Schurian-infinite blocks of type A Hecke algebras

Joint work with Susumu Ariki

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$$\text{representation-finite} \Rightarrow \text{Schurian-finite.}$$

The converse is not true in general – e.g. preprojective algebras of type other than A_n for $1 \leqslant n \leqslant 4$ are representation-infinite, but Schurian-finite.

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We want to determine the Schurian-finiteness of blocks of type A Hecke algebras, using the above proposition.

Hecke algebras

The Iwahori–Hecke algebra of the symmetric group is the unital, associative $\mathbb{F}$-algebra $\mathscr{H}_n$ with generators $T_1, T_2, \dots, T_{n-1}$ and relations

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0 && \text{for all } i, \\ T_i T_j &= T_j T_i && \text{for } |i - j| > 1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leqslant i \leqslant n-2, \end{aligned}$$

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If $e \leqslant n$, the simple modules appear as quotients of the Specht modules: $\{\mathrm{D}^\lambda \mid \lambda \vdash n,\ \lambda \text{ is } e\text{-regular}\}$.

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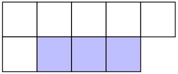
$[\lambda] =$ 

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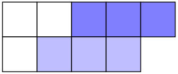
$[\lambda] =$ 

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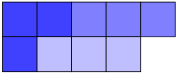
$[\lambda] =$ 

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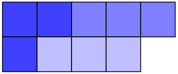
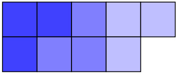
$[\lambda] =$ 



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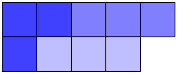
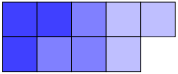
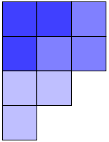
$[\lambda] =$ 



$[\mu] =$ 

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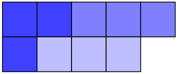
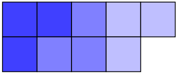
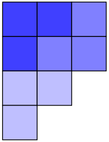
$[\lambda] =$ 



$[\mu] =$ 

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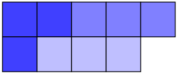
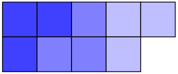
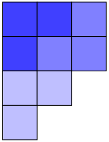
$[\lambda] =$ 



$[\mu] =$ 

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Graded decomposition numbers

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The graded decomposition number $d_{\lambda\mu}^{e,p}(v)$ is defined to the graded composition multiplicity of D^μ in S^λ. In other words

$$d_{\lambda\mu}^{e,p}(v) = [\mathsf{S}^\lambda : \mathsf{D}^\mu]_v = \sum_{d\in\mathbb{Z}} [\mathsf{S}^\lambda : \mathsf{D}^\mu\langle d\rangle] v^d \in \mathbb{N}[v, v^{-1}].$$

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Combining this with an argument involving idempotent truncation, we're able to obtain our main tool for showing that a given block of $\mathscr{H}_n$ is Schurian-infinite.

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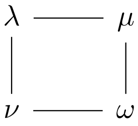

which is $A_3^{(1)} \rightsquigarrow$ the result (in characteristic 0).

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Theorem (Ariki–S.)

Suppose $e \geqslant 3$, and that B is the principal block of $\mathscr{H}_n$ (i.e. the block containing the trivial module) with weight $\geqslant 2$. Then B is Schurian-infinite in any characteristic.

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Theorem (Donkin)

If λ, μ, $\lambda^{(0)}$, $\mu^{(0)}$, $\lambda^{(1)}$, *and* $\mu^{(1)}$ *are as above, then* $d^{e,p}_{\lambda\mu}(1) = d^{e,p}_{\lambda^{(0)}\mu^{(0)}}(1) d^{e,p}_{\lambda^{(1)}\mu^{(1)}}(1)$.

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Example

Let $e = 3$, $\mu = (9, 5, 3, 2)$, and $\lambda = (8, 5, 3^2)$. Then

$[\mu] =$

0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1				
1	2	0	1						
0	1	2							
2									

$[\lambda] =$

0	1	2	0	1	2	0	1	2
2	0	1	2	0	1			
1	2	0	1					
0	1	2	0					
2								

Kleshchev's decomposition numbers

Fill the nodes (r, c) of a Young diagram $[\lambda]$ with residues $c - r \pmod e$.

Theorem (Bowman–S.)

Suppose that $\lambda \lhd \mu$ are partitions (μ: e-regular), that differ by moving a single node of residue i from row r of μ to row $r+s$ of λ, and that there are no addable or removable i-nodes in rows $r+1, r+2, \dots, r+s-1$. Then $d^{e,p}_{\lambda\mu}(v) = v$.

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We can now prove the following result!

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Let's prove (i) and (ii).

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	$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$
$\lambda^{(1)}$	1			
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0	1	$\cdots$	-2	-1	0	1	$\cdots$	-2	-1	$\cdots$
-1	0	$\cdots$	-3	-2						
-2	-1	$\cdots$	-4	-3						

0	1	$\cdots$	-2	-1	0	1	$\cdots$	-2	$\cdots$
-1	0	$\cdots$	-3	-2	-1				
-2	-1	$\cdots$	-4						
-3									

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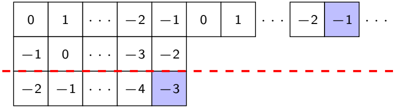

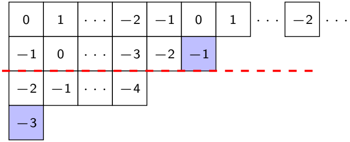

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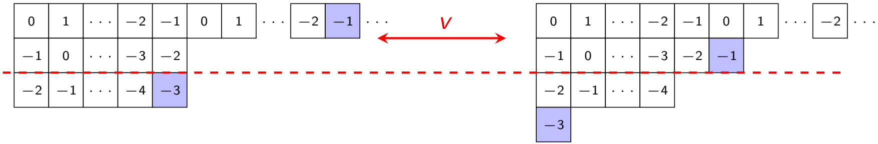

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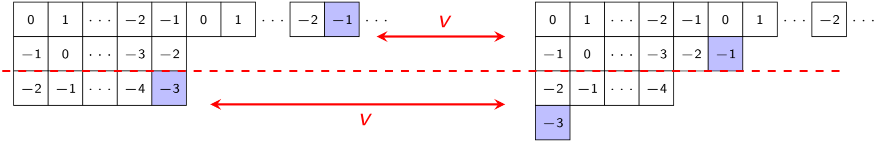

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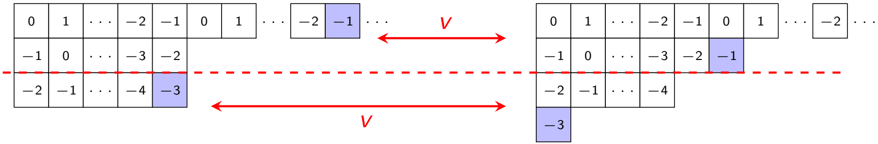

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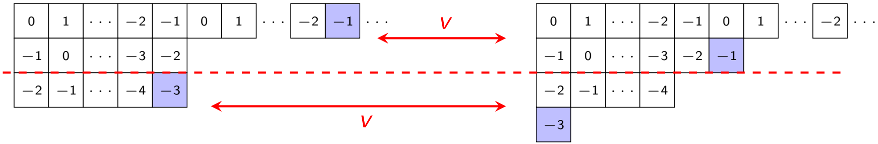

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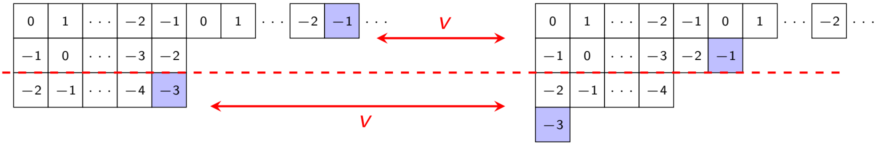

Then the decomposition matrix is (‡).

Recap

(ii) The core is (1), $e \geqslant 3$, and the weight is $w \geqslant 5$.

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Set

$$\lambda^{(1)} = ((w-3)e+1, (e+1)^2, 1^{e-2}) \quad \lambda^{(2)} = ((w-3)e+1, e+1, e, 2, 1^{e-3})$$
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$$\lambda^{(3)} = ((w-3)e, e+2, e+1, 1^{e-2}) \quad \lambda^{(4)} = ((w-3)e, e+2, e, 2, 1^{e-3})$$

	$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$
$\lambda^{(1)}$	1			
$\lambda^{(2)}$	v	1		
$\lambda^{(3)}$	v	0	1	
$\lambda^{(4)}$		v	v	1

$\lambda^{(1)} \longleftrightarrow \lambda^{(2)}, \lambda^{(3)}$: move single node down a row $\rightsquigarrow$ decomp. no. v. Similarly, $\lambda^{(4)} \longleftrightarrow \lambda^{(2)}, \lambda^{(3)}$. $\lambda^{(2)}, \lambda^{(3)}$ incomparable in dominance order.

(ii) The core is (1), $e \geqslant 3$, and the weight is $w \geqslant 5$.

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$\lambda^{(2)}$	v	1		
$\lambda^{(3)}$	v	0	1	
$\lambda^{(4)}$	v^2	v	v	1

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N.B. This doesn't cover all the principal blocks. Filling the gaps requires working a little harder – we must use a combination of techniques, including some Fock space magic.

Theorem (Ariki–S.)

Suppose $e \geqslant 3$, and that B is the principal block of the Hecke algebra $\mathscr{H}_n$, $n \geqslant 4e$. Then B is Schurian-infinite in any characteristic. (Separate arguments for weight 2 and 3 blocks $\Rightarrow$ can weaken assumption to $n \geqslant 2e$.) In particular, if $e \geqslant 3$, then B is Schurian-finite if and only if it is representation-finite.