SPECIAL RELATIVITY HOMEWORK – WEEK 7

We will now upgrade from Maxwell to Yang-Mills theory. To avoid confusion, all fields will be classical – nothing is quantum. The charged matter field ϕ is no longer just a complex number, but a vector in some larger complex space – in a unitary representation of the non-Abelian gauge group. $\phi^\dagger$ is the Hermitian conjugate of this vector. The generators of the group are Hermitian matrices $G = G^\dagger$, with commutators $[G_1, G_2] = iG_3$. The gauge transformation parameter α, the gauge potential A_μ and its field strength $F_{\mu\nu}$ are no longer just real numbers, but elements of this space of generators. To simplify things, we will assume that α is infinitesimal. We no longer need a separate "electric charge" q: the action of gauge transformations on ϕ is already encoded in the generator matrices, i.e. in the structure of the representation. The gauge transformation and the covariant derivative of ϕ read simply:

$$\phi \;\rightarrow\; e^{i\alpha}\phi \approx \phi + i\alpha\phi \; ; \tag{1}$$

$$\nabla_\mu\phi = \partial_\mu\phi - iA_\mu\phi \; . \tag{2}$$

Recall that generators G themselves form a representation of the gauge group – the adjoint representation. We define the gauge transformation and covariant derivative of a generator-valued field as:

$$G \;\rightarrow\; G + i[\alpha, G] \; ; \tag{3}$$

$$\nabla_\mu G = \partial_\mu G - i[A_\mu, G] \; . \tag{4}$$

Exercise 1. *Let us define the gauge transformation of A_μ as:*

$$A_\mu \rightarrow A_\mu + \nabla_\mu\alpha \; . \tag{5}$$

Show that this ensures the proper gauge transformation $\nabla_\mu\phi \rightarrow (1+i\alpha)\nabla_\mu\phi$ for the covariant derivative.

Exercise 2. *As in electromagnetism, the commutator of covariant derivatives takes the form:*

$$[\nabla_\mu, \nabla_\nu]\phi = -iF_{\mu\nu}\phi \; . \tag{6}$$

1. *Express the field strength $F_{\mu\nu}$ in terms of the potential A_μ.*

2. *From the gauge transformation of A_μ, derive the gauge transformation of $F_{\mu\nu}$. There are two way to do this: either using the previous answer and the transformation* (5)*, or more abstractly from* (6)*.*

3. *Prove the identity $\nabla_{[\mu} F_{\nu\rho]} = 0$. Hint: the Jacobi identity on commutators implies $[A_{[\mu}, [A_\nu, A_{\rho]}]] = 0$.*

4. *Prove the identity $\nabla_\mu \nabla_\nu F^{\mu\nu} = 0$. Hint: use* (6)*, with $F_{\mu\nu}$ in the role of ϕ.*

Exercise 3. *The Yang-Mills Lagrangian reads:*

$$\mathcal{L} = -\frac{1}{4g^2} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}) + \mathcal{L}_{matter} \ ; \quad \mathcal{L}_{matter} = -\nabla_\mu \phi^\dagger \nabla^\mu \phi - m^2 \phi^\dagger \phi \ , \tag{7}$$

where, by convention, the coefficient ϵ_0 from electromagnetism is now renamed into $1/g^2$.

1. *Find the expression for the current $J^\mu \equiv \partial \mathcal{L}_{matter} / \partial A_\mu$.*

2. *Derive the conservation law $\nabla_\mu J^\mu = 0$ from the equation of motion for ϕ.*

3. *Derive the equation of motion for A_μ as $\nabla_\nu F^{\mu\nu} = g^2 J^\mu$.*