 * \nu_r^1 * \nu_m^2 * \nu_l^3 * \nu_r^2 * \nu_r^3$. We apply Lemma 4.6 to compute

$$\Psi_2\Psi_1\Psi_2(a_r^1, a_m^2, a_l^3)e(\nu_r^1 * \nu_m^2 * \nu_l^3) - \Psi_1\Psi_2\Psi_1(a_r^1, a_m^2, a_l^3)e(\nu_r^1 * \nu_m^2 * \nu_l^3).$$

Then the result is as follows.

(i) If $i \neq 0$ then

$$\psi_{w^A} g_A^\lambda - \psi_{w^B} g_B^\lambda = \Psi_3\Psi_6\Psi_4\Psi_5\Psi_4(\underline{b}')(X_1 + X_2)\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda,$$

where $X_1 = x_{a_l^1+a_l^2+1} + x_{a_l^1+a_r^1+a_l^2+a_m^2+a_l^3}$ and $X_2 = x_{a_l^1+a_r^1+a_l^2+1} + x_{a_l^1+a_r^1+a_l^2+a_m^2}$.

(ii) If $i = 0$ then

$$\psi_{w^A} g_A^\lambda - \psi_{w^B} g_B^\lambda = \Psi_3\Psi_6\Psi_4\Psi_5\Psi_4(\underline{b}')X_1\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda,$$

where $X_1 = x_{a_l^1+a_l^2+1} + x_{a_l^1+a_r^1+a_l^2+a_l^3+1}$.

As $\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})$ does not touch the fifth block of $\underline{b}$, if $i \neq 0$ then

$$X_2\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda = \Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})X_2m^\lambda = 0.$$ □

Lemma 4.8. *Let*

$$\begin{aligned}\underline{b} &= (a_l^1, 1, a_r^1 - 1, a_l^2, a_m^2, a_r^2, a_l^3 - 1, 1, a_r^3), \\ \underline{b}' &= (a_l^1, a_l^2, 1, a_r^1 - 1, a_m^2, a_l^3 - 1, 1, a_r^2, a_r^3), \\ \underline{c} &= (a_l^1, 1, a_r^1 - 1, a_l^2 - 1, 1, a_m^2, a_r^2, a_l^3 - 1, 1, a_r^3), \\ \underline{d} &= (a_l^1, 1, a_r^1 - 1, a_l^2, a_m^2, 1, a_r^2 - 1, a_l^3 - 1, 1, a_r^3).\end{aligned}$$

(1) If $a_l^2 = 0$, then

$$\Psi_3\Psi_6\Psi_4\Psi_5\Psi_4(\underline{b}')x_{a_l^1+a_l^2+1}\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda = 0.$$

Otherwise,

$$\begin{aligned}&\Psi_3\Psi_6\Psi_4\Psi_5\Psi_4(\underline{b}')x_{a_l^1+a_l^2+1}\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda \\ &= (\Psi_7\Psi_8\Psi_6\Psi_2)\Psi_5\Psi_4\Psi_3\Psi_5\Psi_6\Psi_7(\underline{c})m^\lambda + (\Psi_7\Psi_8\Psi_6\Psi_2)(\Psi_4\Psi_5\Psi_4 - \Psi_5\Psi_4\Psi_5)\Psi_3\Psi_6\Psi_7(\underline{c})m^\lambda.\end{aligned}$$

(2) If $a_r^2 = 0$, then

$$\Psi_3\Psi_6\Psi_4\Psi_5\Psi_4(\underline{b}')x_{a_l^1+a_r^1+a_l^2+a_m^2+a_l^3}\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda = 0.$$

Otherwise,

$$\Psi_3\Psi_6\Psi_4\Psi_5\Psi_4(\underline{b}')x_{a_l^1+a_r^1+a_l^2+a_m^2+a_l^3}\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda$$
$$= -(\Psi_3\Psi_2\Psi_8\Psi_4)\Psi_5\Psi_6\Psi_7\Psi_5\Psi_4\Psi_3(\underline{d})m^\lambda - (\Psi_3\Psi_2\Psi_8\Psi_4)(\Psi_6\Psi_5\Psi_6 - \Psi_5\Psi_6\Psi_5)\Psi_7\Psi_4\Psi_3(\underline{d})m^\lambda.$$

Proof. First, note that $\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})$ is the product of

$$\Psi_7(a_l^1, a_l^2, 1, a_r^1-1, a_m^2, a_l^3-1, \underline{a_r^2, 1}, a_r^3)e(\nu_l^1 * \nu_l^2 * (i+1) * \dot{\nu}_r^1 * \nu_m^2 * \dot{\nu}_l^3 * \underline{\nu_r^2 * (i+1)} * \nu_r^3)$$
$$\Psi_6(a_l^1, a_l^2, 1, a_r^1-1, a_m^2, \underline{a_r^2, a_l^3-1}, 1, a_r^3)e(\nu_l^1 * \nu_l^2 * (i+1) * \dot{\nu}_r^1 * \nu_m^2 * \underline{\nu_r^2 * \dot{\nu}_l^3} * (i+1) * \nu_r^3)$$
$$\Psi_2(a_l^1, \underline{1, a_l^2}, a_r^1-1, a_m^2, a_r^2, a_l^3-1, 1, a_r^3)e(\nu_l^1 * \underline{(i+1) * \nu_l^2} * \dot{\nu}_r^1 * \nu_m^2 * \nu_r^2 * \dot{\nu}_l^3 * (i+1) * \nu_r^3)$$
$$\Psi_3(a_l^1, 1, \underline{a_r^1-1, a_l^2}, a_m^2, a_r^2, a_l^3-1, 1, a_r^3)e(\nu_l^1 * (i+1) * \underline{\dot{\nu}_r^1 * \nu_l^2} * \nu_m^2 * \nu_r^2 * \dot{\nu}_l^3 * (i+1) * \nu_r^3)$$

in this order, where $\nu_r^1 = (i+1) * \dot{\nu}_r^1$, $\nu_l^3 = \dot{\nu}_l^3 * (i+1)$.

(1) Moving $x_{a_l^1+a_l^2+1}$ to the right,

$$x_{a_l^1+a_l^2+1}\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda = \Psi_7\Psi_6(x_{a_l^1+a_l^2+1}\Psi_2 - \Psi_2 x_{a_l^1+1})\Psi_3(\underline{b})m^\lambda.$$

We apply Lemma 2.12(3) to $(x_{a_l^1+a_l^2+1}\Psi_2 - \Psi_2 x_{a_l^1+1})e(\mu)$, where

$$\mu = \nu_l^1 * (i+1) * \nu_l^2 * \dot{\nu}_r^1 * \nu_m^2 * \nu_r^2 * \dot{\nu}_l^3 * (i+1) * \nu_r^3.$$

If $a_l^2 = 0$ then Ψ_2 and Ψ_3 are the identity and

$$x_{a_l^1+a_l^2+1}\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda = x_{a_l^1+a_l^2+1}\Psi_7\Psi_6(\underline{b})m^\lambda = \Psi_7\Psi_6(\underline{b})x_{a_l^1+a_l^2+1}m^\lambda = 0.$$

Now suppose that $a_l^2 \geqslant 1$. Note that $a_l^3 = a_l^2 + 1 \geqslant 2$. We write

$$\nu = \nu_l^1 * (i+1) * \dot{\nu}_r^1 * \dot{\nu}_l^2 * (i+1) * \nu_m^2 * \nu_r^2 * \dot{\nu}_l^3 * (i+1) * \nu_r^3.$$

Then, after applying Lemma 2.12(3), we obtain

$$\Psi_3\Psi_6\Psi_4\Psi_5\Psi_4(\underline{b}')x_{a_l^1+a_l^2+1}\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda = \Psi_4\Psi_7\Psi_5\Psi_6\Psi_5\Psi_8\Psi_7\Psi_2\Psi_4\Psi_3(\underline{c})m^\lambda.$$

Then we can continue as follows.

$$\begin{aligned} &=(\Psi_4\Psi_7)\Psi_8(\Psi_5\Psi_6\Psi_5)(\Psi_2\Psi_4\Psi_3)\Psi_7(\underline{c})m^\lambda \\ &=(\Psi_4\Psi_7\Psi_8)\Psi_5\Psi_6\Psi_5(\underline{c}')e(\nu')\Psi_2\Psi_4\Psi_3\Psi_7(\underline{c})m^\lambda, \end{aligned}$$

where $\nu' = \nu_l^1 * \dot{\nu}_l^2 * (i+1) * (i+1) * \dot{\nu}_r^1 * \nu_m^2 * \dot{\nu}_l^3 * \nu_r^2 * (i+1) * \nu_r^3$ and $\underline{c}' = (a_l^1, a_l^2-1, 1, 1, a_r^1-1, a_m^2, a_l^3-1, a_r^2, 1, a_r^3)$, so that $\Psi_5\Psi_6\Psi_5(\underline{c}')e(\nu') = \Psi_6\Psi_5\Psi_6(\underline{c}')e(\nu')$ and we have

$$\begin{aligned} \Psi_4\Psi_7\Psi_8\Psi_5\Psi_6\Psi_5\Psi_2\Psi_4\Psi_3\Psi_7(\underline{c})m^\lambda &= \Psi_4\Psi_7\Psi_8\Psi_6\Psi_5\Psi_6\Psi_2\Psi_4\Psi_3\Psi_7(\underline{c})m^\lambda \\ &= \Psi_7\Psi_8\Psi_6\Psi_2(\Psi_4\Psi_5\Psi_4)\Psi_3\Psi_6\Psi_7(\underline{c})m^\lambda. \end{aligned}$$

Hence, $\Psi_4\Psi_7\Psi_8\Psi_5\Psi_6\Psi_5\Psi_2\Psi_4\Psi_3\Psi_7(\underline{c})m^\lambda$ is equal to

$$(\Psi_7\Psi_8\Psi_6\Psi_2)\Psi_5\Psi_4\Psi_3\Psi_5\Psi_6\Psi_7(\underline{c})m^\lambda + (\Psi_7\Psi_8\Psi_6\Psi_2)(\Psi_4\Psi_5\Psi_4 - \Psi_5\Psi_4\Psi_5)\Psi_3\Psi_6\Psi_7(\underline{c})m^\lambda.$$

(2) Similarly, if we move $x_{a_l^1+a_r^1+a_l^2+a_m^2+a_l^3}$ to the right,

$$x_{a_l^1+a_r^1+a_l^2+a_m^2+a_l^3}\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda$$
$$= \Psi_2\Psi_3(x_{a_l^1+a_r^1+a_l^2+a_m^2+a_l^3}\Psi_7 - \Psi_7 x_{a_l^1+a_r^1+a_l^2+a_m^2+a_r^2+a_l^3})\Psi_6(\underline{b})m^\lambda,$$

and we apply Lemma 2.12(4) to $(x_{a_l^1+a_r^1+a_l^2+a_m^2+a_l^3}\Psi_7 - \Psi_7 x_{a_l^1+a_r^1+a_l^2+a_m^2+a_r^2+a_l^3})e(\mu)$, where

$$\mu = \nu_l^1 * (i+1) * \dot{\nu}_r^1 * \nu_l^2 * \nu_m^2 * \dot{\nu}_l^3 * \nu_r^2 * (i+1) * \nu_r^3.$$

If $a_r^2 = 0$ then Ψ_6 and Ψ_7 are the identity and

$$x_{a_l^1+a_r^1+a_l^2+a_m^2+a_l^3}\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda = \Psi_2\Psi_3(\underline{b})x_{a_l^1+a_r^1+a_l^2+a_m^2+a_l^3}m^\lambda = 0.$$

Now suppose that $a_r^2 \geqslant 1$. We write $\nu = \nu_l^1*(i+1)*\dot{\nu}_r^1*\nu_l^2*\nu_m^2*(i+1)*\dot{\nu}_r^2*\dot{\nu}_l^3*(i+1)*\nu_r^3$. Then, after applying Lemma 2.12(4), we obtain

$$\Psi_3\Psi_6\Psi_4\Psi_5\Psi_4(\underline{b}')x_{a_l^1+a_r^1+a_l^2+a_m^2+a_l^3}\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda$$
$$= -\Psi_3\Psi_6\Psi_4\Psi_5\Psi_4\Psi_8\Psi_6\Psi_7\Psi_2\Psi_3(\underline{d})m^\lambda$$
$$= -(\Psi_3\Psi_2\Psi_8\Psi_4)(\Psi_6\Psi_5\Psi_6)\Psi_7\Psi_4\Psi_3(\underline{d})m^\lambda.$$

Hence, $-(\Psi_3\Psi_2\Psi_8\Psi_4)(\Psi_6\Psi_5\Psi_6)\Psi_7\Psi_4\Psi_3(\underline{d})m^\lambda$ is equal to

$$-(\Psi_3\Psi_2\Psi_8\Psi_4)\Psi_5\Psi_6\Psi_7\Psi_5\Psi_4\Psi_3(\underline{d})m^\lambda - (\Psi_3\Psi_2\Psi_8\Psi_4)(\Psi_6\Psi_5\Psi_6 - \Psi_5\Psi_6\Psi_5)\Psi_7\Psi_4\Psi_3(\underline{d})m^\lambda. \quad \square$$

Lemma 4.9. *Let $\underline{c}$ and $\underline{d}$ be as in Lemma 4.8. Then*

$$S_7S_8S_6S_2S_5S_4S_3S_5S_6S_7(\underline{c}) \quad \text{and} \quad S_3S_2S_8S_4S_5S_6S_7S_5S_4S_3(\underline{d})$$

are reduced.

Proof. By Corollary 2.9, it suffices to check that they have length 10 if $\underline{c} = \underline{d} = (1,\dots,1)$, which is easy to check. $\square$

Lemma 4.10. *Let $A' = (2, c-1)$, $B' = (1, c'+1)$. Then we have the following.*

(1) Let $a_l^2 \geqslant 1$ and $a_r^2 = 0$. Then

$$\Psi_3\Psi_6\Psi_4\Psi_5\Psi_4(\underline{b}')x_{a_l^1+a_l^2+1}\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda = \psi_w g^\lambda_{A'}$$

for some reduced expression of $w \in \mathfrak{S}_n$ with $\ell(w\mathtt{G}^{A'}) = \ell(w) + \ell(\mathtt{G}^{A'}) < \ell(\mathtt{G}^{A,B})$.

(2) Let $a_r^2 \geqslant 1$ and $a_l^2 = 0$. Then

$$\Psi_3\Psi_6\Psi_4\Psi_5\Psi_4(\underline{b}')x_{a_l^1+a_r^1+a_l^2+a_m^2+a_l^3}\Psi_7\Psi_6\Psi_2\Psi_3(\underline{b})m^\lambda = -\psi_w g^\lambda_{B'}$$

for some reduced expression of $w \in \mathfrak{S}_n$ with $\ell(w\mathtt{G}^{B'}) = \ell(w) + \ell(\mathtt{G}^{B'}) < \ell(\mathtt{G}^{A,B})$.

Proof. The result follows readily if we can show that the second term in each of Lemma 4.8(1) and (2) are zero under the corresponding conditions, since $g^\lambda_{A'} = \Psi_5\Psi_6\Psi_7(\underline{c})m^\lambda$ and $g^\lambda_{B'} = \Psi_5\Psi_4\Psi_3(\underline{d})m^\lambda$. (We note that under the corresponding hypotheses, $\Psi_7(\underline{c})$ in (1) and $\Psi_3(\underline{d})$ in (2) are the identity.)

(1) For the second term from Lemma 4.8(1), it suffices to consider

$$(\Psi_4\Psi_5\Psi_4 - \Psi_5\Psi_4\Psi_5)\Psi_3\Psi_6\Psi_7(\underline{c})m^\lambda = (\Psi_4\Psi_5\Psi_4 - \Psi_5\Psi_4\Psi_5)(\underline{c}')e(\nu')\Psi_3\Psi_6\Psi_7(\underline{c})m^\lambda,$$

where $\nu' = \nu^1_l * (i+1) * \dot{\nu}^2_l * \dot{\nu}^1_r * (i+1) * \dot{\nu}^3_l * \nu^2_m * \nu^2_r * (i+1) * \nu^3_r$ and

$$\underline{c}' = (a^1_l, 1, a^2_l - 1, a^1_r - 1, 1, a^3_l - 1, a^2_m, a^2_r, 1, a^3_r).$$

For this, we need to compute

$$(\Psi_2\Psi_1\Psi_2 - \Psi_1\Psi_2\Psi_1)(a^1_r - 1, 1, a^3_l - 1)e(\dot{\nu}^1_r * (i+1) * \dot{\nu}^3_l).$$

When $a^1_r = 1$, this error term is zero, and we are done. So we may assume that $a^1_r \geqslant 2$ and continue as follows. The above expression is equal to

$$\Psi[2, a^3_l - 2]\Psi_2(2, a^1_r - 2, a^3_l - 1)\Psi_2(1, a^1_r - 2, 1)e(\dot{\nu}^1_r * (i+1) * \dot{\nu}^3_l)$$

by Lemma 2.16, but we need a different formula here: first, we apply Lemma 2.16 to obtain

$$\begin{aligned}(\Psi_2\Psi_1\Psi_2 - \Psi_1\Psi_2\Psi_1)&(a^3_l - 1, 1, a^1_r - 1)e(\dot{\nu}^3_l * (i+1) * \dot{\nu}^1_r)\\ &= \Psi_2(a^1_r - 1, a^3_l - 2, 1)\Psi[a^3_l - 2, a^1_r - 1]\Psi_2(a^3_l - 1, 2, a^1_r - 2)e(\dot{\nu}^3_l * (i+1) * \dot{\nu}^1_r).\end{aligned}$$

Then, we apply the anti-involution of $R(\beta)$ to obtain

$$\begin{aligned}(\Psi_2\Psi_1\Psi_2 - \Psi_1\Psi_2\Psi_1)&(a^1_r - 1, 1, a^3_l - 1)e(\dot{\nu}^1_r * (i+1) * \dot{\nu}^3_l)\\ &= \Psi_2(a^3_l - 1, a^1_r - 2, 2)\Psi[a^1_r - 1, a^3_l - 2]\Psi_2(a^1_r - 1, 1, a^3_l - 2)e(\dot{\nu}^1_r * (i+1) * \dot{\nu}^3_l).\end{aligned}$$

We use this formula to compute the second term. To state the result in this case, we change $\underline{c}$ to

$$\underline{c} = (a^1_l, 1, 1, a^1_r - 2, a^2_l - 1, 1, a^2_m, a^3_l - 2, 1, 1, a^3_r).$$

The second term is then

$$-(\Psi_8\Psi_9)(\Psi_7\Psi_8)(\Psi_2)[(\Psi_7\Psi_6)(\Psi_4\Psi_5)(\Psi_6)](\Psi_8\Psi_7)(\Psi_3\Psi_4)(\underline{c})m^\lambda.$$

We continue as follows.

$$\begin{aligned}&= -\Psi_2\Psi_8\Psi_9(\Psi_7\Psi_8\Psi_7(\underline{c}')e(\nu'))\Psi_6\Psi_4\Psi_5\Psi_6\Psi_3\Psi_4\Psi_8\Psi_7(\underline{c})m^\lambda\\ &= -\Psi_2\Psi_8\Psi_9\Psi_8\Psi_7\Psi_6\Psi_4\Psi_5\Psi_6\Psi_3\Psi_4(\Psi_8^2)\Psi_7(\underline{c})m^\lambda\\ &= -\Psi_2(\Psi_8\Psi_9\Psi_8(\underline{c}'')e(\nu''))\Psi_7\Psi_6\Psi_4\Psi_5\Psi_6\Psi_3\Psi_4\Psi_7(\underline{c})m^\lambda\end{aligned}$$

$= -\Psi_2\Psi_9\Psi_8\Psi_7\Psi_6\Psi_4\Psi_5\Psi_6\Psi_3\Psi_4\Psi_7\Psi_9(\underline{c})m^\lambda$

$= 0$ (because $\Psi_9(\underline{c}) = \psi_{a_l^1+a_r^1+a_l^2+a_m^2+a_l^3-1}$ annihilates m^λ, as $a_l^3 = a_l^2 + 1 \geqslant 2$),

where the third equality above follows from Lemma 2.12(1) and

$$\underline{c}' = (a_l^1, 1, a_l^2 - 1, a_l^3 - 2, 1, 1, a_r^1 - 2, 1, a_m^2, 1, a_r^3),$$
$$\nu' = \nu_l^1 * (i+1) * \dot{\nu}_l^2 * \ddot{\nu}_l^3 * (i+2) * (i+1) * \ddot{\nu}_r^1 * (i+2) * \nu_m^2 * (i+1) * \nu_r^3,$$
$$\underline{c}'' = (a_l^1, 1, a_l^2 - 1, a_l^3 - 2, 1, 1, a_m^2, a_r^1 - 2, 1, 1, a_r^3),$$
$$\nu'' = \nu_l^1 * (i+1) * \dot{\nu}_l^2 * \ddot{\nu}_l^3 * (i+2) * (i+1) * \nu_m^2 * \ddot{\nu}_r^1 * (i+2) * (i+1) * \nu_r^3.$$

Thus, if we write

$$\psi_w = \Psi_7\Psi_8\Psi_6\Psi_2\Psi_5\Psi_4\Psi_3(a_l^1, 1, a_r^1 - 1, a_l^2 - 1, a_l^3 - 1, 1, a_m^2, a_r^2, 1, a_r^3)$$

by abuse of notation, we have the result.

(2) For the second term from Lemma 4.8(2), it suffices to consider

$$(\Psi_6\Psi_5\Psi_6 - \Psi_5\Psi_6\Psi_5)\Psi_7\Psi_4\Psi_3(\underline{d})m^\lambda = (\Psi_6\Psi_5\Psi_6 - \Psi_5\Psi_6\Psi_5)(\underline{d}')e(\nu')\Psi_7\Psi_4\Psi_3(\underline{d})m^\lambda,$$

where $\nu' = \nu_l^1 * (i+1) * \nu_l^2 * \nu_m^2 * \dot{\nu}_r^1 * (i+1) * \dot{\nu}_l^3 * \dot{\nu}_r^2 * (i+1) * \nu_r^3$ and

$$\underline{d}' = (a_l^1, 1, a_l^2, a_m^2, a_r^1 - 1, 1, a_l^3 - 1, a_r^2 - 1, 1, a_r^3).$$

For this, we need to compute

$$(\Psi_2\Psi_1\Psi_2 - \Psi_1\Psi_2\Psi_1)(a_r^1 - 1, 1, a_l^3 - 1)e(\dot{\nu}_r^1 * (i+1) * \dot{\nu}_l^3)$$

again. But, our assumption that $a_l^2 = 0$ is equivalent to $a_l^3 = 1$, which implies that this error term is zero. Thus if we write

$$\psi_w = \Psi_3\Psi_2\Psi_8\Psi_4\Psi_5\Psi_6\Psi_7(a_l^1, 1, a_l^2, a_m^2, 1, a_r^1 - 1, a_r^2 - 1, a_l^3 - 1, 1, a_r^3)$$

by abuse of notation, the result follows. □

Lemma 4.11. *We have*

$$\psi_{w^A} g_A^\lambda \equiv \psi_{w^B} g_B^\lambda \qquad (\mathrm{mod}\ \mathcal{G}^\lambda_{<\ell(\mathtt{G}^{A,B})}).$$

Proof. Lemma 4.8 implies that

$$\psi_{w^A} g_A^\lambda - \psi_{w^B} g_B^\lambda = \psi_w g_{A'}^\lambda - \psi_{w'} g_{B'}^\lambda + \gamma + \delta,$$

for some reduced expressions of $w, w' \in \mathfrak{S}_n$ such that $\ell(w) + \ell(\mathtt{G}^{A'}) < \ell(\mathtt{G}^{A,B})$ and $\ell(w') + \ell(\mathtt{G}^{B'}) < \ell(\mathtt{G}^{A,B})$. Here, γ and δ are the second terms appearing in Lemma 4.8(1) and (2) respectively, and are both zero by Lemma 4.10 unless $a_l^2 \geqslant 1$ and $a_r^2 \geqslant 1$. Lemma 4.9 implies that $\psi_w g_{A'}^\lambda$ and $\psi_{w'} g_{B'}^\lambda$ belong to $\mathcal{G}^\lambda_{<\ell(\mathtt{G}^{A,B})}$; we may also have that $\psi_w g_{A'}^\lambda$ or $\psi_{w'} g_{B'}^\lambda$ are zero, in the degenerate cases that $a_l^2 = 0$ or $a_r^2 = 0$, respectively, by Lemma 4.8.

So it remains to show that $\gamma+\delta \equiv 0 \pmod{\mathcal{G}^{\lambda}_{<\ell(\mathsf{G}^{A,B})}}$ when $a_l^2 \geqslant 1$ and $a_r^2 \geqslant 1$. In fact, we will show that $\gamma+\delta=0$. We continue by further calculation with γ and δ.

As in Lemma 4.8(1), we compute γ by using

$$\begin{aligned}(\Psi_2\Psi_1\Psi_2-\Psi_1\Psi_2\Psi_1)&(a_r^1-1,1,a_l^3-1)e(\dot{\nu}_r^1*(i+1)*\dot{\nu}_l^3)\\ &=\Psi_2(a_l^3-1,a_r^1-2,2)\Psi[a_r^1-1,a_l^3-2]\Psi_2(a_r^1-1,1,a_l^3-2)e(\dot{\nu}_r^1*(i+1)*\dot{\nu}_l^3).\end{aligned}$$

To state the result, we change $\underline{c}$ to

$$\underline{c}=(a_l^1,1,1,a_r^1-2,a_l^2-1,1,a_m^2,1,a_r^2-1,a_l^3-2,1,1,a_r^3),$$

and write $\nu=\nu_l^1*(i+1)*(i+2)*\ddot{\nu}_r^1*\dot{\nu}_l^2*(i+1)*\nu_m^2*(i+1)*\dot{\nu}_r^2*\ddot{\nu}_l^3*(i+2)*(i+1)*\nu_r^3$. Then

$$\begin{aligned}\gamma=&-(\Psi_8\Psi_9)(\Psi_{10}\Psi_{11})(\Psi_7\Psi_8)(\Psi_2)(\Psi_7\Psi_6\Psi_4\Psi_5\Psi_6)(\Psi_3\Psi_4)(\Psi_8\Psi_7)\Psi_9\Psi_{10}\Psi_8\Psi_9(\underline{c})m^\lambda\\ =&-\Psi_8\Psi_9\Psi_{10}\Psi_{11}\Psi_2(\Psi_7\Psi_8\Psi_7(\underline{c}')e(\nu'))\Psi_6\Psi_4\Psi_5\Psi_6\Psi_3\Psi_4\Psi_8\Psi_7\Psi_9\Psi_{10}\Psi_8\Psi_9(\underline{c})m^\lambda\\ =&-\Psi_8\Psi_9\Psi_{10}\Psi_{11}\Psi_2\Psi_8\Psi_7\Psi_6\Psi_4\Psi_5\Psi_6\Psi_3\Psi_4(\Psi_8^2(\underline{c}'')e(\nu''))\Psi_7\Psi_9\Psi_{10}\Psi_8\Psi_9(\underline{c})m^\lambda\\ =&-(\Psi_8\Psi_9\Psi_8(\underline{c}''')e(\nu'''))\Psi_{10}\Psi_{11}\Psi_2\Psi_7\Psi_6\Psi_4\Psi_5\Psi_6\Psi_3\Psi_4\Psi_7\Psi_9\Psi_{10}\Psi_8\Psi_9(\underline{c})m^\lambda,\\ =&-\Psi_9\Psi_8(\Psi_9\Psi_{10}\Psi_9(\underline{c}''')e(\omega))\Psi_{11}\Psi_2\Psi_7\Psi_6\Psi_4\Psi_5\Psi_6\Psi_3\Psi_4\Psi_7\Psi_{10}\Psi_8\Psi_9(\underline{c})m^\lambda,\end{aligned}$$

where

$$\begin{aligned}\underline{c}'&=(a_l^1,1,a_l^2-1,a_l^3-2,1,1,a_r^1-2,1,a_m^2,1,a_r^2-1,1,a_r^3),\\ \nu'&=\nu_l^1*(i+1)*\dot{\nu}_l^2*\ddot{\nu}_l^3*(i+2)*(i+1)*\ddot{\nu}_r^1*(i+2)*\nu_m^2*(i+1)*\dot{\nu}_r^2*(i+1)*\nu_r^3,\\ \underline{c}''&=(a_l^1,1,1,a_r^1-2,a_l^2-1,1,a_l^3-2,a_m^2,1,1,a_r^2-1,1,a_r^3),\\ \nu''&=\nu_l^1*(i+1)*(i+2)*\ddot{\nu}_r^1*\dot{\nu}_l^2*(i+1)*\ddot{\nu}_l^3*\nu_m^2*(i+2)*(i+1)*\dot{\nu}_r^2*(i+1)*\nu_r^3,\\ \underline{c}'''&=(a_l^1,a_l^2-1,1,a_l^3-2,1,1,a_m^2,a_r^1-2,1,1,1,a_r^2-1,a_r^3),\\ \nu'''&=\nu_l^1*\dot{\nu}_l^2*(i+1)*\ddot{\nu}_l^3*(i+2)*(i+1)*\nu_m^2*\ddot{\nu}_r^1*(i+2)*(i+1)*(i+1)*\dot{\nu}_r^2*\nu_r^3,\\ \omega&=\nu_l^1*\dot{\nu}_l^2*(i+1)*\ddot{\nu}_l^3*(i+2)*(i+1)*\nu_m^2*\ddot{\nu}_r^1*(i+1)*(i+2)*(i+1)*\dot{\nu}_r^2*\nu_r^3.\end{aligned}$$

We use

$$(\Psi_2\Psi_1\Psi_2-\Psi_1\Psi_2\Psi_1)(1,1,1)e((i+1)*(i+2)*(i+1))=e((i+1)*(i+2)*(i+1))$$

to continue as follows.

$$\begin{aligned}=&-\Psi_9\Psi_8\Psi_{10}\Psi_9(\Psi_{10}\Psi_{11}\Psi_{10}(\underline{e})e(\omega'))\Psi_2\Psi_7\Psi_6\Psi_4\Psi_5\Psi_6\Psi_3\Psi_4\Psi_7\Psi_8\Psi_9(\underline{c})m^\lambda\\ &+\Psi_9\Psi_8\Psi_{11}\Psi_2\Psi_7\Psi_6\Psi_4\Psi_5\Psi_6\Psi_3\Psi_4\Psi_7\Psi_{10}\Psi_8\Psi_9(\underline{c})m^\lambda\\ =&-\Psi_9\Psi_8\Psi_{10}\Psi_9\Psi_{11}\Psi_{10}\Psi_2\Psi_7\Psi_6\Psi_4\Psi_5\Psi_6\Psi_3\Psi_4\Psi_7\Psi_8\Psi_9\Psi_{11}(\underline{c})m^\lambda\end{aligned}$$

$$
\begin{aligned}
&+\Psi_4\Psi_2\Psi_3\Psi_9\Psi_8\Psi_7\Psi_6\Psi_5\Psi_6\Psi_7\Psi_8\Psi_{11}\Psi_{10}\Psi_9\Psi_4(\underline{c})m^\lambda \\
=&+\Psi_4\Psi_2\Psi_3\Psi_9(\Psi_8\Psi_7\Psi_6\Psi_5\Psi_6\Psi_7\Psi_8)\Psi_{11}\Psi_{10}\Psi_9\Psi_4(\underline{c})m^\lambda, \qquad (\dagger)
\end{aligned}
$$

where, in the final equality, we have used that $\Psi_{11}(\underline{c})m^\lambda = \psi_{a_l^1+a_r^1+a_l^2+a_m^2+a_r^2+a_l^3-1}m^\lambda = 0$ since $a_l^3 \geqslant 2$, and $\underline{e} = (a_l^1, a_l^2-1, 1, a_l^3-2, 1, 1, a_m^2, a_r^1-2, 1, a_r^2-1, 1, 1, a_r^3)$,

$$\omega' = \nu_l^1 * \dot\nu_l^2 * (i+1) * \ddot\nu_l^3 * (i+2) * (i+1) * \nu_m^2 * \ddot\nu_r^1 * (i+1) * \dot\nu_r^2 * (i+2) * (i+1) * \nu_r^3.$$

Similarly, we compute δ as in the proof of Lemma 4.8(2) by using the same equality as above. We replace $\underline{d}$ with $\underline{c}$:

$$\underline{c} = (a_l^1, 1, 1, a_r^1-2, a_l^2-1, 1, a_m^2, 1, a_r^2-1, a_l^3-2, 1, 1, a_r^3)$$

and write $\nu = \nu_l^1 * (i+1) * (i+2) * \ddot\nu_r^1 * \dot\nu_l^2 * (i+1) * \nu_m^2 * (i+1) * \dot\nu_r^2 * \ddot\nu_l^3 * (i+2) * (i+1) * \nu_r^3$. Then

$$
\begin{aligned}
\delta =&-(\Psi_5\Psi_4)(\Psi_3\Psi_2)(\Psi_6\Psi_5)(\Psi_{11})(\Psi_9\Psi_8\Psi_6\Psi_7\Psi_8)(\Psi_{10}\Psi_9)(\Psi_5\Psi_6)\Psi_4\Psi_5\Psi_3\Psi_4(\underline{c})m^\lambda \\
=&-\Psi_5\Psi_4\Psi_3\Psi_2\Psi_9\Psi_8(\Psi_6\Psi_5\Psi_6(\underline{c}')e(\nu'))\Psi_{11}\Psi_7\Psi_8\Psi_{10}\Psi_9\Psi_5\Psi_6\Psi_4\Psi_5\Psi_3\Psi_4(\underline{c})m^\lambda \\
=&-\Psi_5\Psi_4\Psi_5\Psi_3\Psi_2\Psi_9\Psi_8\Psi_6\Psi_7\Psi_8(\Psi_{11}\Psi_{10}\Psi_9)(\Psi_5^2(\underline{c}'')e(\nu''))(\Psi_4\Psi_3)\Psi_6\Psi_5\Psi_4(\underline{c})m^\lambda \\
=&-\Psi_5\Psi_4\Psi_5\Psi_3\Psi_2\Psi_9\Psi_6(\Psi_4\Psi_3)(\Psi_8\Psi_7\Psi_8(\underline{c}''')e(\nu'''))(\Psi_{11}\Psi_{10}\Psi_9)\Psi_6\Psi_5\Psi_4(\underline{c})m^\lambda \\
=&-(\Psi_5\Psi_4\Psi_5(\underline{d})e(\omega))\Psi_3\Psi_2\Psi_4\Psi_3\Psi_9\Psi_6\Psi_7\Psi_8\Psi_7\Psi_{11}\Psi_{10}\Psi_9\Psi_6\Psi_5\Psi_4(\underline{c})m^\lambda \\
=&-\Psi_4\Psi_5(\Psi_4\Psi_3\Psi_4(\underline{d}')e(\omega'))\Psi_2\Psi_3\Psi_9\Psi_6\Psi_7\Psi_8\Psi_7\Psi_{11}\Psi_{10}\Psi_9\Psi_6\Psi_5\Psi_4(\underline{c})m^\lambda \\
=&-\Psi_4\Psi_3\Psi_5\Psi_4(\Psi_3\Psi_2\Psi_3(\underline{d}'')e(\omega''))\Psi_9\Psi_6\Psi_7\Psi_8\Psi_7\Psi_{11}\Psi_{10}\Psi_9\Psi_6\Psi_5\Psi_4(\underline{c})m^\lambda \\
&-\Psi_4\Psi_5\Psi_2\Psi_3\Psi_9\Psi_6\Psi_7\Psi_8\Psi_7(\Psi_{11}\Psi_{10}\Psi_9)(\Psi_6\Psi_5)\Psi_4(\underline{c})m^\lambda \\
=&-\Psi_4\Psi_3\Psi_5\Psi_4\Psi_2\Psi_3\Psi_9\Psi_6\Psi_7\Psi_8\Psi_7\Psi_{11}\Psi_{10}\Psi_9\Psi_6\Psi_5\Psi_4\Psi_2(\underline{c})m^\lambda \\
&-\Psi_4\Psi_2\Psi_3\Psi_9(\Psi_5\Psi_6\Psi_7\Psi_8\Psi_7\Psi_6\Psi_5)\Psi_{11}\Psi_{10}\Psi_9\Psi_4(\underline{c})m^\lambda \\
=&-\Psi_4\Psi_2\Psi_3\Psi_9(\Psi_5\Psi_6\Psi_7\Psi_8\Psi_7\Psi_6\Psi_5)\Psi_{11}\Psi_{10}\Psi_9\Psi_4(\underline{c})m^\lambda \qquad (\ddagger)
\end{aligned}
$$

where we have used Lemma 2.12(2) for the fourth equality, $\Psi_2(\underline{c})m^\lambda = \psi_{a_l^1+1}m^\lambda = 0$ (since $a_r^1 \geqslant a_r^2+1 \geqslant 2$) in the final equality, and

$$
\begin{aligned}
\underline{c}' &= (a_l^1, 1, a_l^2-1, 1, a_m^2, 1, a_l^3-2, a_r^1-2, 1, 1, 1, a_r^2-1, a_r^3), \\
\nu' &= \nu_l^1 * (i+1) * \dot\nu_l^2 * (i+1) * \nu_m^2 * (i+2) * \ddot\nu_l^3 * \ddot\nu_r^1 * (i+1) * (i+2) * (i+1) * \dot\nu_r^2 * \nu_r^3, \\
\underline{c}'' &= (a_l^1, 1, a_l^2-1, 1, 1, a_m^2, a_r^1-2, 1, a_r^2-1, a_l^3-2, 1, 1, a_r^3), \\
\nu'' &= \nu_l^1 * (i+1) * \dot\nu_l^2 * (i+1) * (i+2) * \nu_m^2 * \ddot\nu_r^1 * (i+1) * \dot\nu_r^2 * \ddot\nu_l^3 * (i+2) * (i+1) * \nu_r^3, \\
\underline{c}''' &= (a_l^1, 1, 1, a_l^2-1, 1, a_m^2, a_r^1-2, 1, a_l^3-2, 1, 1, a_r^2-1, a_r^3), \\
\nu''' &= \nu_l^1 * (i+1) * (i+2) * \dot\nu_l^2 * (i+1) * \nu_m^2 * \ddot\nu_r^1 * (i+1) * \ddot\nu_l^3 * (i+2) * (i+1) * \dot\nu_r^2 * \nu_r^3,
\end{aligned}
$$

$$\begin{aligned}
\underline{d} &= (a_l^1, a_l^2-1, 1,1,1,1, a_l^3-2, a_m^2, 1,1, a_r^1-2, 1, a_r^2-1, a_r^3), \quad \underline{d}' = \underline{d}\\
\omega &= \nu_l^1 * \dot\nu_l^2 * (i+1)*(i+1)*(i+2)*\ddot\nu_l^3 * \nu_m^2 *(i+1)*(i+2)*\ddot\nu_r^1 *(i+1)*\dot\nu_r^2*\nu_r^3,\\
\omega' &= \nu_l^1 * \dot\nu_l^2 * (i+1)*(i+2)*(i+1)*\ddot\nu_l^3 * \nu_m^2 *(i+1)*(i+2)*\ddot\nu_r^1 *(i+1)*\dot\nu_r^2*\nu_r^3,\\
\underline{d}'' &= (a_l^1, 1,1, a_l^2-1, 1,1, a_l^3-2, a_m^2, 1,1, a_r^1-2, 1, a_r^2-1, a_r^3),\\
\omega'' &= \nu_l^1 *(i+1)*(i+2)* \dot\nu_l^2 * (i+1)*\ddot\nu_l^3 * \nu_m^2 *(i+1)*(i+2)*\ddot\nu_r^1 *(i+1)*\dot\nu_r^2*\nu_r^3.
\end{aligned}$$

It's easy to see, by applying three further braid relations which don't yield error terms, that (†) and (‡) are negations of each other, so that $\gamma + \delta = 0$, and the proof is complete. □

4.3. **Proof of Theorem 3.19.**

Lemma 4.12. *Let A and B be Garnir nodes of $[\lambda]$. Let w^A and w^B be the fully commutative elements given in Lemma 1.18. Then we have*

$$\psi_{w^A} g_A^\lambda \equiv \psi_{w^B} g_B^\lambda \qquad (\mathrm{mod}\ \mathcal{G}^\lambda_{<\ell(\mathtt{G}^{A,B})})$$

Proof. We may assume that $A=(r,c)$ is to the left of $B=(r',c')$ in $[\lambda]$.

Suppose $r = r'+1$ and $\mathrm{res}(B) = \mathrm{res}(r+1,c)$. Without loss of generality, we may assume that $[\lambda]$ has 3 rows and $r'=1$. Then the assertion holds by Lemma 4.11.

Otherwise, it follows from Proposition 2.2 and Corollary 3.2 that

$$\psi_{w^A} g_A^\lambda - \psi_{w^B} g_B^\lambda = \sum_w a_w \psi_w m^\lambda \quad \text{for some } a_w \in \mathcal{O},$$

where w runs over all elements such that (i) $w \prec w^{\mathtt{G}^{A,B}}$ and (ii) $e(\mathrm{res}(\mathtt{G}^{A,B}))\psi_w m^\lambda = \psi_w m^\lambda$. By Lemma 4.5, we conclude that

$$\psi_{w^A} g_A^\lambda - \psi_{w^B} g_B^\lambda = 0.$$ □

Lemma 4.13. *Let $\mathtt{T} \in \mathsf{Row}(\lambda)$. Suppose that $\mathtt{T} = w\mathtt{G}^A$ for $w \in \mathfrak{S}_n$ and a Garnir node A of $[\lambda]$ with $\ell(\mathtt{T}) = \ell(w) + \ell(\mathtt{G}^A)$.*

(1) If $\mathtt{T} = u\mathtt{G}^B$ for $u \in \mathfrak{S}_n$ and a Garnir node B of $[\lambda]$ with $\ell(\mathtt{T}) = \ell(u)+\ell(\mathtt{G}^B)$, then

$$\psi_w g_A^\lambda \equiv \psi_u g_B^\lambda \qquad (\mathrm{mod}\ \mathcal{G}^\lambda_{<\ell(\mathtt{T})}).$$

(2) We have

$$\begin{cases} x_i \psi_w g_A^\lambda \equiv 0 & \text{for all } i \in \{1,\dots,n\},\\ \psi_j \psi_w g_A^\lambda \equiv 0 & \text{unless } s_j\mathtt{T} \in \mathsf{Row}(\lambda) \text{ and } s_j \mathtt{T} \lhd \mathtt{T}, \end{cases} \qquad (\mathrm{mod}\ \mathcal{G}^\lambda_{<\ell(\mathtt{T})}).$$

(3) For $\sigma \in \mathfrak{S}_n$ with $\ell(\sigma) + \ell(\mathtt{G}^A) < \ell(\mathtt{T})$,

$$\psi_\sigma g_A^\lambda \equiv 0 \qquad (\mathrm{mod}\ \mathcal{G}^\lambda_{<\ell(\mathtt{T})}).$$

Proof. First, we prove (1), (2), and (3) for Garnir tableaux. Let $\mathtt{T} = \mathtt{G}^A$. If $\mathtt{G}^A = u\mathtt{G}^B$ for some $u \in \mathfrak{S}_n$ and some Garnir node B of $[\lambda]$ with $\ell(\mathtt{T}) = \ell(u) + \ell(\mathtt{G}^B)$, then it implies $\mathtt{G}^A \geqslant_L \mathtt{G}^B$, so that $\mathtt{G}^{A,B} = \mathtt{G}^A$ follows. Thus $B = A$ and $u = \mathrm{id}$ by Lemma 1.17, proving (1). Assertion (3) also holds obviously since there is no $\sigma \in \mathfrak{S}_n$ such that $\ell(\sigma) + \ell(\mathtt{G}^A) < \ell(\mathtt{G}^A)$. Assertion (2) follows from Lemma 3.7.

Now we prove (1), (2), and (3) by induction on $l := \ell(\mathtt{T})$. If $l = \min\{\ell(\mathtt{T}) \mid \mathtt{T} \in \mathsf{Row}(\lambda)\}$, then $\mathtt{T}$ is a Garnir tableau and there is nothing to prove. We assume that (1), (2), and (3) hold for all $\mathtt{T}' = w'\mathtt{G}^{A'} \in \mathsf{Row}(\lambda)$ with

$$\ell(\mathtt{T}') = \ell(w') + \ell(\mathtt{G}^{A'}) < l.$$

(1) We consider $\mathtt{T} = w\mathtt{G}^A = u\mathtt{G}^B$ for some $w, u \in \mathfrak{S}_n$ and Garnir nodes A, B of $[\lambda]$ with $\ell(\mathtt{T}) = l = \ell(w) + \ell(\mathtt{G}^A) = \ell(u) + \ell(\mathtt{G}^B)$. By Lemma 1.14, there is $v \in \mathfrak{S}_n$ such that

$$\mathtt{T} = v\mathtt{G}^{A,B} \text{ with } \ell(T) = \ell(v) + \ell(\mathtt{G}^{A,B}),$$

which tells us that

$$\begin{aligned} w &= vw^A \ \text{ with } \ell(w) = \ell(v) + \ell(w^A), \\ u &= vw^B \ \text{ with } \ell(u) = \ell(v) + \ell(w^B), \end{aligned}$$

where w^A and w^B are given in Lemma 1.18. Then, by Proposition 2.2 and the induction hypothesis on (3), we know that if ψ_σ appears as an error term in $(\psi_w - \psi_v\psi_{w^A})e(\mathrm{res}(\mathtt{G}^A))$ then $\psi_\sigma g_A^\lambda \in \mathtt{G}^\lambda_{<\ell(\mathtt{T})}$. Similarly, if ψ_σ appears as an error term in $(\psi_u - \psi_v\psi_{w^B})e(\mathrm{res}(\mathtt{G}^B))$ then $\psi_\sigma g_B^\lambda \in \mathtt{G}^\lambda_{<\ell(\mathtt{T})}$.

By Lemma 4.12 and the induction hypothesis, we have

$$\begin{aligned} \psi_w g_A^\lambda - \psi_u g_B^\lambda &\equiv \psi_v\psi_{w^A} g_A^\lambda - \psi_v\psi_{w^B} g_B^\lambda \\ &\equiv \psi_v(\psi_{w^A} g_A^\lambda - \psi_{w^B} g_B^\lambda) \\ &\equiv 0 \qquad\qquad (\mathrm{mod}\ \mathcal{G}^\lambda_{<\ell(\mathtt{T})}). \end{aligned} \tag{4.2}$$

Thus, (1) holds for $\ell(\mathtt{T}) = l$.

(2) For $i = 1, \ldots, n$, it follows from

$$x_i\psi_w e(\mathrm{res}(\mathtt{G}^A)) = \psi_w x_{w^{-1}(i)} e(\mathrm{res}(\mathtt{G}^A)) + \sum_{w' \prec w} \psi_{w'} f_{w'} e(\mathrm{res}(\mathtt{G}^A)) \quad \text{ for } f_{w'} \in \mathcal{O}[x_1, \ldots, x_n]$$

that

$$x_i\psi_w g_A^\lambda = \psi_w x_{w^{-1}(i)} g_A^\lambda + \sum_{w' \prec w} \psi_{w'} f_{w'} g_A^\lambda \equiv 0 \qquad (\mathrm{mod}\ \mathcal{G}^\lambda_{<\ell(\mathtt{T})})$$

by the induction hypothesis. It remains to prove that $\psi_j\psi_w g_A^\lambda \equiv 0$ unless $s_j\mathtt{T} \in \mathsf{Row}(\lambda)$ and $s_j\mathtt{T} \lhd \mathtt{T}$. There are two cases:

(i) $s_j\mathtt{T} \notin \mathsf{Row}(\lambda)$ (i.e. $s_j\mathtt{T} \notin \mathsf{RowStd}(\lambda)$ or $s_j\mathtt{T} \in \mathsf{Std}(\lambda)$),

(ii) $s_j\mathtt{T} \in \mathsf{Row}(\lambda)$ and $s_j\mathtt{T} \rhd \mathtt{T}$.

(i) If $s_j\mathtt{T} \notin \mathsf{RowStd}(\lambda)$, then there is a node $(r,c) \in [\lambda]$ such that

$$\mathtt{T}(r,c) = j, \quad \mathtt{T}(r,c+1) = j+1.$$

By Lemma 1.13 (2), we can take $B \in [\lambda]$ and $u \in \mathfrak{S}_n$ such that
(a) $\mathtt{G}^B(r,c+1) = \mathtt{G}^B(r,c)+1$,
(b) $\mathtt{T} = u\mathtt{G}^B$,
(c) $s_j u = u s_p$ where $p = \mathtt{G}^B(r,c)$.
Thus, we have

$$\psi_j\psi_u e(\nu) = \psi_u\psi_p e(\nu) + \sum_{u' \prec u} \psi_{u'} f_{u'} e(\nu) \quad \text{ for } f_{u'} \in \mathcal{O}[x_1,\dots,x_n],$$

where $\nu = e(\mathrm{res}(\mathtt{G}^B))$. Since (1) holds for the length l, Proposition 2.2 implies that $\psi_w g_A^\lambda - \psi_u g_B^\lambda \in \mathcal{G}^\lambda_{\leqslant l-3}$, so that $\psi_j\psi_w g_A^\lambda \equiv \psi_j\psi_u g_B^\lambda \pmod{\mathcal{G}^\lambda_{<\ell(\mathtt{T})}}$ by the induction hypothesis on (3). Similarly, Proposition 2.2 and Lemma 3.7 imply that $\psi_j\psi_u g_B^\lambda \equiv \psi_u\psi_p g_B^\lambda \pmod{\mathcal{G}^\lambda_{<\ell(\mathtt{T})}}$, and $\psi_p g_B^\lambda = 0$, so that

$$\psi_j\psi_w g_A^\lambda \equiv \psi_j\psi_u g_B^\lambda \equiv \psi_u\psi_p g_B^\lambda = 0 \qquad \pmod{\mathcal{G}^\lambda_{<\ell(\mathtt{T})}}.$$

Suppose that $s_j\mathtt{T} \in \mathsf{Std}(\lambda)$. Then there is a node $C = (r,c) \in [\lambda]$ such that

$$\mathtt{T}(r,c) = j+1, \quad \mathtt{T}(r+1,c) = j.$$

By Lemma 1.13(1), there is a permutation $v \in \mathfrak{S}_n$ such that
(a) $\mathtt{T} = v\mathtt{G}^C$,
(b) $s_j v = v s_q$ where $q = \mathtt{G}^C(r+1,c)$.
In a similar manner as above, we have

$$\psi_j\psi_w g_A^\lambda \equiv \psi_j\psi_v g_C^\lambda \equiv \psi_v\psi_q g_C^\lambda = 0 \qquad \pmod{\mathcal{G}^\lambda_{<\ell(\mathtt{T})}}.$$

(ii) We assume that $\mathtt{S} := s_j\mathtt{T} \in \mathsf{Row}(\lambda)$ and $\mathtt{S} \rhd \mathtt{T}$. Note that $\ell(\mathtt{T}) = \ell(\mathtt{S})+1$. Then

$$\mathtt{S} = u\mathtt{G}^B \ \text{ and } \ell(\mathtt{S}) = \ell(u) + \ell(\mathtt{G}^B)$$

for some Garnir node $B \in [\lambda]$ and $u \in \mathfrak{S}_n$. As (1) holds for the length l, we have

$$\psi_j\psi_w g_A^\lambda \equiv \psi_j\psi_j\psi_u g_B^\lambda \equiv Q_{\nu_j,\nu_{j+1}}(x_j,x_{j+1})\psi_u g_B^\lambda \equiv 0 \qquad \pmod{\mathcal{G}^\lambda_{<\ell(\mathtt{T})}},$$

where $\nu = (\nu_1,\dots,\nu_n) \in I^n$ is the residue sequence of $u\mathtt{G}^B$.

By (i) and (ii) the assertion (2) holds for $\ell(\mathtt{T}) = l$.

(3) Suppose that $\mathtt{T} = w\mathtt{G}^A$ for $w \in \mathfrak{S}_n$ and a Garnir node A of $[\lambda]$ with $\ell(\mathtt{T}) = \ell(w)+\ell(\mathtt{G}^A)$. Let $\sigma = s_{i_1} \dots s_{i_r}$ be a reduced expression for σ, so that $\psi_\sigma = \psi_{i_1} \dots \psi_{i_r}$ and $\ell(\sigma)+\ell(\mathtt{G}^A) < l$. Note that we do not assume that $\ell(\sigma\mathtt{G}^A) = \ell(\sigma) + \ell(\mathtt{G}^A)$.

If $\ell(\sigma\mathtt{G}^A) = \ell(\sigma) + \ell(\mathtt{G}^A)$, then $\sigma\mathtt{G}^A \in \mathsf{Row}(\lambda)$ by Lemma 1.12. Thus, we are done.

If $\ell(\sigma\mathtt{G}^A) < \ell(\sigma) + \ell(\mathtt{G}^A)$, then there is some k such that $\ell(s_{i_k} \dots s_{i_r}\mathtt{G}^A) = \ell(s_{i_k} \dots s_{i_r}) + \ell(\mathtt{G}^A)$ and $\ell(s_{i_{k-1}} \dots s_{i_r}\mathtt{G}^A) < \ell(s_{i_k} \dots s_{i_r}\mathtt{G}^A)$. Once again, by the induction hypothesis on (2), we have that $\psi_{i_{k-1}} \dots \psi_{i_r} g^\lambda_A \in \mathcal{G}^\lambda_{<\ell(s_{i_k} \dots s_{i_r}\mathtt{G}^A)}$ and may conclude that $\psi_{i_1} \dots \psi_{i_r} g^\lambda_A \in \mathcal{G}^\lambda_{<l}$ by induction.

Thus, assertion (3) holds for $\ell(\mathtt{T}) = l$, which completes the proof. □

Corollary 4.14. *For each $\mathtt{T} \in \mathsf{Row}(\lambda)$, we fix a Garnir node A and $w \in \mathfrak{S}_n$ such that $\mathtt{T} = w\mathtt{G}^A$, and set $g^\lambda_{\mathtt{T}} = \psi_w g^\lambda_A + \mathcal{G}^\lambda_{<\ell(\mathtt{T})}$. Then*

(1) The element $g^\lambda_{\mathtt{T}} \in \mathcal{G}^\lambda/\mathcal{G}^\lambda_{<\ell(\mathtt{T})}$ does not depend on the choice of A or the choice of reduced expression for w.

(2) The $\mathcal{O}$-submodule $\sum_{t>0} \mathcal{G}^\lambda_{<t}$ is an $R(\beta)$-submodule of $\mathcal{M}^\lambda$.

Proof. Part (1) and (2) follows from Lemma 4.13 (1) and Lemma 4.13 (2), respectively. □

Recall that $\mathcal{G}^\lambda = \mathrm{im}\ \mathcal{H}^\lambda$ and $\mathcal{S}^\lambda = q^{\deg(\mathtt{T}^\lambda)}\mathrm{coker}\ \mathcal{H}^\lambda$ for $\lambda \vdash n$.

Theorem 4.15. *(1) The $\mathcal{O}$-submodules $\{\mathcal{G}^\lambda_{<t}\}_{t\in\mathbb{Z}_{>0}}$ give a filtration of $\mathcal{G}^\lambda$.*

(2) For $t \in \mathbb{Z}_{>0}$, $\mathcal{G}^\lambda_{<t+1}/\mathcal{G}^\lambda_{<t}$ is a free $\mathcal{O}$-module with basis

$$\{g^\lambda_{\mathtt{T}} \mid \mathtt{T} \in \mathsf{Row}(\lambda), \ell(T) = t\}.$$

Proof. Corollary 4.14(2) implies that $\mathcal{G}^\lambda = \sum_{t>0} \mathcal{G}^\lambda_{<t}$, which is (1). Then (2) is clear. □

We are now ready to prove Theorem 3.19.

Proof of Theorem 3.19. Let us consider

$$v = a_{\mathtt{T}^1}\psi_{w^{\mathtt{T}_1}} m^\lambda + a_{\mathtt{T}^2}\psi_{w^{\mathtt{T}_2}} m^\lambda + \dots + a_{\mathtt{T}^t}\psi_{w^{\mathtt{T}_t}} m^\lambda \in \mathcal{M}^\lambda$$

for some $a_{\mathtt{T}^1}, \dots, a_{\mathtt{T}^t} \in \mathcal{O}$ and $\mathtt{T}_1, \dots, \mathtt{T}_t \in \mathsf{Std}(\lambda)$. By (3.4) and Theorem 4.15, we have

$$v \in \mathcal{G}^\lambda \text{ if and only if } a_{\mathtt{T}^1} = \dots = a_{\mathtt{T}^t} = 0,$$

which implies that $\{\psi_{w^{\mathtt{T}}}\overline{m}^\lambda \mid \mathtt{T} \in \mathsf{Std}(\lambda)\}$ is linearly independent in $\mathcal{S}^\lambda$. Thus the assertion follows from Theorem 3.12(1). □

5. A conjecture in type $C_\ell^{(1)}$

We end with a conjecture giving the elements g_A^λ explicitly in type $C_\ell^{(1)}$, as well as a basis of $\mathcal{S}^\lambda$ in this type. In Remark 3.6, we noted the similarity between the Garnir elements in type A_∞ and C_∞. Similarly, we expect that the affine type C case resembles that of affine type A. Our constructions in this section closely follow [21, Section 5]. We fix $\lambda \in \mathscr{P}_n^l$ and $\kappa \in \mathbb{Z}^l$ throughout, as well as a Garnir node $A = (r, c, t) \in [\lambda]$. Recall the definition of the Garnir tableau $\mathtt{G}^A$ from earlier, as well as the residue pattern in type $C_\ell^{(1)}$ – in particular the natural projection $p : \mathbb{Z} \to \mathbb{Z}/2\ell\mathbb{Z}$.

Definition 5.1. A *brick* is a set of 2ℓ adjacent nodes in the same row of the Garnir belt $\mathbf{B}^A$, $\{(a, b, t), (a, b+1, t), \dots, (a, b+2\ell-1, t)\}$, such that $p(\kappa_t + b - a) = p(\kappa_t + c - r)$.

We denote by k the number of bricks contained in $\mathbf{B}^A$, and label the bricks $B_1, B_2, \dots, B_k$ from left-to-right along row $r+1$ and then from left-to-right along row r. We now introduce permutations which transpose adjacent bricks. Let d be the smallest entry of B_1 in $\mathtt{G}^A$. Then for $1 \leqslant r < k$, the permutation

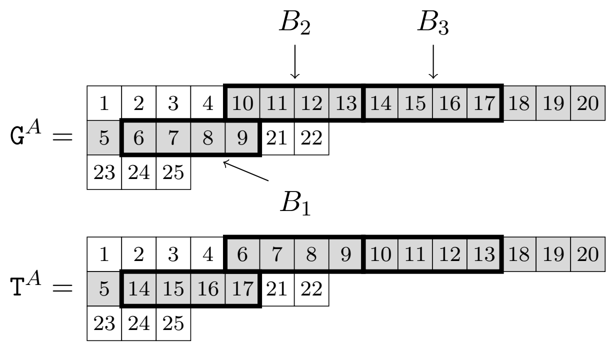
$$w_r = w_r^A := \prod_{a=d+2(r-1)\ell}^{d+2r\ell-1} (a, a+2\ell) \in \mathfrak{S}_n$$

may be thought of as transposing B_r and B_{r+1}. We have the corresponding elements $\sigma_r = \sigma_r^A := (-1)^\ell \psi_{w_r} \in R(\beta)$. We further define $\tau_r = \tau_r^A := \sigma_r + 1$. We should emphasise here that we have a $(-1)^\ell$ in our definition of σ_r, which differs from the definition of *row* Specht modules in [21, Section 5.4]. However a similar minus sign occurs in their definition of corresponding elements in *column* Specht modules [21, Section 7.1]. We suspect that this minus sign is merely an artefact of our choice of the polynomials $Q_{i,j}(u, v)$.

Note that any permutation u of bricks may be written as a reduced expression $u = w_{r_1} \dots w_{r_i}$, and if u is fully commutative we have a well-defined element $\tau_u := \tau_{r_1} \dots \tau_{r_i}$.

We define $\mathtt{T}^A$ to be the tableau obtained from $\mathtt{G}^A$ by rearranging the entries in the bricks $B_1, \dots, B_k$ so that they are in order along row r and then row $r+1$. This is the most dominant row-strict tableau which may be obtained from $\mathtt{G}^A$ by acting by our brick permutations w_r.

Example 5.2. Let $\ell = 2, \lambda = (15, 7, 3) \in \mathscr{P}^1_{25}$, and $A = (1, 5)$. We depict the Garnir tableau $\mathtt{G}^A$ below with Garnir belt shaded and bricks B_1, B_2, B_3 labelled, as well as the tableau $\mathtt{T}^A$.



Note that $\mathtt{G}^A = w_1 w_2 \mathtt{T}^A$.

Conjecture 5.3. *Let $\lambda \in \mathscr{P}^l_n$, and let $A \in [\lambda]$ be a Garnir node. Suppose $\mathbf{B}^A$ contains a bricks in the first row, and b in the second. Then in type $C^{(1)}_\ell$, the Garnir element g^λ_A is*

$$g^\lambda_A = \sum_u \tau^A_u \psi_{w^{\mathtt{T}^A}} m^\lambda,$$

where the sum is over all $u \in \mathfrak{S}_{a+b}/\mathfrak{S}_a \times \mathfrak{S}_b$. Furthermore, Theorem 3.19 and Corollary 3.21 hold in type $C^{(1)}_\ell$, giving a homogeneous basis of $\mathcal{S}^\lambda$ indexed by standard λ-tableaux.

Example 5.4. Continuing from the previous example,

$$\begin{aligned} g^\lambda_A &= \psi_{w^{\mathtt{T}^A}} m^\lambda + \tau_2 \psi_{w^{\mathtt{T}^A}} m^\lambda + \tau_1 \tau_2 \psi_{w^{\mathtt{T}^A}} m^\lambda \\ &= 3\psi_{w^{\mathtt{T}^A}} m^\lambda + 2\sigma_2 \psi_{w^{\mathtt{T}^A}} m^\lambda + \sigma_1 \sigma_2 \psi_{w^{\mathtt{T}^A}} m^\lambda + \sigma_1 \psi_{w^{\mathtt{T}^A}} m^\lambda. \end{aligned}$$

Evidence for our conjecture is provided by many examples we have computed in GAP [11].

Finally, we note that the above form for g^λ_A does not instantly yield a clean expression in terms of basis elements of $\mathcal{M}^\lambda$ – this can be seen in Example 5.4 where $w_1 \mathtt{T}^A m^\lambda$ is not row-strict. Fayers [10] has addressed this problem in type A, and in fact if our conjecture holds, then his work automatically applies to our g^λ_A too.

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