Kleshchev's decomposition numbers for cyclotomic Hecke algebras

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1 Motivation

Fix $\mathbb{F}$ a field of characteristic $p \geqslant 0$ throughout and $e \in \{3,4,\dots\} \cup \{\infty\}$. Let $q \in \mathbb{F}$ be a primitive eth root of unity, (if $e = \infty$ then q is not a root of unity) and $\kappa = (\kappa_1, \dots, \kappa_l) \in (\mathbb{Z}/e\mathbb{Z})^l$. We denote by $\mathscr{H}_n = \mathscr{H}_n(q,\kappa)$ the *cyclotomic Hecke algebra* (of type $G(l,1,n)$) of degree n with parameters q and κ.

Brundan and Kleshchev have shown that $\mathscr{H}_n$ is a $\mathbb{Z}$-graded algebra, and Hu and Mathas have shown that it is in fact a *graded cellular* algebra. The cellular structure agrees with that of the Dipper–James–Mathas construction. The cell modules are indexed by the set $\mathscr{P}_n^l$ of l-multipartitions of n and the simple modules by a certain subset $\Theta \subset \mathscr{P}_n^l$ of these.

Ariki's theorem tells us that in fact there are many different parameterisations $\Theta \subset \mathscr{P}_n^l$ for the simple modules, and the Dipper–James–Mathas setup sees only one of these. We would like different cellular structures for each one of these.

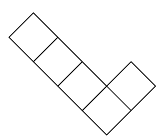
Aim Study graded decomposition numbers for $\mathscr{H}_n$ corresponding to various parameterisations of the simple modules.

In order to do this, we must lift to the setting of quasi-hereditary covers of $\mathscr{H}_n$. We will see that the diagrammatic Cherednik algebra depends on a weighting, θ, which in turn determines which parameterisation Θ we are seeing. (For any θ we get a quasi-hereditary cover of $\mathscr{H}_n$, and different covers may yield different parameterisations of simples.)

2 The diagrammatic Cherednik algebra

We will now discuss the combinatorics of Webster's diagrammatic Cherednik algebra.

We take a mirrored-Russian convention for drawing our Young diagrams. For example, the Young diagram for the partition $(4,1)$ is drawn as

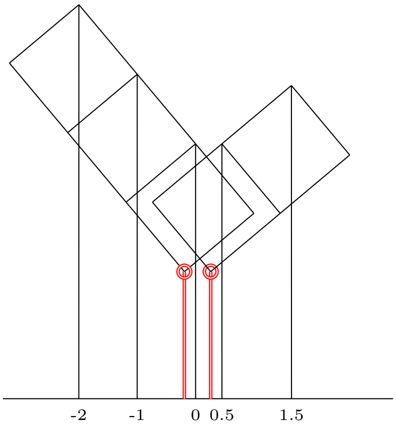


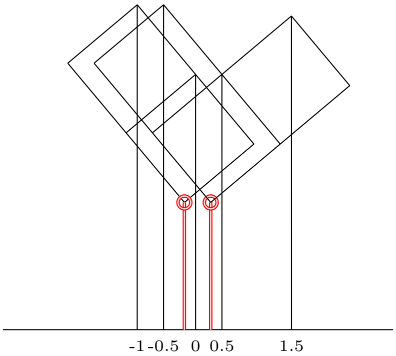
Given a *weighting* $\theta \in \mathbb{R}^l$, we have an associated diagrammatic Cherednik algebra $A(n,\theta,\kappa)$ which is a quasi-hereditary cover of the cyclotomic Hecke algebra $\mathscr{H}_n$.

Definition 2.1. Given a weighting θ and some $\lambda \in \mathscr{P}^l_n$, we draw the Young diagram $[\lambda]$ of λ by placing the first node of the jth component at point θ_j on the x-axis, with all boxes have diagonals of length 2. We tilt our Young diagrams ever so slightly clockwise.

The *loading* i_λ is the n-tuple of real numbers given by projecting the top vertices of boxes of $[\lambda]$ onto the real line, along with the residue associated to each box.

Example. Suppose $\theta = (0, 0.5)$. If $\lambda = ((3), (1^2))$ and $\mu = ((2), (2, 1))$, then we draw $[\lambda]$ and $[\mu]$ as below, with the loadings given by projections onto the real line.





Definition 2.2. We write $\lambda \trianglerighteq_\theta \mu$, and say λ *θ-dominates* μ, if for every real number a and every $j \in \mathbb{Z}/e\mathbb{Z}$, there are at least as many j-nodes to the left of a in i_λ as in i_μ.

If $l = 1$ this order is a coarsening of the usual dominance order.

Definition 2.3. Let $\lambda, \mu \in \mathscr{P}^l_n$. A semistandard tableau of shape λ and weight μ is a map $\mathtt{T} : [\lambda] \to i_\mu$ which respects residues and for all admissible r, c, k,

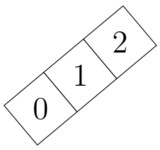
- $T(1, 1, k) \geqslant \theta_k$,
- $T(r, c, k) \geqslant T(r - 1, c, k) + 1$,
- $T(r, c, k) \geqslant T(r, c - 1, k) - 1$.

We denote the set of semistandard tableaux of shape λ and weight μ by $\mathrm{SStd}(\lambda, \mu)$.

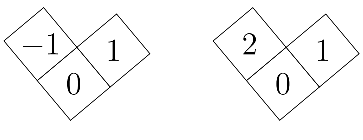
Note that $\mathrm{SStd}(\lambda, \mu) = \emptyset$ unless $\lambda \trianglerighteq_\theta \mu$.

Examples. 1. Let $e = 3$, $l = 1$, $n = 3$ and $\theta = \kappa = (0)$.

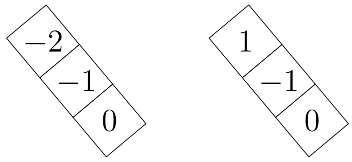
The only semistandard tableau of shape (1^3) is



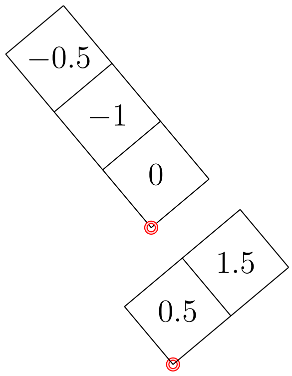
There is one semistandard tableau in $\mathrm{SStd}((2, 1), (2, 1))$ and one in $\mathrm{SStd}((2, 1), (1^3))$.



There is one semistandard tableau in $\mathrm{SStd}((3),(3))$, and one in $\mathrm{SStd}((3),(2,1))$.



2. Suppose $e = 3$, $\theta = (0, 0.5)$ and $\kappa = (0, 1)$. With λ and μ as before, the only element of $\mathrm{SStd}(\lambda, \mu)$ is



Theorem 2.4 (Webster). *The diagrammatic Cherednik algebra $A(n, \theta, \kappa)$ is a graded cellular algebra with respect to the θ-dominance order and a basis indexed by $\mathrm{SStd}(\lambda, \mu)$ as λ and μ range over $\mathscr{P}^l_n$.*

In particular we have graded standard modules $\Delta(\lambda) = \langle C_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{SStd}(\lambda, -)\rangle_{\mathbb{F}}$ with graded simple heads $\mathrm{L}(\lambda)$ forming a complete set of graded simple modules, up to grading shift.

Over $\mathbb{C}$, the module category of $A(n, \theta, \kappa)$ is equivalent to category $\mathcal{O}$ for the rational cyclotomic Cherednik algebra. If θ is well-separated (that is, $\theta_j - \theta_k >> 0$ for all j and k), then $A(n, \theta, \kappa)$ is Morita equivalent to the q-Schur algebra of Dipper–James–Mathas over arbitrary fields.

We would like to compute the graded decomposition numbers $d_{\lambda\mu}(v)$ for $A(n, \theta, \kappa)$.

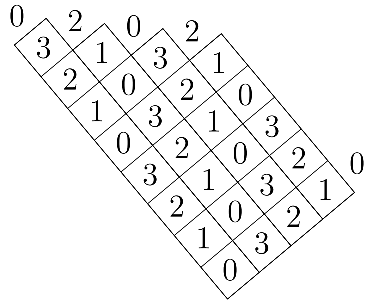
Example. With the level 1 example for $e = n = 3$, the graded decomposition numbers are obtained from the semistandard tableaux, almost for free.

3 Subquotients of $A(n, \theta, \kappa)$

Pick a set $S \subset \mathbb{Z}/e\mathbb{Z}$ of residues which is adjacency-free; that is, if $i \in S$ then $i \pm 1 \notin S$.

Fix $\gamma \in \mathscr{P}^l_n$ with no removable i-nodes for any $i \in S$, and let $\mathcal{M}$ denote a multiset of residues in S. Now let $\Gamma_{\mathcal{M}}$ denote the set of multipartitions obtained from γ by adding nodes of residues in $\mathcal{M}$.

Example. Let $e = 4$, $S = \{0, 2\}$ and $\gamma = (8, 7, 6, 5)$. If $\mathcal{M} = \{0, 0, 2\}$ then we could have for instance, $\lambda = (9,8,7,5), \mu = (8,7^2,6,1) \in \Gamma_{\mathcal{M}}$.



Fact: The set $\Gamma_{\mathcal{M}}$ is an interval in the θ-dominance order, with maximal element given by placing all nodes as far left as possible, and minimal element given by placing all nodes as far right as possible. Thus we may take a subquotient $A_{\Gamma_{\mathcal{M}}}$ whose standard modules are indexed by all $\lambda \in \Gamma_{\mathcal{M}}$.

Theorem 3.1 (Bowman, S)**.** *If $\mathcal{M}$ is adjacency-free and $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \dots \cup \mathcal{M}_{e-1}$ is a decomposition of $\mathcal{M}$ into disjoint residues, then*

$$A_{\Gamma_{\mathcal{M}}} \cong A_{\Gamma_{\mathcal{M}_0}} \otimes A_{\Gamma_{\mathcal{M}_1}} \otimes \dots \otimes A_{\Gamma_{\mathcal{M}_{e-1}}}$$

as graded $\mathbb{F}$-algebras.

Thus, we may from now on consider the subquotient A_{Γ_m}, corresponding to partitions obtained from some γ by adding m i-nodes for some fixed residue i.

Example. Continuing with the previous example, we have that

$$d_{\lambda\mu} = d_{(9,7^2,5)(8,7^2,5,1)} \times d_{(8^2,6,5)(8,7,6^2)}.$$

4 Isomorphisms and decomposition numbers

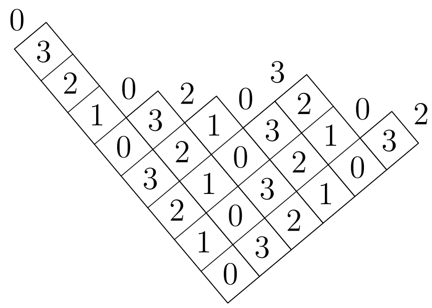
Theorem 4.1 (Bowman, S)**.** *Let $e, \bar{e} \in \{3,4,\dots\} \cup \{\infty\}$ with $i \in \mathbb{Z}/e\mathbb{Z}$ and $\bar{i} \in \mathbb{Z}/\bar{e}\mathbb{Z}$, and suppose γ (respectively $\bar{\gamma}$) is a multipartition with no removable i-nodes (respectively $\bar{i}$-nodes) and x addable i-nodes (respectively $\bar{i}$-nodes) for some x. Then the corresponding subquotients A_{Γ_m} and $A_{\bar{\Gamma}_m}$ are isomorphic as graded vector space over $\mathbb{F}$.*

Moreover, if $\mathbb{F} = \mathbb{C}$, we have

$$d_{\lambda\mu}(v) = d_{\bar{\lambda}\bar{\mu}}(v)$$

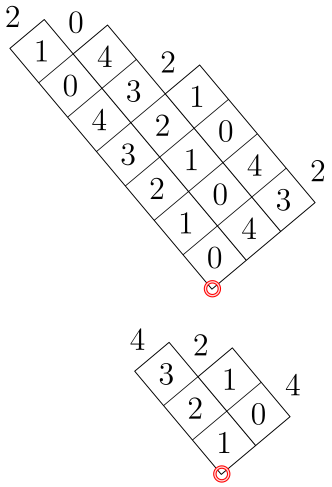
for $\lambda, \mu \in \Gamma_m$ and $\bar{\lambda}, \bar{\mu} \in \bar{\Gamma}_m$.

Example. Let $e = 4$ and $\bar{e} = 5$. Take $\gamma = (8,5,4,3^2,1)$.



Then γ has no removable 0-nodes and four addable 0-nodes.

Let $\bar{\kappa} = (0, 1)$, $\bar{\theta} = (0, 0.5)$ and $\bar{\gamma} = ((7, 6, 4), (3, 2))$.



Then $\bar{\gamma}$ has no removable 2-nodes and four addable 2-nodes. Thus we can compare decomposition numbers in A_{Γ_m} with those in $A_{\bar{\Gamma}_m}$, where Γ_m is the set of partitions obtained from γ by adding m 0-nodes, and $\bar{\Gamma}_m$ is the set of bipartitions obtained from $\bar{\gamma}$ by adding m 2-nodes.

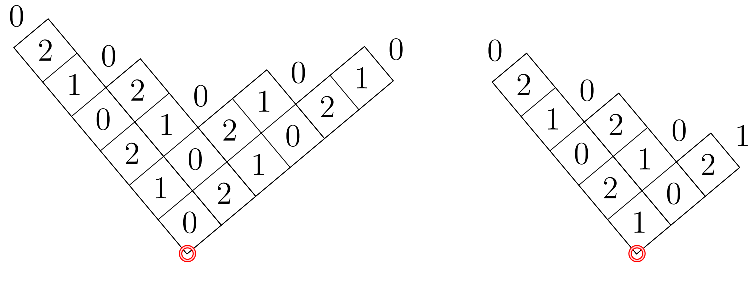
Our proof of the graded vector space isomorphism is by an explicit construction, which we verify by examining *i-diagonals* in $[\gamma]$.

Theorem 4.2 (Bowman, S)**.** *Take $e, \bar{e}, i, \bar{i}, \gamma, \bar{\gamma}$ as before. Under some extra conditions on the i-diagonals of γ and $\bar{\gamma}$, the isomorphism of the previous theorem is an isomorphism of graded algebras.*

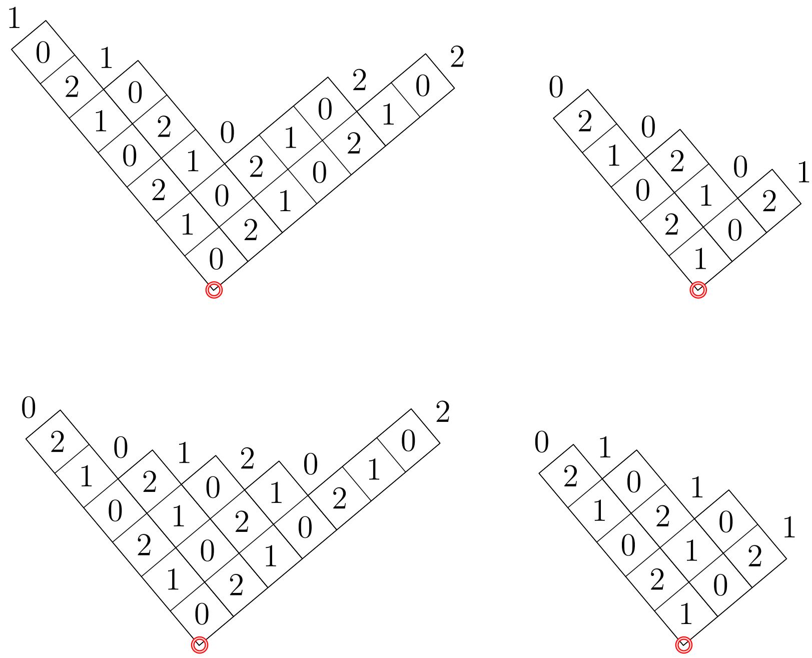
In particular, if κ contains at most one instance of the residue i, the graded decomposition numbers are parabolic Kazhdan–Lusztig polynomials, and can be calculated explicitly by a combinatorial result of Tan and Teo.

Remark. Tan and Teo's combinatorics just uses the sequences of addable and removable i-nodes in λ and μ.

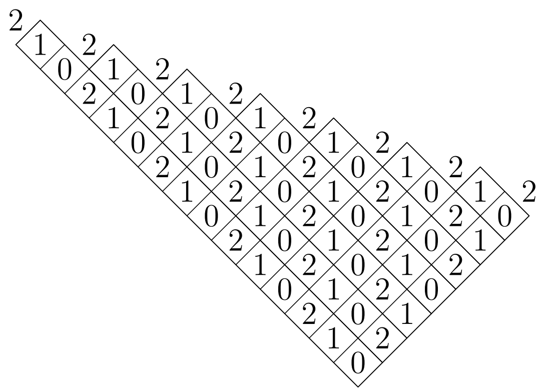
Example. Let $\kappa = (0, 1)$ and $e = 3$, $i = 0$. For $\theta = (0, 20)$, $\gamma = ((6, 4, 2^2, 1^2), (5, 3, 1))$. Then $[\gamma]$ is



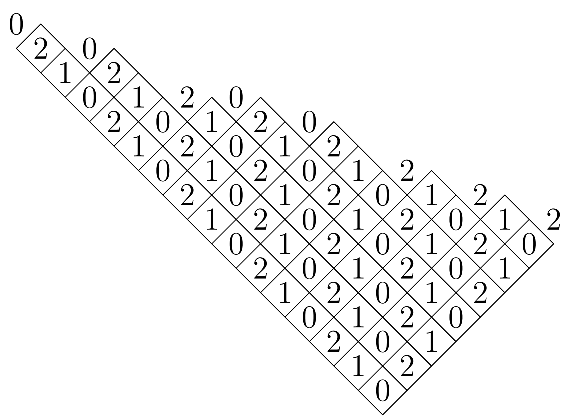
Setting $\lambda = ((7, 5, 2^3, 1^2), (5, 3, 1))$ and $\mu = ((6, 4, 3, 2, 1^3), (5, 4, 2))$, we have the Young diagrams below.

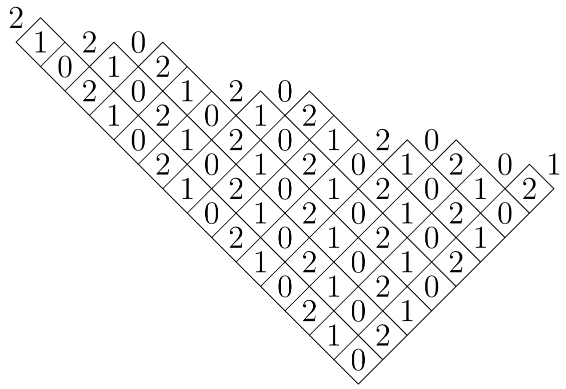


Now take $\bar{e} = 3$, $\bar{i} = 2$ and $\bar{\gamma} = (14, 12, 10, 8, 6, 4, 2)$. The Young diagram $[\bar{\gamma}]$ with residues is

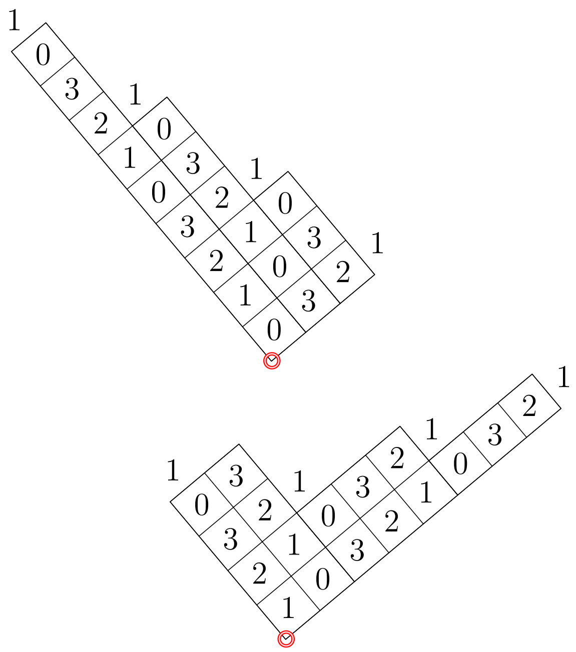


Setting $\bar{\lambda} = (15, 13, 10, 9, 7, 4, 2)$ and $\bar{\mu} = (14, 12, 11, 8, 7, 4, 3, 1)$, we have the Young diagrams below.

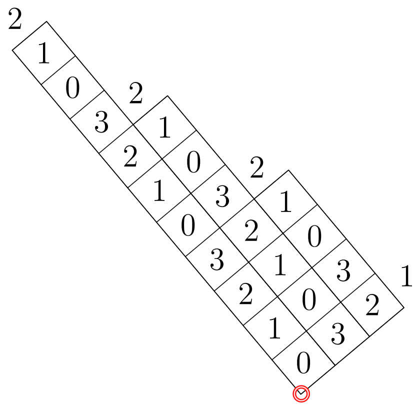


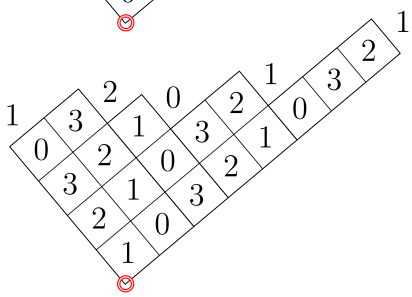


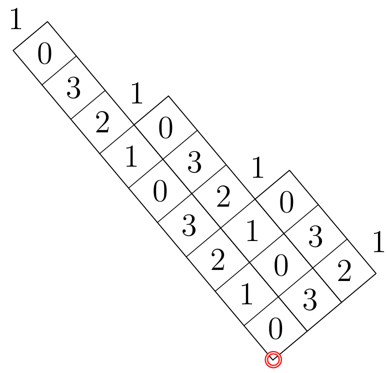
Finally, take $\bar{\bar{e}} = 4$, $\bar{\bar{i}} = 1$, $\bar{\bar{\theta}} = (0, 0.5)$, $\bar{\bar{\gamma}} = ((9,6,3),(4^2,2^3,1^3))$. Then $[\bar{\bar{\gamma}}]$ is

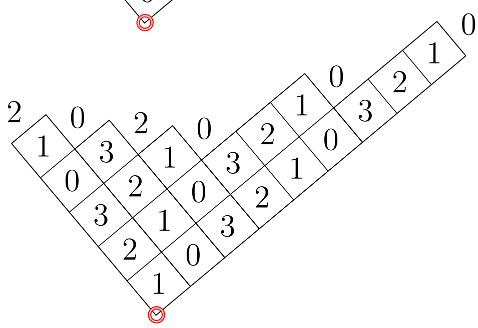


Taking $\bar{\bar{\lambda}} = ((10,7,4),(4^2,3,2^2,1^3))$ and $\bar{\bar{\mu}} = ((9,6,3),(5,4,3,2^3,1^3))$, we get the Young diagrams below.









All three examples involve the same sequences of addable and removable nodes, and we can compute that $d_{\lambda\mu}(v) = v^{11} + 2v^9 + 2v^7 + v^5$ in all three cases.