Schurian-infinite blocks of type A Hecke algebras

Joint work with Susumu Ariki and Sinéad Lyle

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Schurian-finiteness

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \geqslant 0$ throughout. For any $\mathbb{F}$-algebra A, we say that an A-module M is Schurian if $\mathrm{End}_A(M) \cong \mathbb{F}$. We say that A is Schurian-finite if there are only finitely many isomorphism classes of Schurian A-modules, and Schurian-infinite otherwise.

Schurian modules must be indecomposable, so clearly

$$\text{representation-finite} \Rightarrow \text{Schurian-finite.}$$

The converse is not true in general – e.g. preprojective algebras of type other than A_n for $1 \leqslant n \leqslant 4$ are representation-infinite, but Schurian-finite.

Schurian-finiteness

A result of Demonet, Iyama and Jasso (2019) yields that A is Schurian-finite if and only if it is τ-tilting finite.

So we can use established results for τ-tilting (in)finite algebras to determine when algebras are Schurian-(in)finite. In particular, we make heavy use of the following reduction result.

Proposition

If the Gabriel quiver of a finite-dimensional $\mathbb{F}$-algebra A contains the quiver of an affine Dynkin diagram with zigzag orientation (i.e. every vertex is a sink or a source) as a subquiver, then A is Schurian-infinite.

We want to determine the Schurian-finiteness of blocks of type A Hecke algebras, using the above proposition.

Hecke algebras

The Iwahori–Hecke algebra of the symmetric group is the unital, associative $\mathbb{F}$-algebra $\mathscr{H}_n$ with generators $T_1, T_2, \dots, T_{n-1}$ and relations

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0 && \text{for all } i, \\ T_i T_j &= T_j T_i && \text{for } |i-j| > 1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leqslant i \leqslant n-2, \end{aligned}$$

where $q \in \mathbb{F}$ is a primitive eth root of unity.
$\mathscr{H}_n$ is semisimple if $e > n$.

The *Specht modules* $\{\mathrm{S}^\lambda \mid \lambda \vdash n\}$ over $\mathscr{H}_n$ are the ordinary irreducible $\mathscr{H}_n$-modules, indexed by partitions λ of n.

If $e \leqslant n$, the simple modules appear as quotients of the Specht modules: $\{\mathrm{D}^\lambda \mid \lambda \vdash n,\ \lambda \text{ is } e\text{-regular}\}$.

Blocks

Two Specht modules S^λ and S^μ (or simple modules D^λ and D^μ) are in the same block of $\mathscr{H}_n$ if and only if λ and μ *have the same core* and the same *e-weight*.

The *e-weight* of a partition has a simple combinatorial definition, and may be seen roughly as a measure of how complicated a block is.

e.g. weight 0 blocks are simple algebras, and weight 1 blocks are Brauer tree algebras associated to a line with no exceptional vertex.

Graded decomposition numbers

Results of Brundan, Kleshchev, and Wang $\rightsquigarrow \mathscr{H}_n$ is isomorphic to a cyclotomic KLR algebra, and its Specht modules and simple modules may be graded.

The graded decomposition number $d_{\lambda\mu}^{e,p}(v)$ is defined to be the graded composition multiplicity of D^μ in S^λ. In other words

$$d_{\lambda\mu}^{e,p}(v) = [\mathsf{S}^\lambda : \mathsf{D}^\mu]_v = \sum_{d\in\mathbb{Z}} [\mathsf{S}^\lambda : \mathsf{D}^\mu\langle d\rangle] v^d \in \mathbb{N}[v, v^{-1}].$$

Using a result of Shan on Jantzen filtrations and radical filtrations of Weyl modules for q-Schur algebras, we can deduce the following.

Lemma

Suppose that $e \geqslant 3$, $p = 0$, and λ, μ are e-regular partitions of n. If the coefficient of v in $d^{e,0}_{\lambda\mu}(v)$ is nonzero, then

$$\mathrm{Ext}^1(\mathrm{D}^\lambda, \mathrm{D}^\mu) = \mathrm{Ext}^1(\mathrm{D}^\mu, \mathrm{D}^\lambda) \neq 0.$$

Combining this with an argument involving idempotent truncation, we're able to obtain our main tool for showing that a given block of $\mathscr{H}_n$ is Schurian-infinite.

Key Proposition (Ariki–S.)

Suppose $e \geqslant 3$ & $p \geqslant 0$. If the char 0 graded decomposition matrix has one of the following as a submatrix, and $d^{e,p}_{\lambda\mu}(1) = d^{e,0}_{\lambda\mu}(1) \in \{0,1\}$ for all row labels λ, μ of the submatrix, then the block is Schurian-infinite.

$$\begin{pmatrix} 1 & & & \\ v & 1 & & \\ 0 & v & 1 & \\ v & v^2 & v & 1 \end{pmatrix} \qquad (\dagger) \qquad\qquad \begin{pmatrix} 1 & & & \\ v & 1 & & \\ v & 0 & 1 & \\ v^2 & v & v & 1 \end{pmatrix} \qquad (\ddagger)$$

Why these matrices? Take the matrix $(\ddagger)$, with rows and columns labelled by four e-regular partitions $\lambda, \mu, \nu, \omega$. Then if $p = 0$, the previous lemma gives subquiver

$$\begin{array}{ccc} \lambda & \text{———} & \mu \\ | & & | \\ \nu & \text{———} & \omega \end{array}$$

which is $A_3^{(1)} \rightsquigarrow$ the result (in characteristic 0).

Main results

(It is known that a block of $\mathscr{H}_n$ of weight 0 or 1 is representation-finite and therefore Schurian-finite.)

Theorem (Ariki–S.)

Suppose $e \geqslant 3$, and that B is any block of weight 2 or 3. Then B is Schurian-infinite in any characteristic.

Hidden in this theorem is A LOT of work. Ingredients include James–Mathas's runner removal, LLT algorithm, a graded analogue of Scopes equivalences, work on (graded) decomposition numbers and Ext^1 by Richards, Fayers, Fayers–Tan, analysis of Specht homomorphisms, ...

Theorem (Lyle–S.)

Suppose $e \geqslant 3$, and that B is any block of $\mathscr{H}_n$ with weight $\geqslant 4$. Then B is Schurian-infinite in any characteristic.