Semisimple Specht modules indexed by bihooks

Joint work with Rob Muth and Louise Sutton

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$$\begin{aligned} s_i^2 &= 1 && \text{for all } i, \\ s_i s_j &= s_j s_i && \text{for } |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } 0 \leqslant i \leqslant n-2, \end{aligned}$$

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For $p = 2$: if n is odd and $n \geqslant 2b$, then S^λ is indecomposable iff $a - b - 1 \equiv 0 \pmod{2^L}$, where $2^{L-1} \leqslant b < 2^L$.

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S^λ is decomposable iff $S^{\lambda'}$ is $\rightsquigarrow$ complete classification of which Specht modules indexed by hook partitions are decomposable.

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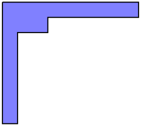

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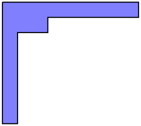

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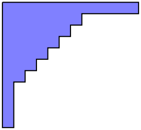
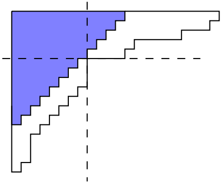
For $\lambda = (a, m, m-1, m-2, \ldots, 2, 1^b)$, the decomposition of S^λ into indecomposable summands (as Young modules) is given.



Theorem (Bessenrodt, Bowman, & Sutton, 2019)

Over $\mathbb{C}$, if λ is 2-separated, then S^{λ} is semisimple, and all composition factors are given.



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Theorem (Brundan & Kleshchev, 2009)

The (integral) cyclotomic Hecke algebra in quantum characteristic $e \geqslant 2$ is isomorphic to a level ℓ cyclotomic Khovanov–Lauda–Rouquier algebra $\mathscr{R}_n^\Lambda$ of type $A_{e-1}^{(1)}$ if $e < \infty$, or A_∞ if $e = \infty$

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KLR algebras

The cyclotomic KLR algebra $\mathscr{R}_n^\Lambda$ is a unital, associative $\mathbb{F}$-algebra with generators

$$\{e(\mathbf{i}) \mid \mathbf{i} \in (\mathbb{Z}/e\mathbb{Z})^n\} \cup \{y_1, y_2, \ldots, y_n\} \cup \{\psi_1, \psi_2, \ldots \psi_{n-1}\}$$

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This algebra is naturally $\mathbb{Z}$-graded, which leads us to studying the graded representation theory of cyclotomic Hecke algebras.

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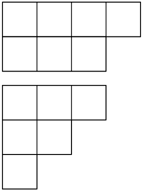

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and $\mathbf{i}^\lambda = (0,2,1,1,0,2,0,2,1,0,2,1,0)$.

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Theorem (Rouquier, 2008, Fayers–S., 2016)

If $e \neq 2$ and $\kappa_i \neq \kappa_j$ for all $i \neq j$, or if λ is a conjugate Kleshchev multipartition, then S^λ is indecomposable.

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In previous work, we found large families of decomposable Specht modules indexed by bihooks.

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  - *(i) if $j = 1$ or $k = 1$, then S^λ is decomposable iff $p \nmid j + k$.*
  - *(ii) if $j, k > 1$, and $j + k$ is even and $p \neq 2$, or if $j + k$ is odd, then S^λ is decomposable.*

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- *Using Brundan & Kleshchev's i-induction & i-restriction functors, can extend the results from* $\lambda = ((ke), (je))$ *to* $\lambda = ((ke+a, 1^b), (je+a, 1^b))$*, for any* $0 < a \leqslant e$ *and* $0 \leqslant b < e$ *with* $a + b \neq e$*.*

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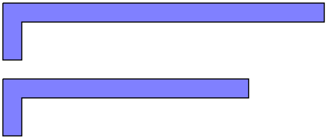
$[\lambda] =$ 

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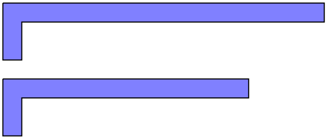
$[\lambda] =$ 

We conjectured that when $e \neq 2$ and $p \neq 2$, the above (+ their conjugates) provide a complete list of decomposable Specht modules indexed by bihooks.

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The direct summands of $\mathrm{S}^{((ke),(je))}$ are self-dual, up to a grading shift.

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In particular, $S^{((8e),(3e))}$ cannot be semisimple in characteristic 3.

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When $p > 11$, $\mathrm{S}^{((8e),(3e))}$ is semisimple – a direct sum of 4 simples.

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Inductive argument $\rightsquigarrow$ this 'final new summand' $L(2^3,1^5)$ corresponds to $\mathrm{D}^{((8e,2e+1),(e-1))}$ coming from the (degree 3) hom $\gamma : \mathrm{S}^{((8e,2e+1),(e-1))} \to \mathrm{S}^{((8e),(3e))} \rightsquigarrow \mathrm{D}^{((8e,2e+1),(e-1))}\langle 3\rangle$ is a summand.

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Inductive argument $\rightsquigarrow$ this 'final new summand' $L(2^3,1^5)$ corresponds to $\mathrm{D}^{((8e,2e+1),(e-1))}$ coming from the (degree 3) hom $\gamma : \mathrm{S}^{((8e,2e+1),(e-1))} \to \mathrm{S}^{((8e),(3e))} \rightsquigarrow \mathrm{D}^{((8e,2e+1),(e-1))}\langle 3\rangle$ is a summand.

$$\begin{aligned}\mathrm{S}^{((8e),(3e))} \cong\ & \mathrm{D}^{((11e),\varnothing)}\langle 3\rangle \oplus \mathrm{D}^{((10e,1),(e-1))}\langle 3\rangle\\ &\oplus \mathrm{D}^{((9e,e+1),(e-1))}\langle 3\rangle \oplus \mathrm{D}^{((8e,2e+1),(e-1))}\langle 3\rangle.\end{aligned}$$

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Labels and gradings as in the semisimple case $\rightsquigarrow$

$$S^{((8e),(3e))} \cong \begin{array}{c} D^{((11e),\varnothing)}\langle 3\rangle \\ | \\ D^{((8e,2e+1),(e-1))}\langle 3\rangle \\ | \\ D^{((11e),\varnothing)}\langle 3\rangle \end{array} \;\oplus\; \begin{array}{c} D^{((10,1),(e-1))}\langle 3\rangle \\ | \\ D^{((9e,e+1),(e-1))}\langle 3\rangle \\ | \\ D^{((10,1),(e-1))}\langle 3\rangle \end{array}$$

Main Theoren

The general situation looks similar to the previous examples when p divides none of, or exactly one of, the integers $k+j$, $k+j-1$, $\ldots$, $k-j+2$.

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Suppose $p \neq 2$ and $k \geqslant j \geqslant 1$. If p divides none of $k+j$, $k+j-1$, ..., $k-j+2$, then

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If p divides **exactly one** *of those integers, each summand of $S^{((ke),(je))}$ is either one of the simples above (incl. degree shift by j), or a uniserial module $D^{\mu}\langle j\rangle \mid D^{\nu}\langle j\rangle \mid D^{\mu}\langle j\rangle$, for μ and ν among the above bipartitions.*