Specht modules for the KLR algebras of type C

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Abstract

The KLR algebras were introduced almost a decade ago to categorify the negative half of a quantum group. In type A, Brundan and Kleshchev showed that cyclotomic quotients of KLR algebras are isomorphic to cyclotomic Hecke algebras, which has spurred on the development of their graded representation theory, in particular with a theory of Specht modules. We will report on recent joint work with Susumu Ariki and Euiyong Park, in which we have defined a family of Specht modules for the KLR algebras in type C. We will outline some of their basic properties and explain why they are interesting objects to study. We will finally discuss how we used these Specht modules to classify which cyclotomic quotients of the KLR algebras of type C are semisimple.

1 Tableaux and type C residues

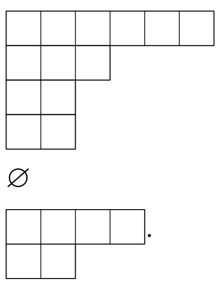
Fix a field $\mathbb{F}$. Let $\ell \in \{2,3,\dots\} \cup \{\infty\}$ and $I = \mathbb{Z}_{\geqslant 0}$ if $\ell = \infty$ or $I = \{0,1,2,\dots,\ell\}$ otherwise. We have the root datum of type C_∞ if $\ell = \infty$, or type $C_\ell^{(1)}$ if $\ell < \infty$. In particular we have the Cartan matrix $A = (a_{ij})_{i,j\in I}$, simple roots $\{\alpha_i \mid i \in I\}$ and fundamental weights $\{\Lambda_i \mid i \in I\}$ in P, and an invariant symmetric bilinear form on P satisfying $(\alpha_i, \alpha_j) = d_i a_{ij}$ where $d = (2,1,1,\dots)$ if $\ell = \infty$ and $d = (2,1,\dots,1,2)$ if $\ell < \infty$.

Fix a *level* $l \in \mathbb{Z}_{>0}$ and a *multicharge* $\kappa = (\kappa_1, \dots, \kappa_l) \in \mathbb{Z}^l$.

Definition 1.1. An *l-multipartition of n* is an l-tuple of partitions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$ such that $\sum |\lambda^{(i)}| = n$. We denote the set of l-multipartitions by $\mathscr{P}_n^l$.

We may draw the *Young diagram* $[\lambda]$ of $\lambda \in \mathscr{P}_n^l$ as depicted in the following example.

Example. Let $\lambda = ((6,3,2,2), \varnothing, (4,2))$. Then

$[\lambda] =$ 

Next, we describe the type C residue pattern. To each *node* in a Young diagram, we may associate a residue as follows. Let $A = (r, c, m) \in [\lambda]$.

If $\ell = \infty$, then we define the residue of A to be $\operatorname{res} A := \overline{\kappa_m + c - r} = |\kappa_m + c - r|$.

If $\ell < \infty$, we define $p : \mathbb{Z} \to \mathbb{Z}/2\ell\mathbb{Z}$ to be the natural projection, and $f_\ell : \mathbb{Z} \to I$ the function determined by

$$f_\ell(0+2\ell\mathbb{Z}) = 0, \quad f_\ell(\ell+2\ell\mathbb{Z}) = \ell, \quad f_\ell(k+2\ell\mathbb{Z}) = f_\ell(2\ell-k+2\ell\mathbb{Z}) = k \text{ for } 1 \leqslant k \leqslant \ell-1.$$

Then $\operatorname{res} A := \overline{\kappa_m + c - r}$, where $\overline{} : f_\ell \circ p : \mathbb{Z} \to I$ is the composition of these maps.

Example. Let $\ell = 3$, $\kappa = (2, 0, -1)$, and $\lambda = ((6, 3, 2, 2), \varnothing, (4, 2))$ as above.

$$\begin{array}{|c|c|c|c|c|c|} \hline 2&3&2&1&0&1\\ \hline 1&2&3 \\ \cline{1-3} 0&1\\ \cline{1-2} 1&0\\ \cline{1-2} \end{array}$$

$$\varnothing$$

$$\begin{array}{|c|c|c|c|} \hline 1&0&1&2\\ \hline 2&1\\ \cline{1-2} \end{array}$$

If instead we let $\ell = \infty$, then the residue pattern is as follows.

$$\begin{array}{|c|c|c|c|c|c|} \hline 2&3&4&5&6&7\\ \hline 1&2&3 \\ \cline{1-3} 0&1\\ \cline{1-2} 1&0\\ \cline{1-2} \end{array}$$

$$\varnothing$$

$$\begin{array}{|c|c|c|c|} \hline 1&0&1&2\\ \hline 2&1\\ \cline{1-2} \end{array}$$

Definition 1.2. Let $\lambda \in \mathscr{P}^l_n$. A λ-tableau is a filling of $[\lambda]$ with $\{1, 2, \ldots, n\}$ without repeats. A λ-tableau is called *row-strict* if its entries increase along rows within each component. If the entries also increase down columns within each component, we call the tableau *standard*. We denote the sets of row-strict and standard λ-tableaux by $\operatorname{RowStd}(\lambda)$ and $\operatorname{Std}(\lambda)$, respectively.

The initial λ-tableau $\mathtt{T}^\lambda$ is obtained by filling the entries along each row in order down the Young diagram (beginning with the first component).

The *residue sequence* of a λ-tableau $\mathtt{T}$ is $\operatorname{res} \mathtt{T} = (i_1, \ldots, i_n)$ where i_r is the residue of the node occupied by r in $\mathtt{T}$.

Example. For $\lambda = ((6, 3, 2, 2), \varnothing, (4, 2))$ and $\kappa = (2, 0, -1)$ as before, the initial tableau $\mathtt{T}^\lambda$ is

$$\begin{array}{|c|c|c|c|c|c|} \hline 1&2&3&4&5&6\\ \hline 7&8&9 \\ \cline{1-3} 10&11\\ \cline{1-2} 12&13\\ \cline{1-2} \end{array}$$

$$\varnothing$$

$$\begin{array}{|c|c|c|c|} \hline 14&15&16&17\\ \hline 18&19\\ \cline{1-2} \end{array}\ .$$

If $\ell = 3$, then $\operatorname{res} \mathtt{T}^\lambda = (2, 3, 2, 1, 0, 1, 1, 2, 3, 0, 1, 1, 0, 1, 0, 1, 2, 2, 1)$. If $\ell = \infty$, then $\operatorname{res} \mathtt{T}^\lambda = (2, 3, 4, 5, 6, 7, 1, 2, 3, 0, 1, 1, 0, 1, 0, 1, 2, 2, 1)$.

For a λ-tableau $\mathtt{T}$, we denote by $w^{\mathtt{T}} \in \mathfrak{S}_n$ the permutation such that $w^{\mathtt{T}}\mathtt{T}^{\lambda}$.

If $A = (r, c, m) \in [\lambda]$ and $(r+1, c, m) \in [\lambda]$, we call A a *Garnir node*. The corresponding *Garnir belt* $\mathbf{B^A}$ is the set of nodes

$$\{(r, a, m) \in [\lambda] \mid c \leqslant a \leqslant \lambda_r^{(m)}\} \cup \{(r+1, a, m) \in [\lambda] \mid 1 \leqslant a \leqslant c\}.$$

The *Garnir tableau* $\mathtt{G}^A$ is the λ-tableau which agrees with $\mathtt{T}^\lambda$ outside of $\mathbf{B^A}$ and has the entries of $\mathbf{B^A}$ appearing in order along the bottom row, then along the top row.

Example. For $\lambda = ((6, 3, 2, 2), \varnothing, (4, 2))$ and $A = (2, 2, 1)$,

$$\mathtt{G}^A = \begin{array}{l} \begin{array}{|c|c|c|c|c|c|} \hline 1 & 4 & 5 & 6 & 7 & 8 \\ \hline 2 & 3 & 9 \\ \cline{1-3} 10 & 11 \\ \cline{1-2} 12 & 13 \\ \cline{1-2} \end{array} \\ \\ \varnothing \\ \\ \begin{array}{|c|c|c|c|} \hline 14 & 15 & 16 & 17 \\ \hline 18 & 19 \\ \cline{1-2} \end{array} \; . \end{array}$$

2 KLR algebras and Specht modules

We fix a system of polynomials $Q_{i,j}(u, v) \in \mathbb{F}[u, v]$ for $i, j \in I$ of the form

$$Q_{i,j}(u, v) = \begin{cases} \sum_{p(\alpha_i,\alpha_i)+q(\alpha_j,\alpha_j)+2(\alpha_i,\alpha_j)=0} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

where $t_{i,j;p,q} \in \mathbb{F}$ are such that $t_{i,j;-a_{ij},0} \in \mathbb{F}^\times$ and $Q_{i,j}(u, v) = Q_{j,i}(v, u)$.

Here, we choose a specific system of polynomials as follows. If $\ell < \infty$ then, for $i < j$,

$$Q_{i,j}(u, v) = \begin{cases} u + v^2 & \text{if } (i, j) = (0, 1), \\ u + v & \text{if } i \neq 0, j = i + 1, j \neq \ell, \\ u^2 + v & \text{if } (i, j) = (\ell - 1, \ell), \\ 1 & \text{otherwise.} \end{cases}$$

If $\ell = \infty$ then, for $i < j$,

$$Q_{i,j}(u, v) = \begin{cases} u + v^2 & \text{if } (i, j) = (0, 1), \\ u + v & \text{if } i \neq 0, j = i + 1, \\ 1 & \text{otherwise.} \end{cases}$$

We note that if every element of $\mathbb{F}$ has a square root, then any other choice of polynomials yields an isomorphic algebra.

Definition 2.1. The *Khovanov–Lauda–Rouquier (KLR) algebra* $R(n)$ is the unital $\mathbb{F}$-algebra generated by

$$\{e(\nu) \mid \nu \in I^n\} \cup \{x_1, \ldots, x_n\} \cup \{\psi_1, \ldots, \psi_{n-1}\}$$

subject to the following relations.

$$e(\nu)e(\nu') = \delta_{\nu,\nu'} e(\nu);$$

$$\begin{aligned}
\sum_{\nu\in I^n} e(\nu) &= 1; \\
x_r e(\nu) &= e(\nu)x_r; \\
\psi_r e(\nu) &= e(s_r\nu)\psi_r; \\
x_r x_s &= x_s x_r; \\
\psi_r x_s &= x_s\psi_r && \text{if } s\neq r,r+1; \\
\psi_r\psi_s &= \psi_s\psi_r && \text{if } |r-s|>1; \\
x_r\psi_r e(\nu) &= (\psi_r x_{r+1} - \delta_{\nu_r,\nu_{r+1}})e(\nu); \\
x_{r+1}\psi_r e(\nu) &= (\psi_r x_r + \delta_{\nu_r,\nu_{r+1}})e(\nu); \\
\psi_r^2 e(\nu) &= Q_{\nu_r,\nu_{r+1}}(x_r,x_{r+1})e(\nu); \\
(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r)e(\nu) &= \begin{cases} \frac{Q_{\nu_r,\nu_{r+1}}(x_r,x_{r+1}) - Q_{\nu_r,\nu_{r+1}}(x_{r+2},x_{r+1})}{x_r - x_{r+2}} e(\nu) & \text{if } \nu_r = \nu_{r+2}, \\ 0 & \text{otherwise;} \end{cases}
\end{aligned}$$

for all admissible r, s, ν, ν'.

Let $\Lambda = \Lambda_{\overline{\kappa_1}} + \cdots + \Lambda_{\overline{\kappa_l}}$.

The *cyclotomic KLR algebra* $R^\Lambda(n)$ is the quotient of $R(n)$ by the relations

$$x_1^{\#\{i\mid\overline{\kappa_i}=\nu_1\}} e(\nu) = 0 \quad \text{for } \nu\in I^n.$$

We give a $\mathbb{Z}$-grading on $R(n)$ and $R^\Lambda(n)$ by setting

$$\deg(e(\nu)) = 0, \qquad \deg(x_r e(\nu)) = (\alpha_{\nu_r},\alpha_{\nu_r}), \qquad \deg(\psi_s e(\nu)) = -(\alpha_{\nu_s},\alpha_{\nu_{s+1}})$$

for all admissible r, s and ν.

For $k\in\mathbb{Z}$ and $d\in\mathbb{Z}_{>0}$, we define

$$\nu_{(k;d)} = (\overline{k}, \overline{k+1}, \ldots, \overline{k+d-1}) \in I^d.$$

Then $\mathcal{L}_{(k;d)} = \mathbb{F}l_{(k;d)}$ is the graded $R(d)$-module concentrated in degree 0 with

$$x_r l_{(k;d)} = \psi_s l_{(k;d)} = 0, \quad e(\nu)l_{(k;d)} = \delta_{\nu,\nu_{(k;d)}} l_{(k;d)},$$

for all r, s, ν.

For $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_r)\in\mathscr{P}_n^1$ and $\kappa\in\mathbb{Z}$, we define the graded $R(n)$-module $\mathrm{M}_\kappa^\lambda = \mathrm{M}^\lambda$ to be

$$\mathrm{M}^\lambda := \mathcal{L}_{(\kappa;\lambda_1)} \circ \mathcal{L}_{(\kappa-1;\lambda_2)} \circ \cdots \circ \mathcal{L}_{(\kappa-r+1;\lambda_r)},$$

where $\circ$ denotes the convolution product. For $\lambda = (\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(l)})\in\mathscr{P}_n^l$ and $\kappa = (\kappa_1,\ldots,\kappa_l)\in\mathbb{Z}^l$, we define the graded $R(n)$-module $\mathrm{M}_\kappa^\lambda = \mathrm{M}^\lambda$ to be

$$\mathrm{M}^\lambda := \mathrm{M}_{\kappa_1}^{\lambda^{(1)}} \circ \mathrm{M}_{\kappa_2}^{\lambda^{(2)}} \circ \cdots \circ \mathrm{M}_{\kappa_l}^{\lambda^{(l)}}.$$

In other words, M^λ is the module with generator m^λ subject to the relations

1. $e(\nu)m^\lambda = \delta_{\nu,\mathrm{res}\,\mathtt{T}^\lambda} m^\lambda$,
2. $x_r m^\lambda = 0$,
3. $\psi_r m^\lambda = 0$ if r and $r+1$ lie in the same row of $\mathtt{T}^\lambda$.

For each $w \in \mathfrak{S}_n$, fix a reduced expression $w = s_{i_1} \dots s_{i_r}$. We define $\psi_w = \psi_{i_1} \dots \psi_{i_r}$.

Lemma 2.2. M^λ *has a homogeneous* $\mathbb{F}$*-basis* $\{\psi_{w^{\mathtt{T}}} m^\lambda \mid \mathtt{T} \in \mathrm{RowStd}(\lambda)\}$.

For each Garnir node $A = (r, c, m) \in [\lambda]$, we define

$$\begin{aligned}\mathrm{M}^\lambda_{\kappa,A} = {}& \mathrm{M}^{\lambda^{(1)}}_{\kappa_1} \circ \dots \circ \mathrm{M}^{\lambda^{(m-1)}}_{\kappa_{m-1}} \circ \mathcal{L}_{(\kappa_m;\lambda^{(m)}_1)} \circ \dots \circ \mathcal{L}_{(\kappa_m - r+2;\lambda^{(m)}_{r-1})} \circ \\ & \mathcal{L}_{(\kappa_m - r+1;c-1)} \circ \mathcal{L}_{(\kappa_m - r;\lambda^{(m)}_r + 1)} \circ \mathcal{L}_{(\kappa_m - r+c;\lambda^{(m)}_{r+1} - c)} \circ \\ & \circ \mathcal{L}_{(\kappa_m - r-1;\lambda^{(m)}_{r+2})} \circ \dots \circ \mathcal{L}_{(\kappa_m - t+1;\lambda^{(m)}_t)} \circ \mathrm{M}^{\lambda^{(m+1)}}_{\kappa_{m+1}} \circ \dots \circ \mathrm{M}^{\lambda^{(l)}}_{\kappa_l}\end{aligned}$$

where $\lambda^{(m)} = (\lambda^{(m)}_1, \dots, \lambda^{(m)}_t)$.

Now, we have homomorphisms $\mathrm{M}^\lambda_{\kappa,A}\langle d(\lambda, A)\rangle \to \mathrm{M}^\lambda$, mapping the generator of (a degree shifted copy of) $\mathrm{M}^\lambda_{\kappa,A}$ to a *Garnir element* $g^\lambda_{\kappa,A} \in \mathrm{M}^\lambda$.

The *Specht module* $\mathrm{S}^\lambda = \mathrm{S}^\lambda_\kappa$ is the quotient of $\mathrm{M}^\lambda\langle \deg \mathtt{T}^\lambda\rangle$ (some degree shifted copy of M^λ) by the submodule generated by all Garnir elements $g^\lambda_{\kappa,A}$. We denote by $\overline{m}^\lambda$ the image of m^λ under the projection $\mathrm{M}^\lambda\langle \deg \mathtt{T}^\lambda\rangle \to \mathrm{S}^\lambda$.

Lemma 2.3. *If* $\ell = \infty$, $g^\lambda_{\kappa,A} = \psi_{w^{\mathtt{G}^A}} m^\lambda$. *If* $\ell < \infty$, $g^\lambda_{\kappa,A} = \psi_{w^{\mathtt{G}^A}} m^\lambda + \sum_{w<w^{\mathtt{G}^A}} a_w \psi_w m^\lambda$.

Theorem 2.4 [1, Theorem 3.12 and Corollary 3.13]. *Let* $\lambda \in \mathscr{P}^l_n$. *Then* $\mathrm{S}^\lambda_\kappa$ *is a graded* $R^\Lambda(n)$*-module, spanned by the homogeneous elements* $\{\psi_{w^{\mathtt{T}}} \overline{m}^\lambda \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$.

Theorem 2.5 [1, Theorem 3.19 and Corollary 3.21]. *Suppose* $\ell = \infty$ *and* $\lambda \in \mathscr{P}^l_n$. *Then the set* $\{\psi_{w^{\mathtt{T}}} \overline{m}^\lambda \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ *is an* $\mathbb{F}$*-basis of* S^λ. *Moreover, we have the graded character formula*

$$\mathrm{ch}_q \mathrm{S}^\lambda = \sum_{\mathtt{T} \in \mathrm{Std}(\lambda)} q^{\deg \mathtt{T}} \operatorname{res} \mathtt{T}.$$

We conjectured that the above result remains true when $\ell < \infty$, and conjectured an explicit form for $g^\lambda_{\kappa,A}$ in this case.

Let $A \in [\lambda]$. We define λ_A to be the multipartition obtained from λ by removing A.

Corollary 2.6 [1, Corollaries 3.20 and 3.21]. *Let* E^Λ_i *denote the* i*-restriction functor on* $R^\Lambda(n)$*-modules. Then we have the following branching rule in the Grothendieck group of* $R^\Lambda(n-1)$.

$$[E^\Lambda_i \mathrm{S}^\lambda] = \sum_A q^{d_A(\lambda)} [\mathrm{S}^{\lambda_A}],$$

where the sum runs over all removable i*-nodes of* $[\lambda]$.

We have a similar result for i-induction, with a slightly messier formula for the grading shift.

3 Semisimplicity

For $i \in I$ and $k \in \mathbb{Z}_{>0}$, we define the positive root $\alpha_{i,n} = \alpha_i + \alpha_{i+1} + \dots + \alpha_{i+k-1}$, where we take indices module $\ell + 1$ if $\ell < \infty$.

We note the following conditions which we will refer back to.

(SS1) For all $i \in I$, $(\Lambda, \alpha_{i,n}) \leqslant 1$.

(SS2) For all $1 \leqslant j \leqslant l$, $\frac{n-1}{2} \leqslant \overline{\kappa_j} \leqslant \ell - \frac{n-1}{2}$.

Roughly, condition (SS1) says that the $\overline{\kappa_i}$ are "far enough apart" from each other, while condition (SS2) says they they are far enough away from 0 and ℓ, around which the residue pattern is symmetric. The reason for wanting such conditions is the following lemma, which is key to our semisimplicity argument.

Lemma 3.1 [2, **Lemma 3.1**]. *Suppose that conditions (SS1) and (SS2) hold. Let $\lambda, \mu \in \mathscr{P}_n^l$ and $\mathtt{S} \in \operatorname{Std}(\lambda)$, $\mathtt{T} \in \operatorname{Std}(\mu)$. Then $\mathtt{S} = \mathtt{T}$ if and only if $\operatorname{res} \mathtt{S} = \operatorname{res} \mathtt{T}$.*

Examples. 1. If $\kappa = (0)$ and $n = 2$, then condition (SS2) fails. We have the following residue patterns for partitions (2) and (1^2).

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array} \quad \text{and} \quad \begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline\end{array}$$

The tableaux

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \quad \text{and} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$$

thus have the same residue sequences.

2. If $\kappa = (1, 2)$ and $n = 2$, then condition (SS1) fails. We have the following residue patterns for bipartitions $((2), \varnothing)$ and $((1), (1))$.

$$\begin{array}{c} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \\ \\ \varnothing \end{array} \qquad\qquad \begin{array}{c} \begin{array}{|c|}\hline 1 \\ \hline\end{array} \\ \\ \begin{array}{|c|}\hline 2 \\ \hline\end{array} \end{array}$$

The tableaux

$$\begin{array}{c} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \\ \\ \varnothing \end{array} \qquad\qquad \begin{array}{c} \begin{array}{|c|}\hline 1 \\ \hline\end{array} \\ \\ \begin{array}{|c|}\hline 2 \\ \hline\end{array} \end{array}$$

thus have the same residue sequences.

Theorem 3.2 [2, **Theorem 3.3**]. *Suppose that conditions (SS1) and (SS2) hold, and let $\lambda \in \mathscr{P}_n^l$. Then the Specht module S^λ is concentrated in degree 0, has basis $\{\psi_{w^{\mathtt{T}}} \overline{m}^\lambda \mid \mathtt{T} \in \operatorname{Std}(\lambda)\}$, and the $R^\Lambda(n)$-action on the basis is given by*

$$e(\nu)\psi_{w^{\mathtt{T}}}\overline{m}^\lambda = \delta_{\nu, \operatorname{res}\mathtt{T}}\psi_{w^{\mathtt{T}}}\overline{m}^\lambda, \ x_r\psi_{w^{\mathtt{T}}}\overline{m}^\lambda = 0, \ \psi_r\psi_{w^{\mathtt{T}}}\overline{m}^\lambda = \begin{cases} \psi_{w^{s_r\mathtt{T}}}\overline{m}^\lambda & \text{if } s_r\mathtt{T} \text{ is standard,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, S^λ is an irreducible graded $R^\Lambda(n)$-module.

Examples. Suppose conditions (SS1) and (SS2) hold.

1. Let $n = 3$. Then we have three Specht modules $\mathrm{S}^{(3)}$, $\mathrm{S}^{(2,1)}$, and $\mathrm{S}^{(1^3)}$, with bases $\{\overline{m}^{(3)}\}$, $\{\overline{m}^{(2,1)}, \psi_2\overline{m}^{(2,1)}\}$, and $\{\overline{m}^{(1^3)}\}$, respectively. All x_r and ψ_r generators kill all of these basis elements, with the exception of ψ_2 acting on $\overline{m}^{(2,1)}$.

2. Let $\lambda = (3,2)$. Then S^λ has basis $\{\overline{m}^\lambda, \psi_3\overline{m}^\lambda, \psi_2\psi_3\overline{m}^\lambda, \psi_4\psi_3\overline{m}^\lambda, \psi_2\psi_4\psi_3\overline{m}^\lambda\}$. All x_r generators kill all basis elements, and it is easy to check whether ψ_r kills a basis element or takes it to another basis element.

Theorem 3.3 [**2, Corollary 3.9**]**.** *Suppose that conditions (SS1) and (SS2) hold. Then $R^\Lambda(n)$ is semisimple.*

Proof. The idea of the proof is to show that $\{\psi_{(w^{\mathtt{S}})^{-1}} e(\mathrm{res}\,\mathtt{T}^\lambda)\psi_{w^{\mathtt{T}}} \mid \mathtt{S}, \mathtt{T} \in \mathrm{Std}(\lambda),\ \lambda \in \mathscr{P}^l_n\}$ is a graded cellular basis of $R^\Lambda(n)$, and is a basis of matrix units. □

Theorem 3.4 [**2, Theorem 3.10**]**.** *Suppose at least one of the conditions (SS1) and (SS2) fails. Then $R^\Lambda(n)$ is not semisimple.*

Proof. In the degenerate cases where some $\overline{\kappa_j} = 0$ or ℓ, or where $\overline{\kappa_j} = \overline{\kappa_{j'}}$ for some $j \neq j'$, we explicitly construct a two-dimensional uniserial module with a one-dimensional submodule. In all other cases, the idea of the proof is to find a Specht module with a one-dimensional submodule with no complement. □

Example. If $\kappa = (0)$ and $n = 2$, $R^\Lambda(n) \cong \mathbb{F}[x]/[x^2]$, and the irreducible modules $\mathrm{S}^{(2)}$ and $\mathrm{S}^{(1^2)}$ are isomorphic. In fact, in this case, the two-dimensional module we construct in our proof is isomorphic to $R^\Lambda(n) \cong \mathbb{F}[x]/[x^2]$ as a module.

References

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[2] L. Speyer, *On the semisimplicity of the cyclotomic quiver Hecke algebra of type C*, arXiv:1704.07655, 2017, preprint. [Pages 6, 7.]