Graded decomposition matrices for type C KLR algebras

Joint work with Chris Chung and Andrew Mathas

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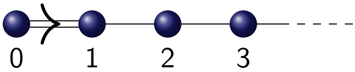

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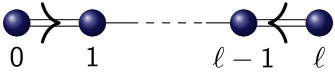

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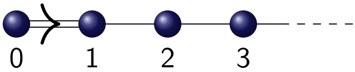

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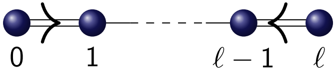

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We have simple roots $\{\alpha_i \mid i \in I\}$, fundamental weights $\{\Lambda_i \mid i \in I\}$, etc.

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$$y_r \psi_r e(\mathbf{i}) = (\psi_r y_{r+1} - \delta_{i_r, i_{r+1}}) e(\mathbf{i}); \qquad y_{r+1} \psi_r e(\mathbf{i}) = (\psi_r y_r + \delta_{i_r, i_{r+1}}) e(\mathbf{i});$$

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$$y_1^{\langle \alpha_{i_1}^\vee, \Lambda \rangle} e(\mathbf{i}) = 0;$$

for all admissible $r, s, \mathbf{i}, \mathbf{j}$.

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- Chung–Hudak (2022): Defect 1 blocks of $\mathscr{R}_n^{\Lambda_k}$ are Morita equivalent to Brauer line algebras. (Graded decomposition matrices identical to type A case.)

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iii) $\psi_r z^\lambda = 0$ if r and $r+1$ lie in the same row of $\mathtt{T}^\lambda$,

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The Specht modules arise as the cell modules in the work of Evseev–Mathas, but also have an earlier construction (Ariki–Park–S.).

Definition/Example

Let $\mathtt{T}^\lambda$ denote the row-initial λ-tableau. e.g. if $\lambda = (5,3,2)$, then

1	2	3	4	5
6	7	8		
9	10			

The Specht module $\mathrm{S}^\lambda = \mathrm{S}^\lambda_k$ is the homogeneous $\mathscr{R}_n^{\Lambda_k}$-module generated by z^λ (of degree $\deg \mathtt{T}^\lambda$) subject to the relations

i) $e(\mathbf{i})z^\lambda = \delta_{\mathbf{i},\mathrm{res}\,\mathtt{T}^\lambda} z^\lambda$,

ii) $y_r z^\lambda = 0$ for all r,

iii) $\psi_r z^\lambda = 0$ if r and $r+1$ lie in the same row of $\mathtt{T}^\lambda$,

iv) For each Garnir node $A \in [\lambda]$, we have a Garnir relation $g^\lambda_{k,A} z^\lambda = 0$.

Specht modules

Let $w^{\mathtt{T}} \in \mathfrak{S}_n$ be such that $\mathtt{T} = w^{\mathtt{T}}\mathtt{T}^{\lambda}$.

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Theorem (Ariki–Park–S. (2019), Evseev–Mathas (2022))

For any partition λ, the Specht module S^{λ} has homogeneous $\mathbb{F}$-basis $\{v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$, and $\deg(v_{\mathtt{T}}) = \deg \mathtt{T}$.

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(This all works for multipartitions, and any Λ, not just the level 1 situation mentioned above. APS proves that this set always spans the Specht module, and is a basis if all Garnir relations have just a single term – i.e. is of the form $\psi_w z^\lambda = 0$.)

Graded characters

Definition

The *graded character* of an $\mathscr{R}^{\Lambda}_{n}$-module M is

$$\mathrm{ch}\, M = \sum_{\mathbf{i}\in I^n} \dim_v(e(\mathbf{i})M)\mathbf{i}.$$

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In particular,

$$\operatorname{ch} \mathrm{S}^\lambda = \sum_{\mathtt{T} \in \mathrm{Std}(\lambda)} v^{\deg \mathtt{T}} \operatorname{res} \mathtt{T}.$$

Graded characters

Example

Let $\ell = 3$, $\Lambda = \Lambda_1$, $\lambda = (3, 1^3)$.

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$$\mathtt{T}_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \quad \mathtt{T}_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \quad \mathtt{T}_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \quad \mathtt{T}_4 = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \quad \mathtt{T}_5 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}$$

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$\operatorname{ch} S^\lambda = (1, 2, 3, 0, 1, 2)$

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$$\operatorname{ch} S^\lambda = (1,2,3,0,1,2) + \color{red}{(1,2,0,3,1,2)}$$

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$$T_1 = \begin{array}{ccc} 1 & 2 & 3 \\ 4 \\ 5 \\ 6 \end{array} \quad T_2 = \begin{array}{ccc} 1 & 2 & 4 \\ 3 \\ 5 \\ 6 \end{array} \quad T_3 = \begin{array}{ccc} 1 & 2 & 5 \\ 3 \\ 4 \\ 6 \end{array} \quad {\color{red} T_4 = \begin{array}{ccc} 1 & 2 & 6 \\ 3 \\ 4 \\ 5 \end{array}} \quad T_5 = \begin{array}{ccc} 1 & 3 & 4 \\ 2 \\ 5 \\ 6 \end{array}$$

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Graded characters

Example

Let $\ell = 3$, $\Lambda = \Lambda_1$, $\lambda = (3, 1^3)$. Then $[\lambda] = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 0 \\ \cline{1-1} 1 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$ and the basis is

$$\mathtt{T}_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} \quad \mathtt{T}_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} \quad \mathtt{T}_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} \quad \mathtt{T}_4 = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \quad \mathtt{T}_5 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$$

$$\mathtt{T}_6 = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} \quad \mathtt{T}_7 = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \quad \mathtt{T}_8 = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} \quad \mathtt{T}_9 = \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \quad \mathtt{T}_{10} = \begin{array}{|c|c|c|}\hline 1 & 5 & 6 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

$$\begin{aligned}\operatorname{ch} S^\lambda = {} & (1,2,3,0,1,2) + (1,2,0,3,1,2) + (1,2,0,1,3,2) + v^2(1,2,0,1,2,3) + (1,0,2,3,1,2) \\ & + (1,0,2,1,3,2) + v^2(1,0,2,1,2,3) + v(1,0,1,2,3,2) + v(1,0,1,2,2,3)\end{aligned}$$

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Graded decomposition numbers

By Evseev–Mathas's graded cellularity result, we have irreducible $\mathscr{R}_n^\Lambda$-modules given as heads of certain Specht modules (upto grading shift and isomorphism).

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Definition

The graded decomposition number $d_{\lambda\mu} = [\mathsf{S}^\lambda : \mathsf{D}^\mu]_v$ is the graded multiplicity of D^μ in S^λ. i.e.

$$d_{\lambda\mu} = \sum_{k\in\mathbb{Z}} [\mathsf{S}^\lambda : \mathsf{D}^\mu\langle k\rangle] v^k.$$

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(Again, this is unsurprising, and is the case for all defect 1 blocks, by Chung–Hudak.)

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Repeat the previous tactic – work row-by-row computing the graded characters of Specht modules, and tear off non-bar-invariant pieces.

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$4,1^2$	v^2	·	v	1	·
$5,1$	·	·	·	v^2	1
3^2	·	·	v	·	·
$4,2$	·	·	v^2	v	·
6	·	·	·	·	v^2

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It is known by work of Shan that in type A, the radical filtration and grading filtration coincide.
But we see that it is already not quite working here: the radical of $\mathrm{S}^{(5,1)}$ is $\mathrm{D}^{(4,1^2)}\langle 2\rangle$ (there is no factor shifted by degree 1).

An overview of canonical bases for $U_q(\widehat{\mathfrak{sp}_\ell})$-modules

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Unlike in type A, the canonical basis element $G(\lambda)$ should not in general correspond to the projective cover of the simple module D^λ.

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$3^2,2$	v^2	·	·	·	v^2	·
$4,2,1^2$	v^2	·	·	v^3+v	v^2	v^2
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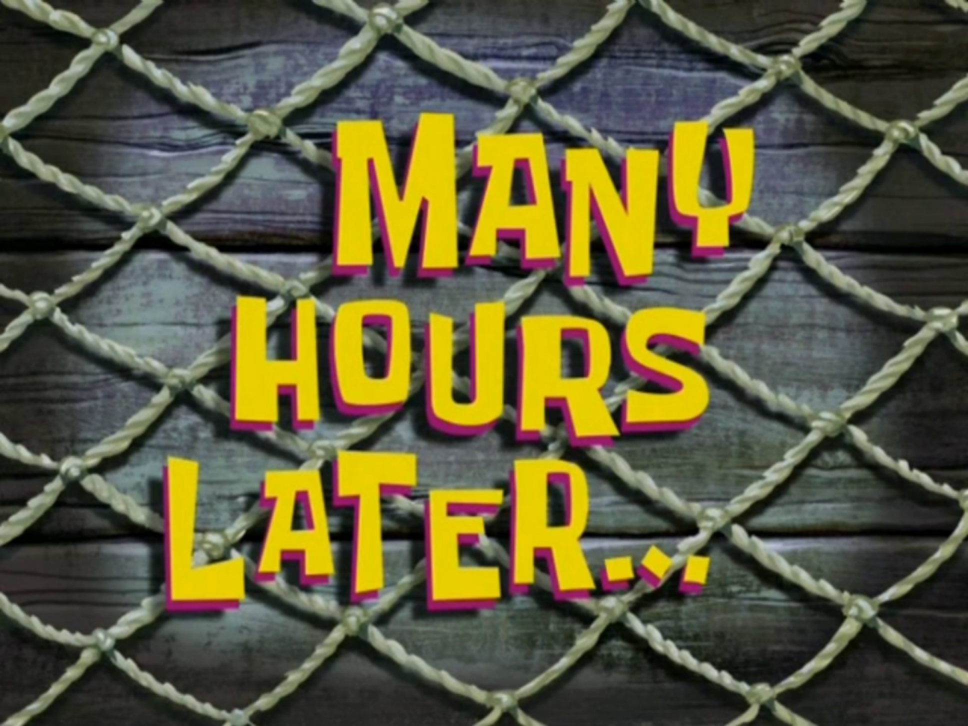
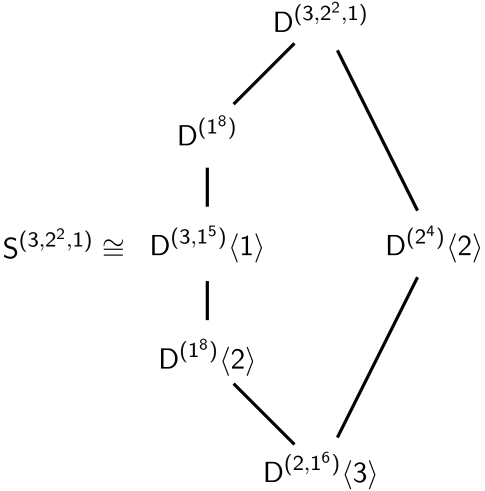
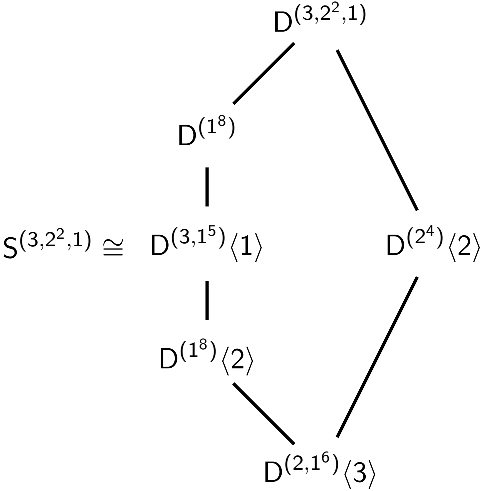
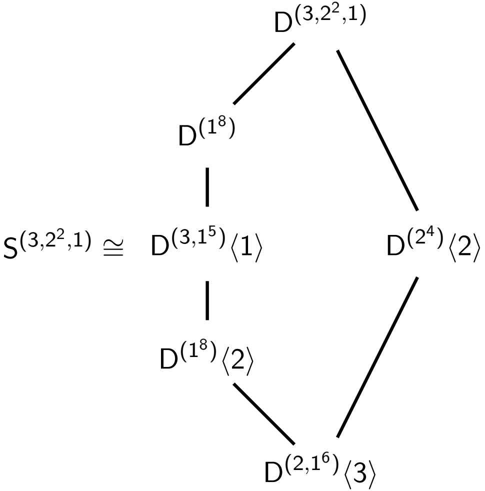
Next, we can do some more brute force computation in this block...


MANY
HOURS
LATER...


$D^{(3,2^2,1)}$
$D^{(1^8)}$
$S^{(3,2^2,1)} \cong$
$D^{(3,1^5)}\langle 1\rangle$
$D^{(2^4)}\langle 2\rangle$
$D^{(1^8)}\langle 2\rangle$
$D^{(2,1^6)}\langle 3\rangle$



So we also get our first non-uniserial module,



So we also get our first non-uniserial module, and an example (in any characteristic!) where the grading filtration and radical filtration do not match.

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