A new construction of simple modules for type A KLR algebras

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Cuspidal systems for KLR algebras

- $e \in \mathbb{Z}_{>1}$: a fixed integer.
- $U_q(\widehat{\mathfrak{sl}}_e(\mathbb{C}))$: quantum group of type $\mathtt{A}_{e-1}^{(1)}$.
- $I = \{\alpha_0, \ldots, \alpha_{e-1}\}$: the set of simple roots.
- $\delta = \alpha_0 + \cdots + \alpha_{e-1}$: the null root.
- Φ_+^{re}: the set of real roots.
- $\Phi_+^{\mathrm{im}} = \{d\delta \mid d \in \mathbb{Z}_{>0}\}$: the set of imaginary roots.
- $\Phi_+ = \Phi_+^{\mathrm{re}} \sqcup \Phi_+^{\mathrm{im}}$: the positive root system of type $\mathtt{A}_{e-1}^{(1)}$.
- $\Psi := \Phi_+^{\mathrm{re}} \sqcup \{\delta\}$: the set of indivisible roots.
- $\mathbb{F}$: an arbitrary field.
- R_ω: the type $\mathtt{A}_{e-1}^{(1)}$ KLR algebra over $\mathbb{F}$, for $\omega \in \mathbb{Z}_{\geqslant 0}I$. These algebras categorify the positive part of $U_q(\widehat{\mathfrak{sl}}_e(\mathbb{C}))$, and their representation theory is studied via *cuspidal systems*, which are associated with PBW bases for the quantum group.

Cuspidal systems for KLR algebras

- $\succcurlyeq$: fixed convex preorder on Φ_+.
- A *Kostant partition* of $\omega \in \mathbb{Z}_{\geqslant 0} I$ is a tuple of non-negative integers $\boldsymbol{K} = (K_\beta)_{\beta \in \Psi}$ with $\sum_{\beta \in \Psi} K_\beta \beta = \omega$.
  If $\beta_1 \succ \cdots \succ \beta_t$ are the elements of Ψ such that $K_{\beta_i} \neq 0$, then we write $\boldsymbol{K}$ in the form $\boldsymbol{K} = (\beta_1^{K_{\beta_1}} \mid \cdots \mid \beta_t^{K_{\beta_t}})$.
- A *root partition* of $\omega \in \mathbb{Z}_{\geqslant 0} I$ is a pair $\pi = (\boldsymbol{K}, \boldsymbol{\nu})$, where $\boldsymbol{K} = (\beta_1^{K_{\beta_1}} \mid \cdots \mid \beta_u^{K_{\beta_u}} \mid \delta^{K_\delta} \mid \beta_{u+1}^{K_{\beta_{u+1}}} \mid \cdots \mid \beta_t^{K_{\beta_t}})$ is a Kostant partition of ω and $\boldsymbol{\nu} = (\nu^{(1)} \mid \cdots \mid \nu^{(e-1)})$ is an $(e-1)$-multipartition of K_δ.
- $\Pi(\omega)$: the set of all root partitions of ω.

Cuspidal systems for KLR algebras

Definition

Let $m \in \mathbb{Z}_{>0}$, $\beta \in \Psi$. We say an $R_{m\beta}$-module M is *semicuspidal* provided that for all $0 \neq \theta_1, \theta_2 \in \mathbb{Z}_{\geqslant 0} I$ with $\theta_1 + \theta_2 = m\beta$, we have $\mathrm{Res}^{R_{m\beta}}_{R_{\theta_1} \otimes R_{\theta_2}} M \neq 0$ only if θ_1 is a sum of positive roots $\preccurlyeq \beta$ and θ_2 is a sum of positive roots $\succcurlyeq \beta$.
We say moreover that M is *cuspidal* if $m = 1$ and the comparisons above are strict.

Cuspidal and semicuspidal modules are key building blocks in the representation theory of R_ω.

To each $\beta \in \Phi_+^{\mathrm{re}}$, we associate a simple *cuspidal* R_β-module $L(\beta)$, and to each $(e-1)$-multipartition $\boldsymbol{\nu}$ of $d \in \mathbb{Z}_{>0}$, we associate a simple *semicuspidal* $R_{d\delta}$-module $L(\boldsymbol{\nu})$.

Cuspidal systems for KLR algebras

Then, to each $\pi = (\boldsymbol{K}, \boldsymbol{\nu}) \in \Pi(\omega)$, with $\boldsymbol{K} = (\beta_1^{K_{\beta_1}} \mid \cdots \mid \beta_u^{K_{\beta_u}} \mid \delta^{K_\delta} \mid \beta_{u+1}^{K_{\beta_{u+1}}} \mid \cdots \mid \beta_t^{K_{\beta_t}})$ and $\boldsymbol{\nu} = (\nu^{(1)} \mid \cdots \mid \nu^{(e-1)})$ an $(e-1)$-multipartition of K_δ, we associate the *proper standard module*

$$\bar{\Delta}(\pi) = L(\beta_1)^{\circ K_{\beta_1}} \circ \cdots \circ L(\beta_u)^{\circ K_{\beta_u}} \circ L(\boldsymbol{\nu}) \circ L(\beta_{u+1})^{\circ K_{\beta_{u+1}}} \circ \cdots \circ L(\beta_t)^{\circ K_{\beta_t}},$$

which has a self-dual simple head $L(\pi)$, and $\{L(\pi) \mid \pi \in \Pi(\omega)\}$ is a complete and irredundant set of simple R_ω-modules up to isomorphism and grading shift.

Previous constructions of simple semicuspidal modules $L(\boldsymbol{\nu})$ (and therefore $\bar{\Delta}(\pi)$ and $L(\pi)$) were *implicit*, with their existence established via categorification. Here, we use *skew Specht modules* to render a more direct combinatorial description of semicuspidal and simple R_ω-modules.

Skew Specht modules

For each skew diagram $\boldsymbol{\tau}$ of content $\omega \in \mathbb{Z}_{\geqslant 0} I$, we can construct an associated skew Specht module, $\mathbf{S}^{\boldsymbol{\tau}}$. This is an R_ω-module, generalising Specht modules indexed by multipartitions; it has a presentation via generators and relations, and an explicit basis indexed by standard $\boldsymbol{\tau}$-tableaux.

Specht modules are key objects in the representation theory of *cyclotomic* KLR algebras, Hecke algebras and symmetric groups.

Muth et al. showed that, for all $\beta \in \Phi_+^{\mathrm{re}}$, there exists an explicit ribbon $\zeta(\beta)$ of content β such that $\mathbf{S}^{\zeta(\beta)} \cong L(\beta)$.

We established an analogous result for the *imaginary* simple semicuspidal modules.

Skew Specht modules

To each $(e-1)$-multipartition $\boldsymbol{\nu}$ of d, we construct a skew diagram $\zeta(\boldsymbol{\nu})$ of content $d\delta$, and show that $L(\boldsymbol{\nu}) \cong \mathrm{hd}\mathbf{S}^{\zeta(\boldsymbol{\nu})}$.

More generally, for each $\pi = (\boldsymbol{K}, \boldsymbol{\nu}) \in \Pi(\omega)$, we construct a skew diagram $\zeta(\pi)$ of content ω by concatenating semicuspidal skew diagrams with multiplicities determined by $\boldsymbol{K}$:

$$\zeta(\pi) = \left(\zeta(\beta_1)^{K_{\beta_1}} \mid \cdots \mid \zeta(\beta_u)^{K_{\beta_u}} \mid \zeta(\boldsymbol{\nu}) \mid \zeta(\beta_{u+1})^{K_{\beta_{u+1}}} \mid \cdots \mid \zeta(\beta_t)^{K_{\beta_t}}\right).$$

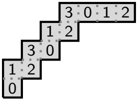
Theorem (Muth–Nicewicz–S.–Sutton, 2024)

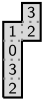
Let $\pi = (\boldsymbol{K}, \boldsymbol{\nu}) \in \Pi(\omega)$. Then the skew Specht module $\mathbf{S}^{\zeta(\pi)}$ is indecomposable with simple head $\mathrm{hd}(\mathbf{S}^{\zeta(\pi)}) \cong L(\pi)$, and $\{\mathrm{hd}(\mathbf{S}^{\zeta(\pi)}) \mid \pi \in \Pi(\omega)\}$ gives a complete irredundant set of simple R_ω-modules up to grading shift.

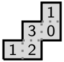
Example

Take $e = 4$, and fix a certain convex preorder $\succcurlyeq$ on Φ_+ (see [MNSS]).

To each $\beta \in \Phi_+^{\mathrm{re}}$, we associate the ribbon $\zeta(\beta)$ of content β via an algorithm, and have $\mathbf{S}^{\zeta(\beta)} \cong L(\beta)$.
For instance, we have:

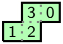
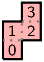
$\zeta(2\delta+\alpha_0+\alpha_1+\alpha_2) =$ 

$\zeta(\delta+\alpha_2+\alpha_3) =$ 

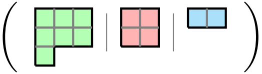
$\zeta(\delta+\alpha_1) =$ 

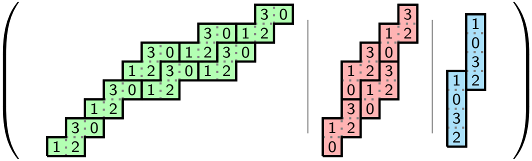
Example

We construct distinct ribbons ζ_i of content δ:

$\zeta_1 =$  $\zeta_2 =$  $\zeta_3 =$ 

and to any multipartition $\boldsymbol{\nu} = (\nu^{(1)} \mid \nu^{(2)} \mid \nu^{(3)})$, we associate the skew diagram $\zeta(\boldsymbol{\nu})$ by 'dilating' nodes in $\nu^{(i)}$ by the ribbon ζ_i. e.g. for $\boldsymbol{\nu} = ((3^2, 1) \mid (2^2) \mid (2))$, we have

$\zeta(\boldsymbol{\nu}) = \zeta$ 

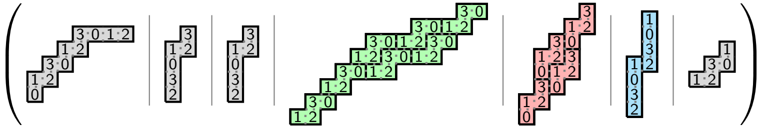
$=$  .

Example

Then $\mathbf{S}^{\zeta(\boldsymbol{\nu})}$ is an indecomposable semicuspidal $R_{13\delta}$-module with $\mathrm{hd}(\mathbf{S}^{\zeta(\boldsymbol{\nu})}) \cong L(\boldsymbol{\nu})$.

Now take $\pi \in \Pi(19\alpha_0 + 20\alpha_1 + 21\alpha_2 + 20\alpha_3)$ defined as

$$\pi = \left(\left(2\delta + \alpha_0 + \alpha_1 + \alpha_2 \mid (\delta + \alpha_2 + \alpha_3)^2 \mid \delta^{13} \mid \delta + \alpha_1\right), \boldsymbol{\nu}\right).$$ Then

$\zeta(\pi) =$ 

Our theorem says that $\mathbf{S}^{\zeta(\pi)}$ is an indecomposable $R_{19\alpha_0+20\alpha_1+21\alpha_2+20\alpha_3}$-module with simple head $L(\pi)$.