The graded representation theory of the symmetric group and dominated homomorphisms

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Joint work with Matthew Fayers.

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For each $\lambda \vdash n$, we construct an $\mathbb{F}\mathfrak{S}_n$-module S_λ, called a Specht module.

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If $p > 0$ and λ is a *p-regular partition* of n, S_λ has a simple head, which we denote D_λ. These form a complete set of pairwise non-isomorphic simple $\mathbb{F}\mathfrak{S}_n$-modules.

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If M is finitely generated, then $\mathrm{Hom}_A(M, N)$ is a graded vector space.
In particular, $\mathrm{Hom}_A(M, N)$ has a homogeneous basis.

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$$y_r y_s = y_s y_r;$$

$$\psi_r y_s = y_s \psi_r \qquad \text{if } s \neq r, r+1;$$

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y_r \psi_r e(i) &= (\psi_r y_{r+1} - \delta_{i_r, i_{r+1}}) e(i); \\
y_{r+1} \psi_r e(i) &= (\psi_r y_r + \delta_{i_r, i_{r+1}}) e(i);
\end{aligned}$$

$$\psi_r^2 e(i) = \begin{cases} 0 & i_r = i_{r+1}, \\ e(i) & i_{r+1} \neq i_r, i_r \pm 1, \\ (y_{r+1} - y_r)e(i) & i_r = i_{r+1}+1, \\ (y_r - y_{r+1})e(i) & i_r = i_{r+1}-1, \\ (y_{r+1} - y_r)(y_r - y_{r+1})e(i) & i_r = i_{r+1}+1, p = 2; \end{cases}$$

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$$\psi_r\psi_{r+1}\psi_r e(i) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1}+1)e(i) & i_{r+2} = i_r = i_{r+1}+1, \\ (\psi_{r+1}\psi_r\psi_{r+1}-1)e(i) & i_{r+2} = i_r = i_{r+1}-1, \\ (\psi_{r+1}\psi_r\psi_{r+1}+y_r-2y_{r+1}+y_{r+2})e(i) & \text{", } p = 2, \\ (\psi_{r+1}\psi_r\psi_{r+1})e(i) & \text{otherwise;} \end{cases}$$

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$$y_1 = 0;$$
$$e(i) = 0 \qquad \text{if } i_1 \neq 0;$$

for all admissible r, s, i, j.

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Corollary

If $p > 0$, $\mathbb{F}\mathfrak{S}_n$ can be non-trivially $\mathbb{Z}$-graded.

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, we have $i_{\mathtt{T}} = (012201)$.

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Example: Let $p = 3$, $\mathtt{T} = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$. Recursively, $\boxed{0}$

$\deg(\mathtt{T}_{<2}) = 0$

Degrees of tableaux

Definition

For $\lambda \vdash n$, A an i-node of λ, we define

$$\deg^A(\lambda) = \#\{\text{addable } i\text{-nodes of } \lambda \text{ strictly above } A\} - \#\{\text{removable } i\text{-nodes of } \lambda \text{ strictly above } A\}.$$

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Example: Let $p = 3$, $\mathtt{T} =$

1	2	5
3	6	
4		

. Recursively,

0	1
2	
1	

$\deg(\mathtt{T}_{<5}) = 0 + 0 + 1 + -1$

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4		

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- *some Garnir relations involving ψ generators*.

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For each $w \in \mathfrak{S}_n$, fix a reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_r}$ in the Coxeter generators of $\mathfrak{S}_n$.

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Example: Take $\lambda = (3, 1)$.

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Example: Take $\lambda = (3, 1)$. We have three standard λ-tableaux
$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$.

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For each $\mathtt{T} \in \mathrm{Std}(\lambda)$, define $v_{\mathtt{T}} := \psi_{\mathtt{T}} z_\lambda \in \mathrm{S}_\lambda$.

Example: Take $\lambda = (3, 1)$. We have three standard λ-tableaux $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$. The corresponding elements $v_{\mathtt{T}}$ are given by z_λ, $\psi_2 z_\lambda$ and $\psi_3 \psi_2 z_\lambda$ respectively.

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Example: $\{z_\lambda, \psi_2 z_\lambda, \psi_3\psi_2 z_\lambda\}$ is a homogeneous basis of $S_{(3,1)}$.

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We can check that $\mathtt{T} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$ is the only standard μ-tableau with residue sequence i_λ. So we check if $v_\mathtt{T}$ satisfies the same relations as z_λ ($y_r s$ all annihilate it, etc). It does, so there's a (degree 1) homomorphism $\mathrm{S}_\lambda \to \mathrm{S}_\mu$ given by $z_\lambda \mapsto v_\mathtt{T}$.

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One of these has degree -2.

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One of these has degree -2. Some have degree 0. Some degree 2.

Homomorphisms

Example 2: $\lambda = (5,4,3)$, $\mu = (5,4,2,1)$. $\{\mathrm{T} \in \mathrm{Std}(\mu) \mid i_{\mathrm{T}} = i_\lambda\} =$

$$\left\{ \begin{array}{ccccc} 1&3&6&9&12\\ 2&5&8&11&\\ 4&7&&&\\ 10&&&& \end{array}, \begin{array}{ccccc} 1&3&6&10&12\\ 2&5&8&11&\\ 4&7&&&\\ 9&&&& \end{array}, \begin{array}{ccccc} 1&3&7&9&12\\ 2&5&8&11&\\ 4&6&&&\\ 10&&&& \end{array}, \begin{array}{ccccc} 1&3&7&10&12\\ 2&5&8&11&\\ 4&6&&&\\ 9&&&& \end{array}, \begin{array}{ccccc} 1&4&6&9&12\\ 2&5&8&11&\\ 3&7&&&\\ 10&&&& \end{array}, \right.$$

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One of these has degree -2. Some have degree 0. Some degree 2. One has degree 4.

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Compute KLR generator actions on $v_{\mathtt{T}}$s for the above tableaux.

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Compute KLR generator actions on $v_{\mathtt{T}}$s for the above tableaux. Splitting across degrees means computation involves linear algebra in 1, 1, 3 and 3 variables, rather than 8. Can compute that there is only one homomorphism, of degree 1 ($\mathtt{T}_\lambda$ has degree 3).

Homomorphisms

Example 3:

Homomorphisms

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$$\left\{
\begin{array}{l}
\begin{array}{llllll}1&3&6&9&12&15\\2&5&8&11\\4&7&14\\10&13&16\end{array},
\begin{array}{llllll}1&3&6&9&12&16\\2&5&8&11\\4&7&14\\10&13&15\end{array},
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\begin{array}{llllll}1&3&6&10&12&15\\2&5&8&11\\4&7&14\\9&13&16\end{array},
\begin{array}{llllll}1&3&6&10&12&16\\2&5&8&11\\4&7&14\\9&13&15\end{array},
\begin{array}{llllll}1&3&6&10&13&15\\2&5&8&11\\4&7&14\\9&12&16\end{array},
\begin{array}{llllll}1&3&6&10&13&16\\2&5&8&11\\4&7&14\\9&12&15\end{array},\\
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\begin{array}{llllll}1&3&7&9&13&15\\2&5&8&11\\4&6&14\\10&12&16\end{array},
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\begin{array}{llllll}1&4&6&9&13&15\\2&5&8&11\\3&7&14\\10&12&16\end{array},
\begin{array}{llllll}1&4&6&9&13&16\\2&5&8&11\\3&7&14\\10&12&15\end{array},
\begin{array}{llllll}1&4&6&10&12&15\\2&5&8&11\\3&7&14\\9&13&16\end{array},
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\begin{array}{llllll}1&4&6&10&13&15\\2&5&8&11\\3&7&14\\9&12&16\end{array},
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$$\left\{ \begin{smallmatrix}1&3&6&9&12&15\\2&5&8&11\\4&7&14\\10&13&16\end{smallmatrix}, \begin{smallmatrix}1&3&6&9&12&16\\2&5&8&11\\4&7&14\\10&13&15\end{smallmatrix}, \begin{smallmatrix}1&3&6&9&13&15\\2&5&8&11\\4&7&14\\10&12&16\end{smallmatrix}, \begin{smallmatrix}1&3&6&9&13&16\\2&5&8&11\\4&7&14\\10&12&15\end{smallmatrix}, \begin{smallmatrix}1&3&6&10&12&15\\2&5&8&11\\4&7&14\\9&13&16\end{smallmatrix}, \begin{smallmatrix}1&3&6&10&12&16\\2&5&8&11\\4&7&14\\9&13&15\end{smallmatrix}, \begin{smallmatrix}1&3&6&10&13&15\\2&5&8&11\\4&7&14\\9&12&16\end{smallmatrix}, \begin{smallmatrix}1&3&6&10&13&16\\2&5&8&11\\4&7&14\\9&12&15\end{smallmatrix},\right.$$

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32 tableaux. Degrees split them into sets of size 1, 1, 5, 5, 10, 10.

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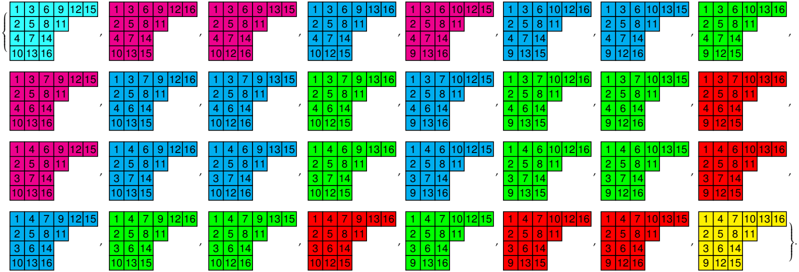

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Computation is getting tough...

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Let $\lambda, \mu \vdash n$. A μ-tableau $\mathtt{T}$ is *λ-dominated* if every entry of $\mathtt{T}$ appears at least as far left as it does in $\mathtt{T}_\lambda$.

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Let $\varphi \in \mathrm{Hom}_{R_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$ & suppose $\varphi(z_\lambda) = \sum_{\mathtt{T}\in\mathrm{Std}(\mu)} a_{\mathtt{T}} v_{\mathtt{T}}$. Say φ is *dominated* if $\mathtt{T}$ is λ-dominated whenever $a_{\mathtt{T}} \neq 0$. Denote set of all such homs by $\mathrm{DHom}_{R_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$.

Example: Let $p = 3, \lambda = (4,3), \mu = (4,2,1)$. There is a homomorphism $\mathrm{S}_\lambda \to \mathrm{S}_\mu$ given by $z_\lambda \mapsto v_{\mathtt{T}}$ for $\mathtt{T} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$.

Theorem (Fayers–S, ’14)

Dominated homomorphisms

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Theorem (Fayers–S, ’14)

If $p \neq 2$, then $\mathrm{DHom}_{R_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) = \mathrm{Hom}_{R_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu)$.

Dominated homomorphisms

Example 2 revisited:

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$\{\mathtt{T} \in \mathrm{Std}(\mu) \mid i_{\mathtt{T}} = i_\lambda\} =$

$$\left\{ \begin{array}{ccccc} 1 & 3 & 6 & 9 & 12 \\ 2 & 5 & 8 & 11 & \\ 4 & 7 & & & \\ 10 & & & & \end{array},\ \begin{array}{ccccc} 1 & 3 & 6 & 10 & 12 \\ 2 & 5 & 8 & 11 & \\ 4 & 7 & & & \\ 9 & & & & \end{array},\ \begin{array}{ccccc} 1 & 3 & 7 & 9 & 12 \\ 2 & 5 & 8 & 11 & \\ 4 & 6 & & & \\ 10 & & & & \end{array},\ \begin{array}{ccccc} 1 & 3 & 7 & 10 & 12 \\ 2 & 5 & 8 & 11 & \\ 4 & 6 & & & \\ 9 & & & & \end{array},\ \begin{array}{ccccc} 1 & 4 & 6 & 9 & 12 \\ 2 & 5 & 8 & 11 & \\ 3 & 7 & & & \\ 10 & & & & \end{array},\right.$$

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Much easier to find the degree 1 homomorphism now!

Dominated homomorphisms

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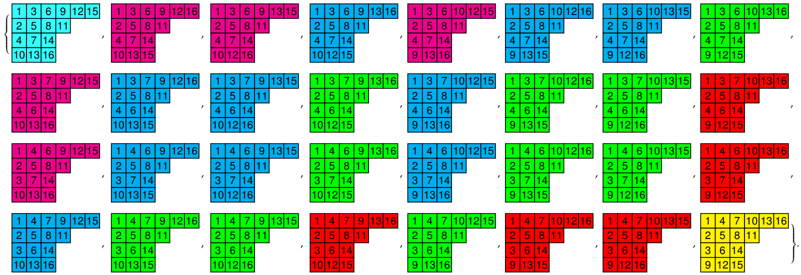

Dominated homomorphisms

Example 3 revisited: $\lambda = (6,5,5)$, $\mu = (6,4,3,3)$.
$\{\mathtt{T} \in \mathrm{Std}(\mu) \mid i_{\mathtt{T}} = i_\lambda,\ \mathtt{T}$ is λ-dominated$\} =$

$$\left\{ \begin{array}{|c|c|c|c|c|c|} \hline 1 & 4 & 7 & 10 & 13 & 16 \\ \hline 2 & 5 & 8 & 11 \\ \cline{1-4} 3 & 6 & 14 \\ \cline{1-3} 9 & 12 & 15 \\ \cline{1-3} \end{array} \right\}.$$

Dominated homomorphisms

Example 3 revisited: $\lambda = (6,5,5)$, $\mu = (6,4,3,3)$.
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Checking the relations is now much easier, and it's not too difficult to show that there is a homomorphism $z_\lambda \mapsto v_{\mathtt{T}}$.

Another application

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If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, define $\bar{\lambda} := (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1)$. i.e. $\bar{\lambda}$ is the result of removing the first column from λ.

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Theorem (Fayers–S,'14)

If $\lambda, \mu \vdash n$ and λ and μ both have a first column of size k, then

$$\mathrm{DHom}_{R_n}(\mathrm{S}_\lambda, \mathrm{S}_\mu) \cong \mathrm{DHom}_{R_{n-k}}(\mathrm{S}_{\bar{\lambda}}, \mathrm{S}_{\bar{\mu}})$$

as graded vector spaces over $\mathbb{F}$.

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Corollary

If $p \neq 2$, $\lambda, \mu \vdash n$ and λ and μ both have a first column of size k, then

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