# Graded decomposition numbers for the diagrammatic Cherednik algebra

Liron Speyer
Osaka University, Suita, Osaka 565-0871, Japan
l.speyer@ist.osaka-u.ac.jp

## 1 Motivation

Fix $\mathbb{F}$ a field of characteristic $p \geqslant 0$ throughout and $e \in \{3, 4, \dots\} \cup \{\infty\}$. Let $q \in \mathbb{F}$ be a primitive $e$th root of unity, (if $e = \infty$ then $q$ is not a root of unity) and $\kappa = (\kappa_1, \dots, \kappa_l) \in (\mathbb{Z}/e\mathbb{Z})^l$. We denote by $\mathscr{H}_n = \mathscr{H}_n(q, \kappa)$ the *cyclotomic Hecke algebra* (of type $G(l, 1, n)$) of degree $n$ with parameters $q$ and $\kappa$.

Brundan and Kleshchev have shown that $\mathscr{H}_n$ is a $\mathbb{Z}$-graded algebra, and Hu and Mathas have shown that it is in fact a *graded cellular* algebra. The cellular structure agrees with that of the Dipper–James–Mathas construction. The cell modules are indexed by the set $\mathscr{P}_n^l$ of $l$-multipartitions of $n$ and the simple modules by a certain subset $\Theta \subset \mathscr{P}_n^l$ of these.

Ariki's theorem tells us that in fact there are many different parameterisations $\Theta \subset \mathscr{P}_n^l$ for the simple modules, and the Dipper–James–Mathas setup sees only one of these. We would like different cellular structures for each one of these.

**Aim** Study graded decomposition numbers for $\mathscr{H}_n$ corresponding to various parameterisations of the simple modules.

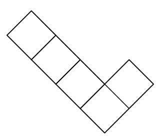
In order to do this, we must lift to the setting of quasi-hereditary covers of $\mathscr{H}_n$. We will see that the diagrammatic Cherednik algebra depends on a weighting, $\theta$, which in turn determines which parameterisation $\Theta$ we are seeing. (For any $\theta$ we get a quasi-hereditary cover of $\mathscr{H}_n$, and different covers may yield different parameterisations of simples.)

## 2 The diagrammatic Cherednik algebra

We will now discuss the combinatorics of Webster's diagrammatic Cherednik algebra.

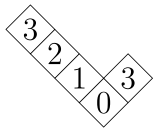
**Definition 2.1.** A *partition* of $n$ is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $|\lambda| = \sum \lambda_i = n$. An *$l$-multipartition* of $n$ is an $l$-tuple of partitions $(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(l)})$ such that $\sum |\lambda^{(i)}| = n$.

We take a mirrored-Russian convention for drawing our Young diagrams. For example, the Young diagram for the partition $(4, 1)$ is drawn as



**Definition 2.2.** The *residue* of a node $(r, c, m) \in [\lambda]$ (i.e. the node in the $r$th row and $c$th column of the Young diagram $[\lambda^{(m)}]$) is defined to be $\kappa_m + c - r \mod e$.

**Example.** Suppose $\kappa = (0)$, $e = 4$ and $\lambda = (4, 1)$. Then the residues are given below.

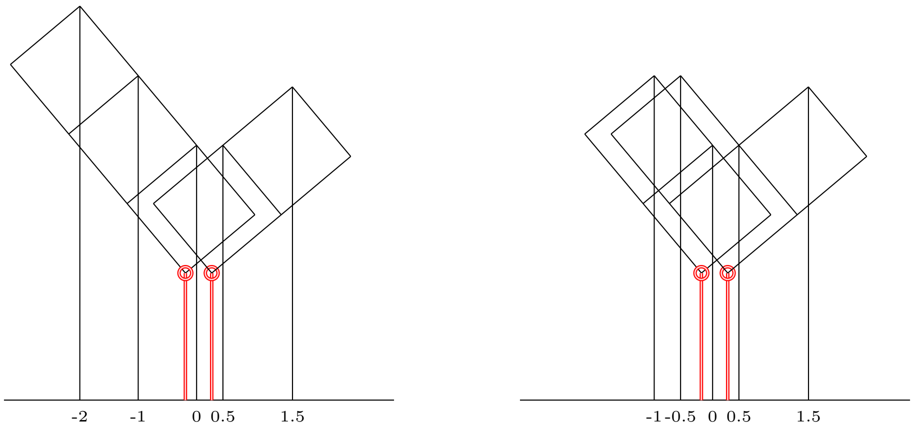



Given a *weighting* $\theta \in \mathbb{R}^l$, we have an associated diagrammatic Cherednik algebra $A(n, \theta, \kappa)$ which is a quasi-hereditary cover of the cyclotomic Hecke algebra $\mathscr{H}_n$.

**Definition 2.3.** Given a weighting $\theta$ and some $\lambda \in \mathscr{P}^l_n$, we draw the Young diagram $[\lambda]$ of $\lambda$ by placing the first node of the $j$th component at point $\theta_j$ on the $x$-axis, with all boxes have diagonals of length 2. We tilt our Young diagrams ever so slightly clockwise, to ensure that the top vertex of each box is at a different $x$-coordinate.

(The *loading* $i_\lambda$ is the $n$-tuple of real numbers given by projecting the top vertices of boxes of $[\lambda]$ onto the real line, along with the residue associated to each box.)

**Example.** Suppose $\theta = (0, 0.5)$. If $\lambda = ((3), (1^2))$ and $\mu = ((2), (2, 1))$, then we draw $[\lambda]$ and $[\mu]$ as below, with the loadings given by projections onto the real line.




There is a partial order on $\mathscr{P}^l_n$, which we denote by $\trianglerighteq_\theta$ and call the $\theta$-*dominance order*, which is a coarsening of the usual dominance order when $l = 1$. For $l > 1$, there is some subtle dependence on $\theta$.

Let $\lambda, \mu \in \mathscr{P}^l_n$. We have a notion of *semistandard tableaux* of shape $\lambda$ and weight $\mu$, which has a technical definition but is analogous to the usual notion.

We denote the set of semistandard tableaux of shape $\lambda$ and weight $\mu$ by $\mathrm{SStd}(\lambda, \mu)$.

**Theorem 2.4** (Webster)**.** *The diagrammatic Cherednik algebra $A(n, \theta, \kappa)$ is a graded cellular algebra with respect to the $\theta$-dominance order and a basis indexed by* $\mathrm{SStd}(\lambda, \mu)$ *as $\lambda$ and $\mu$ range over $\mathscr{P}^l_n$.*

*In particular we have graded standard modules* $\Delta(\lambda) = \langle C_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{SStd}(\lambda, -)\rangle_{\mathbb{F}}$ *with graded simple heads* $\mathrm{L}(\lambda)$ *forming a complete set of graded simple modules, up to grading shift.*

Over $\mathbb{C}$, the module category of $A(n, \theta, \kappa)$ is equivalent to category $\mathcal{O}$ for the rational cyclotomic Cherednik algebra. If $\theta$ is *well-separated* (that is, $\theta_j - \theta_k >> 0$ for all $j$ and $k$), then $A(n, \theta, \kappa)$ is Morita equivalent to the $q$-Schur algebra of Dipper–James–Mathas over arbitrary fields.

We would like to compute the graded decomposition numbers $d_{\lambda\mu}(v)$ for $A(n, \theta, \kappa)$.

# 3 Decomposition number results for $A(n, \theta, \kappa)$

**Definition 3.1.** We call a weighting $\theta \in \mathbb{R}^l$ *FLOTW* if $0 < \theta_j - \theta_i < 1$ for all $i < j$.

**Lemma 3.2.** *If $\theta \in \mathbb{R}^l$ is a FLOTW weighting, then the set of one-column $l$-multipartitions (i.e. the set of $l$-multipartitions $\lambda$ such that $\lambda_i^{(m)} = 0$ or $1$ for all $1 \leqslant m \leqslant l$ and for all $i$) is saturated in the $\theta$ dominance order. Therefore, for any such $\lambda$, we have that the graded decomposition number $[\Delta(\lambda) : L(\mu)]_v \neq 0$ only if $\mu$ is also a one-column multipartition. (We may thus restrict our attention to a subcategory of modules whose simple constituents are labelled by one-column multipartitions.)*

**Theorem 3.3** (Bowman, Cox, S.)**.** *Suppose $\mathbb{F} = \mathbb{C}$ and let $\lambda$ and $\mu$ be one-column $l$-multipartitions of $n$. The graded decomposition number $[\Delta(\lambda) : L(\mu)]_v$ is an affine Kazhdan–Lusztig polynomial of affine type $\hat{A}_{l-1}$. We can compute these efficiently using an algorithm of Soergel's in an alcove geometry.*

Note that when $l = 2$, we recover a result for the Temperley–Lieb algebra of type $B$, sometimes also called the blob algebra. In this case, we have also recovered the submodule structure of all cell modules $\Delta(\lambda)$.

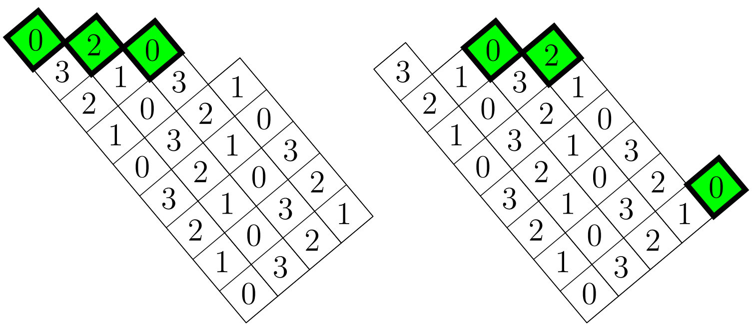
Now let $\mathbb{F}$ be a field of arbitrary characteristic again. We have the following result, which is a generalisation of results of Kleshchev, Chuang–Miyachi–Tan and Tan–Teo.

**Theorem 3.4** (Bowman, S.)**.** *Suppose $\kappa$ contains $i$ with multiplicity at most one, and that $\lambda$ and $\mu$ are $l$-multipartitions of $n$ such that $\lambda \trianglerighteq_\theta \mu$ and one can be obtained from the other by moving nodes of residue $i$. Then $[\Delta(\lambda) : L(\mu)]_v$ is a Kazhdan–Lusztig polynomial of type $A_r \times A_s \subseteq A_{r+s}$. Moreover, we can explicitly compute $[\Delta(\lambda) : L(\mu)]_v$ using a result of Tan–Teo.*

*Remark.*

- In fact, we do better than this. So long as $S = \{i_1, \ldots, i_j\}$ is an adjacency free set of residues, $\lambda$ may differ from $\mu$ by moving nodes of any residues in $S$.
- The computation of $[\Delta(\lambda) : L(\mu)]_v$ relies only on the order of addable and removable $i$-nodes in $\lambda$ and $\mu$, reading from left to right.
- Our results are at the level of isomorphisms of graded algebras (certain subquotients of $A(n, \theta, \kappa)$), and we are able to reduce the problem to that solved by Tan and Teo.

**Example.** Let $e = 4$. In level 1, we could have $\lambda = (9, 8, 7, 5), \mu = (8, 7^2, 6, 1)$.



**Definition 3.5.** Suppose for some $a \in \mathbb{R}$ we can draw a vertical line at $x = a$ such that it intersects the same nodes in both $\lambda$ and $\mu$, and the number of nodes to the left (and

therefore right) of the line $x = a$ is the same in both $\lambda$ and $\mu$. Then we say that the pair $(\lambda, \mu)$ *admits a $\theta$-diagonal cut at $x = a$.*

In this case we define $\mu^L$ to be the smallest multipartition containing all nodes which intersect the line $x = a$ as well all nodes to the left. Similarly, we define $\mu^R$ to be the smallest multipartition containing all nodes which intersect the line $x = a$ as well all nodes to the right. Likewise we may define $\lambda^L$ and $\lambda^R$.

**Theorem 3.6** (Bowman, S.)**.** *Suppose $\lambda \trianglerighteq_\theta \mu$ and $(\lambda, \mu)$ admits a diagonal cut at $x = a$ for some $a \in \mathbb{R}$. Then*

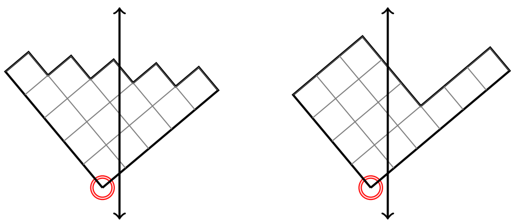
$$d_{\lambda\mu}(v) = d_{\lambda^L\mu^L}(v) \times d_{\lambda^R\mu^R},$$

*and, furthermore,*

$$\mathrm{Ext}^k_{A(n,\theta,\kappa)}(\Delta(\lambda), \Delta(\mu)) \cong \bigoplus_{i+j=k} \mathrm{Ext}^i_{A(n_L,\theta,\kappa)}(\Delta(\lambda^L), \Delta(\mu^L)) \otimes \mathrm{Ext}^j_{A(n_R,\theta,\kappa)}(\Delta(\lambda^R), \Delta(\mu^R)),$$

*where $n_L = |\lambda^L| = |\mu^L|$ and $n_R = |\lambda^R| = |\mu^R|$.*

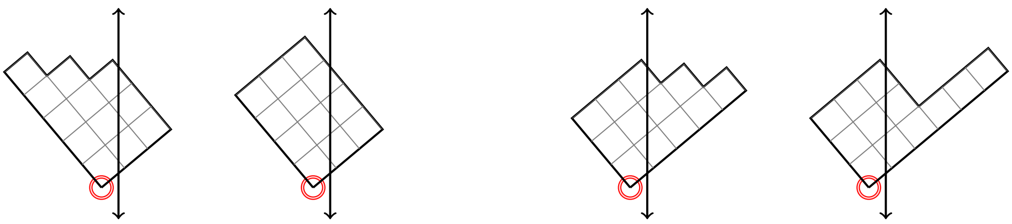
**Examples.** 1. Fix $e = 3$ and $\theta = 0$. The partitions $(5, 4, 3, 2, 1)$ and $(4^3, 1^3)$ admit a diagonal cut at $x = 0.5$.



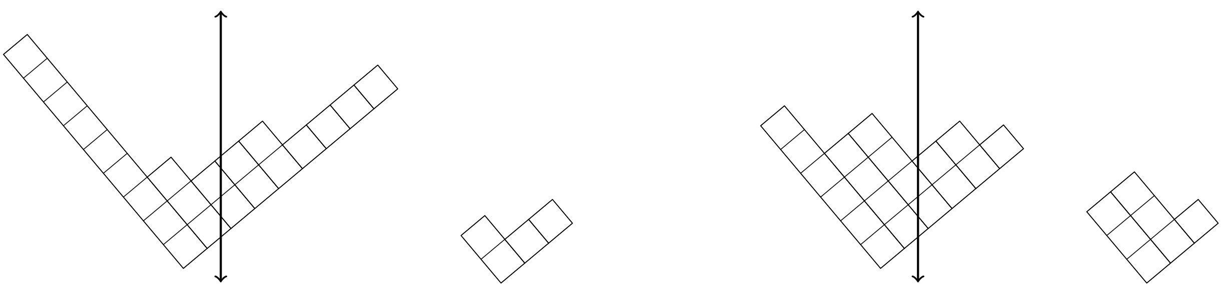
We have

$$\lambda^L = (5, 4, 3), \quad \mu^L = (4^3), \quad \lambda^R = (3^3, 2, 1), \quad \mu^R = (3^3, 1^3).$$

These are depicted below.

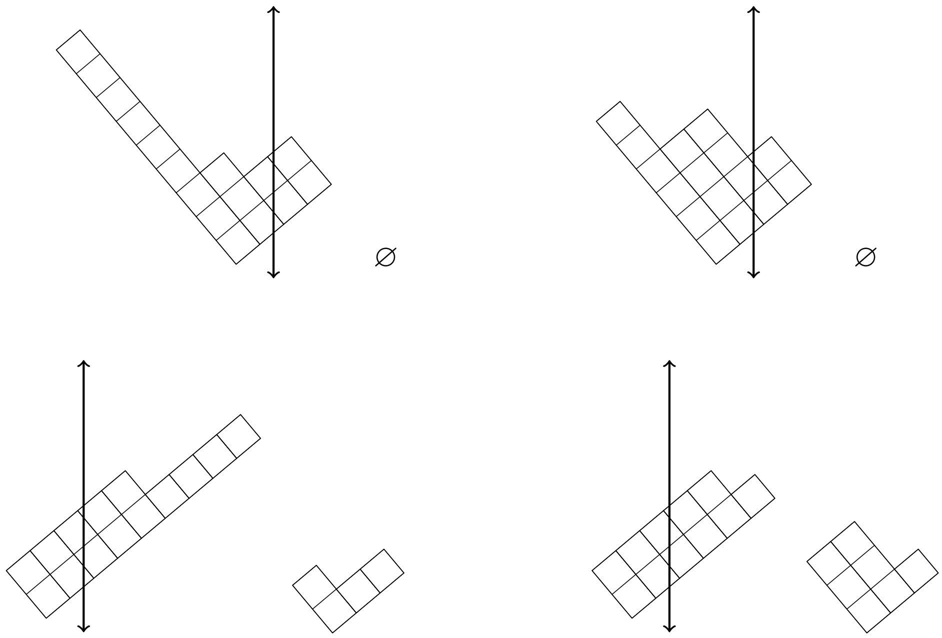


2. Let $\theta = (0, 50)$, $\lambda = ((9, 3, 2^3, 1^4), (2, 1^2))$ and $\mu = ((6, 4^2, 2^2, 1), (3^2, 1))$. Then $(\lambda, \mu)$ admits a $\theta$-diagonal cut at $x = 1.5$.

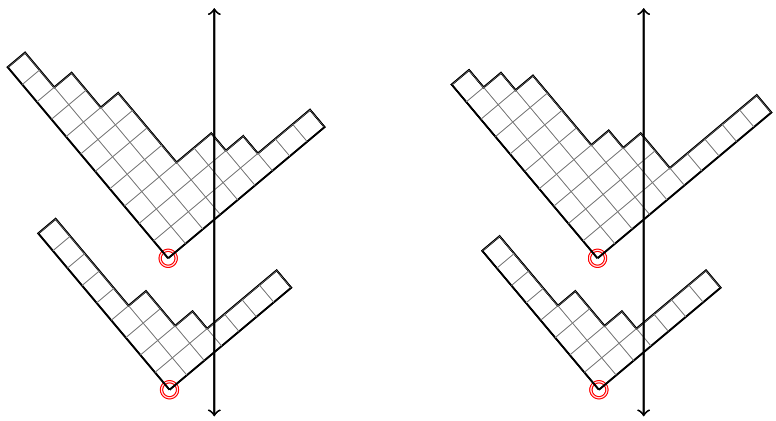


Then we have

$$\lambda^L = ((9,3,2^2),(\varnothing)), \qquad \lambda^R = ((2^5,1^4),(2,1^2)),$$
$$\mu^L = ((6,4^2,2),(\varnothing)), \qquad \mu^R = ((2^5,1),(3^2,1)).$$




3. Let $\theta = (0, 0.5)$, $\lambda = ((11,9,7,3^2,2,1^3),(9,4,2,1^4))$ and $\mu = ((10,9,8,4,3,1^5),(8,4,2,1^4))$. Then $(\lambda,\mu)$ admits a $\theta$-diagonal cut at $x = 2.6$.



Then we have

$$\lambda^L = ((11,9,7,3^2),(9,4,2)), \qquad \lambda^R = ((3^5,2,1^3),(1^7)),$$
$$\mu^L = ((10,9,8,4,3),(8,4,2)), \qquad \mu^R = ((3^5,1^5),(1^7)).$$