# Decomposable Specht modules

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## 1 Decomposable Specht modules in level 1

Let $\mathbb{F}$ be a field of characteristic $p \geqslant 0$ throughout.

The Specht modules $\{\mathrm{S}^\lambda \mid \lambda \vdash n\}$ over $\mathfrak{S}_n$ are the ordinary irreducible $\mathfrak{S}_n$-modules, indexed by partitions $\lambda$ of $n$.

We have the following fundamental fact about Specht modules.

**Theorem 1.1 [7, Corollary 13.18].** *If $p \neq 2$ or $\lambda$ is 2-regular, then $\mathrm{S}^\lambda$ is indecomposable.*

When $p = 2$ and $\lambda$ is 2-singular, it is a difficult problem to determine whether or not $\mathrm{S}^\lambda$ is decomposable. However, some special cases are very tractable.

**Theorem 1.2 [9, Theorems 4.1 and 4.5].** *Let $\lambda = (a, 1^b)$, with $a + b = n$. If $n$ is even, then $\mathrm{S}^\lambda$ is indecomposable.*

*If $n$ is odd and $n \geqslant 2b$, then $\mathrm{S}^\lambda$ is indecomposable if and only if $a-b-1 \equiv 0 \pmod{2^L}$, where $2^{L-1} \leqslant b < 2^L$.*

Given that $\mathrm{S}^\lambda$ is decomposable if and only if $\mathrm{S}^{\lambda'}$ is, where $\lambda'$ is the conjugate of $\lambda$, the restriction that $a \geqslant b$ is in fact not a problem, and Murphy's result gives a complete classification of which Specht modules indexed by hook partitions are decomposable.

30 years later, Dodge and Fayers found the first new examples of decomposable Specht modules since Murphy:

**Theorem 1.3 [5, Theorem 3.1].** *Suppose $\lambda = (a, 3, 1^b)$ with $a \geqslant 4$ and $b \geqslant 2$. Then $\mathrm{S}^\lambda$ is decomposable if at least one of the following holds:*

- *$a + b \equiv 0$ or $2 \pmod 8$, $a \geqslant 6$ and $b \geqslant 4$;*
- *$a + b \equiv 2 \pmod 4$ and $\binom{a+b-3}{a-3}$ is odd;*
- *$a + b \equiv 0 \pmod 4$ and $\binom{a+b-9}{a-5}$ is odd.*

A natural generalisation of this problem is to instead consider Specht modules over the Iwahori–Hecke algebra of the symmetric group. This is the unital, associative $\mathbb{F}$-algebra $\mathscr{H}_n$ with generators $T_1, T_2, \dots, T_{n-1}$ and relations

$$(T_i - q)(T_i + 1) = 0 \qquad \text{for all } i,$$

$$T_iT_j = T_jT_i \qquad \text{for } |i-j|>1,$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \qquad \text{for } 0 \leqslant i \leqslant n-2,$$

where $q \in \mathbb{F}$ is a primitive $e$th root of unity.

Now the Specht modules $\{\mathrm{S}^\lambda \mid \lambda \vdash n\}$ over $\mathscr{H}_n$ are the ordinary irreducible $\mathscr{H}_n$-modules, indexed by partitions $\lambda$ of $n$.

As for symmetric groups, we have (following [3, Theorem 3.5]) that $\mathrm{S}^\lambda$ is decomposable if and only if $\mathrm{S}^{\lambda'}$ is and:

**Theorem 1.4 [4, Corollary 8.7].** *If $e \neq 2$ or $\lambda$ is 2-regular, then $\mathrm{S}^\lambda$ is indecomposable.*

Once again, when $e = 2$ (i.e. $q = -1$), and $\lambda$ is 2-singular, it is difficult to determine whether or not $\mathrm{S}^\lambda$ is decomposable.

Shortly after Dodge and Fayers obtained their results, we extended Murphy's result to $\mathscr{H}_n$.

**Theorem 1.5 [11, Theorem 6.12].** *Suppose $p \neq 2$ and $\lambda = (a, 1^b)$. Then $\mathrm{S}^\lambda$ is indecomposable if and only if $n$ is even or $b = 2$ or 3 with $p \mid \lceil \frac{a}{2} \rceil$.*

# 2 KLR algebras

We may further generalise our setting to cyclotomic Hecke algebras, deformations of the complex reflection groups $G(l,1,n) = \mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n$. For our purposes, the following theorem of Brundan and Kleshchev will provide the perspective we take in looking for decomposable Specht modules.

**Theorem 2.1 [1, Main Theorem].** *The (integral) cyclotomic Hecke algebra in quantum characteristic $e \geqslant 2$ is isomorphic to a level $l$ cyclotomic Khovanov–Lauda–Rouquier algebra $\mathscr{R}_n^\Lambda$ of type $A_{e-1}^{(1)}$ if $e < \infty$, or $A_\infty$ if $e = \infty$ (i.e. corresponding to dominant weight $\Lambda = \Lambda_{\kappa_1} + \Lambda_{\kappa_2} + \cdots + \Lambda_{\kappa_l}$).*

The cyclotomic KLR algebra $\mathscr{R}_n^\Lambda$ is a unital, associative $\mathbb{F}$-algebra with generators

$$\{e(i) \mid i \in (\mathbb{Z}/e\mathbb{Z})^n\} \cup \{y_1, y_2, \ldots, y_n\} \cup \{\psi_1, \psi_2, \ldots \psi_{n-1}\}$$

subject to a long list of relations. This algebra is naturally $\mathbb{Z}$-graded, which leads us to studying the graded representation theory of cyclotomic Hecke algebras.

## 2.1 Specht modules over $\mathscr{R}_n^\Lambda$

There is a theory of Specht modules over cyclotomic Hecke algebras which naturally lead to Specht modules over $\mathscr{R}_n^\Lambda$, which are the ordinary irreducibles.

Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)})$ be an $l$-multipartition of $n$ and let $\mathtt{T}^\lambda$ denote the *column initial* $\lambda$-tableau, and denote by $i^\lambda$ its residue sequence modulo $e$.

**Example.** Let $\lambda = ((4,3),(3,2,1))$. Then

$$\mathtt{T}^\lambda = \begin{array}{|c|c|c|c|}\hline 7 & 9 & 11 & 13 \\ \hline 8 & 10 & 12 \\ \cline{1-3} \end{array}$$
$$\begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

For a $\lambda$-tableau $\mathtt{T}$, define the permutation $w^{\mathtt{T}}$ to be the permutation satisfying $w^{\mathtt{T}}\mathtt{T}^\lambda = \mathtt{T}$.

Following [8], the Specht module $\mathrm{S}^\lambda$ is the cyclic $\mathscr{R}_n^\Lambda$-module with homogeneous generator $z^\lambda$ subject to the following relations.

(i) $e(i)z^\lambda = \delta_{i,i^\lambda} z^\lambda$;

(ii) $y_r z^\lambda = 0$ for all $r$;

(iii) $\psi_r z^\lambda = 0$ whenever $r$ and $r+1$ are in the same column of $\mathtt{T}^\lambda$;

(iv) Garnir relations.

For each $w \in \mathfrak{S}_n$, fix a reduced expression $w = s_{i_1} \dots s_{i_r}$, and define the corresponding element $\psi_w := \psi_{i_1} \dots \psi_{i_r} \in \mathscr{R}_n^\Lambda$. In general these elements depend on the choice of reduced expression, since the $\psi$ generators do not satisfy braid relations! Finally, let $\psi^{\mathtt{T}} = \psi_{w^{\mathtt{T}}}$.

**Theorem 2.2** ([2, 8])**.** *Let $\lambda$ be an $l$-multipartition of $n$. The Specht module $\mathrm{S}^\lambda$ is graded, with homogeneous basis*

$$\{v^{\mathtt{T}} := \psi^{\mathtt{T}} z^\lambda \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}.$$

**Theorem 2.3** ([10, 6])**.** *If $e \neq 2$ and $\kappa_i \neq \kappa_j$ for all $i \neq j$, or if $\lambda$ is a* conjugate Kleshchev multipartition*, then $\mathrm{S}^\lambda$ is indecomposable.*

It is natural to now look for decomposable Specht modules in higher levels. Our presentation and basis allow us to calculate endomorphisms of Specht modules, as any $\varphi \in \mathrm{End}(\mathrm{S}^\lambda)$ satisfies

$$\varphi(z^\lambda) = \sum a_{\mathtt{T}} v^{\mathtt{T}} \text{ for some } a_{\mathtt{T}} \in \mathbb{F},$$

where we sum over all $\mathtt{T} \in \mathrm{Std}(\lambda)$ such that $\mathrm{res}\,\mathtt{T} = i^\lambda$, and the right-hand side must satisfy the defining relations of $\mathrm{S}^\lambda$.

# 3 Decomposable Specht modules in level 2

We now fix $l = 2$, so that $\Lambda = \Lambda_{\kappa_1} + \Lambda_{\kappa_2}$ and $\mathscr{R}_n^\Lambda$ is isomorphic to a Hecke algebra of type $B$.

For now, we fix $e \geqslant 3$ and $\kappa = (0,0)$ (so $\Lambda = 2\Lambda_0$). We study Specht modules indexed by *bihooks* $\lambda = ((a,1^b),(c,1^d))$, a natural generalisation of hooks in level 1.

**Theorem 3.1** ([12])**.** *[Small bihooks] Let $n \leqslant 2e$. Then $\mathrm{S}^\lambda$ is decomposable if and only if $n = 2e$ and $\lambda = ((a,1^b),(a,1^b))$ for some $a, b$.*

*'Proof'.* It is easy to check that if $\lambda = ((a,1^b),(a,1^b))$ is a bipartition of $n < 2e$, $\lambda$ is conjugate Kleshchev, and is thus indecomposable. In all other cases, we deduce indecomposability by looking at the few tableaux of the correct residue and showing that there cannot be a non-trivial endomorphism. If $n = 2e$ and $\lambda = ((a,1^b),(a,1^b))$ for some $a, b$, there is an endomorphism given by swapping the two components of $\mathtt{T}^\lambda$. A long calculation shows that this endomorphism is an idempotent. □

**Theorem 3.2** ([12])**.** *Let $\lambda = ((ke+a,1^b),(je+a,1^b))$ or $((a,1^{je+b}),(a,1^{ke+b}))$, for some $0 < a \leqslant e$ and $0 \leqslant b < e$ with $a+b \neq e$, or for $a = b = 0$.*

*(i) If $j, k > 1$, and $j+k$ is even and $p \neq 2$, or if $j+k$ is odd, then $\mathrm{S}_\lambda$ is decomposable.*

*(ii) If $j = 1$ or $k = 1$, then $\mathrm{S}_\lambda$ is decomposable if and only if $p \nmid j+k$.*

**Conjecture 3.3.** *When $e \neq 2$, Theorem 3.2 provides a complete list of decomposable Specht modules indexed by bihooks.*

In order to prove Theorem 3.2, we use some tricks with $i$-induction and $i$-restriction to show the following reduction.

**Theorem 3.4** ([12])**.** *Let $k \geqslant 1$, $0 < a \leqslant e$ and $0 \leqslant b < e$ with $a + b \neq e$. The Specht module $\mathrm{S}^{((ke),(je))}$ is decomposable if and only if $\mathrm{S}^{((ke+a,1^b),(je+a,1^b))}$ is.*

**Example.** Let $e = 3$ and $\lambda = ((6),(6))$. There are six standard $\lambda$-tableaux with residue sequence $i^\lambda$, obtained by permuting the $e$-bricks.

$\mathtt{T}^\lambda =$

| | | | | | |
|---|---|---|---|---|---|
| 7 | 8 | 9 | 10 | 11 | 12 |
| 1 | 2 | 3 | 4 | 5 | 6 |

$\mathtt{R} =$

| | | | | | |
|---|---|---|---|---|---|
| 4 | 5 | 6 | 10 | 11 | 12 |
| 1 | 2 | 3 | 7 | 8 | 9 |

$\mathtt{S} =$

| | | | | | |
|---|---|---|---|---|---|
| 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 2 | 3 | 10 | 11 | 12 |

$\mathtt{T} =$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 10 | 11 | 12 |
| 4 | 5 | 6 | 7 | 8 | 9 |

$\mathtt{U} =$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 7 | 8 | 9 |
| 4 | 5 | 6 | 10 | 11 | 12 |

$\mathtt{W} =$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 7 | 8 | 9 | 10 | 11 | 12 |

.

There is a homomorphism given by $\varphi(z^\lambda) = 4v^{\mathtt{R}} + 2v^{\mathtt{S}} + 2v^{\mathtt{T}} + v^{\mathtt{U}}$.

It can be shown that $v^{\mathtt{S}} - v^{\mathtt{T}}$ and $v^{\mathtt{W}}$ are eigenvectors for this endomorphism, with eigenvalues $-4$ and $-6$. Thus there are at least two distinct generalised eigenspaces, and $\mathrm{S}^\lambda$ is decomposable.

**Theorem 3.5** ([12])**.** *Let $e = 2$, $\kappa = (0,1)$ or $(0,0)$, and let $\mu$ be a hook partition of $n$ such that $\mathrm{S}_\mu$ is a decomposable $\mathscr{R}_n^{\Lambda_0}$-module (cf. Theorems 1.2 and 1.5). Then for any partition $\nu$ of $m$, the Specht modules $\mathrm{S}_{(\mu,\nu)}$ and $\mathrm{S}_{(\nu,\mu)}$ are decomposable $\mathscr{R}_{m+n}^{\Lambda}$-modules.*

**Theorem 3.6** ([12])**.** *Let $e = 2$ and $\kappa = (0,0)$. Then the decomposable Specht modules arising from Theorem 3.2 are decomposable in this case too.*

**Conjecture 3.7.** *Theorems 3.5 and 3.6 gives a complete list of decomposable Specht modules indexed by bihooks when $e = 2$.*

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