SPECIAL RELATIVITY HOMEWORK – WEEK 6

Exercise 1. *Define the Fourier transform of functions on* $\mathbb{R}^3$ *as:*

$$f(\mathbf{r}) = \int d^3\mathbf{k}\, \tilde{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}} \quad \Longleftrightarrow \quad \tilde{f}(\mathbf{k}) = \int \frac{d^3\mathbf{r}}{(2\pi)^3}\, f(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}} \ . \tag{1}$$

Find the Fourier transform $\tilde{f}(\mathbf{k})$ *of* $f(\mathbf{r}) = 1/r$. *Hint: take the Fourier transform of the equation* $\nabla^2 f(\mathbf{r}) = -4\pi\delta^3(\mathbf{r})$.

Exercise 2. *There are 4 linearly-independent ways to complete the function* $1/(x_\mu x^\mu)$ *into a well-defined distribution on spacetime:*

$$\frac{1}{x^\mu x^\mu + i\varepsilon}\ , \quad \frac{1}{x^\mu x^\mu - i\varepsilon}\ , \quad \frac{1}{x^\mu x^\mu + i\varepsilon\,\mathrm{sign}(t)}\ , \quad \frac{1}{x^\mu x^\mu - i\varepsilon\,\mathrm{sign}(t)}\ . \tag{2}$$

There is a neat way to diagnose functions $f(t)$ *that contain only Fourier modes* $e^{-i\omega t}$ *with positive frequencies* $\omega > 0$*: these are the functions that have no singularities at complex values of* t *with* $\mathrm{Im}\, t < 0$. *Which of the "functions" (2) contains only positive frequencies? Which one contains only negative frequencies?*

Exercise 3. *A free massless quantum field on* $\mathbb{R}^{3,1}$ *takes the form:*

$$\hat{\phi}(x^\mu) = \int \frac{d^3\mathbf{k}}{2|\mathbf{k}|}\left(\hat{a}(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|t)} + \hat{a}^\dagger(\mathbf{k})\, e^{-i(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|t)}\right)\ , \tag{3}$$

where the creation and annihilation operators satisfy the commutator algebra:

$$[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = 2|\mathbf{k}|\delta^3(\mathbf{k}-\mathbf{k}')\ ; \quad [\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = 0\ . \tag{4}$$

The vacuum state $|0\rangle$ *is defined via* $\hat{a}(\mathbf{k})\,|0\rangle = 0$ *(i.e. you can't remove any more particles from it).*

1. *Find the vacuum expectation value* $\langle 0|\,\hat{\phi}(x^\mu)\hat{\phi}(0)\,|0\rangle$ *for* $x^\mu = (t, \vec{0})$. *Hint: reduce the calculation to a single* $d^3\mathbf{k}$ *integral, then compute it in spherical coordinates.*
2. *Find* $\langle 0|\,\hat{\phi}(x^\mu)\hat{\phi}(0)\,|0\rangle$ *for* $x^\mu = (0, \mathbf{r})$. *Hint: use the answer to Exercise 1.*
3. *Using Lorentz symmetry, deduce* $\langle 0|\,\hat{\phi}(x^\mu)\hat{\phi}(0)\,|0\rangle$ *for any non-lightlike* x^μ.
4. *Using the answer to Exercise 2, write the full expression for* $\langle 0|\,\hat{\phi}(x^\mu)\hat{\phi}(0)\,|0\rangle$ *as a distribution on spacetime.*
5. *Find the vacuum expectation values of the opposite ordering* $\hat{\phi}(0)\hat{\phi}(x^\mu)$, *of the commutator* $[\hat{\phi}(x), \hat{\phi}(0)]$, *and of the time-ordered product* $T\hat{\phi}(x^\mu)\hat{\phi}(0) \equiv \theta(t)\hat{\phi}(x^\mu)\hat{\phi}(0) + \theta(-t)\hat{\phi}(0)\hat{\phi}(x^\mu)$.