MHV Graviton Scattering Amplitudes & Current Algebra on the Celestial Sphere

Recent Developments in S-Matrix Theory, ICTS
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Celestial Amplitudes

- The Lorentz group in *4-d* Minkowski space acts on the celestial sphere at null infinity as *2-d* *conformal transformations*. S-matrix elements can be expressed in a basis which makes this manifest.

[*S. Pasterski, S.H. Shao, A. Strominger; hep-th/1701.00049*]

[*S. Pasterski, S.H. Shao; hep-th/1705.01027*]

[*see Stephan Stieberger's Talk @ RDST, ICTS, 2020*]

  - For massless particles this basis change is implemented by a *Mellin transform*.

$$\langle \prod_{i=1}^{n} \mathcal{O}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i)\rangle = \int_0^\infty \prod_{i=1}^{n} d\omega_i\, \omega_i^{\Delta_i-1} \mathcal{A}_n(\sigma_i;\omega_i,z_i,\bar{z}_i)\,\delta^{(4)}\left(\sum_{i=1}^{n}\epsilon_i\omega_i q_i^\mu\right)$$

S-matrix

$\Delta = 1 + i\lambda,\ \lambda\in\mathbb{R}$ *: Scaling dimensions*

$p^\mu = \omega(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z})$

σ *: Spin (helicity)*

- By construction the L.H.S. transforms in the same fashion as a *n*-pt. function of primary operators in a *2-d* CFT.

- Provides a setup for understanding holography for quantum theories of gravity in asymptotically flat spacetimes.

Celestial OPE

- We would like to understand the properties of this putative dual celestial CFT.
- A fundamental aspect of any CFT is the existence of an operator product expansion (*OPE*).
- The celestial OPE can be extracted from collinear limits of scattering amplitudes. In Feynman diagrams collinear limit (*for massless particles*) gives singularities of the form

$$p_1 || p_2 \qquad \frac{1}{2p_1.p_2} \rightarrow \frac{1}{z_{12}}$$

$$p^\mu = \omega(1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z})$$

[*W. Fan, A. Fotopoulos, T. R. Taylor, hep-th/1903.01676*]

[*A. Fotopoulos, S. Stieberger, T. R. Taylor, B. Zhu; hep-th/1912.10973*]

- Asymptotic symmetries impose powerful constraints on the OPE. Global symmetries associated to subleading soft gluon and sub-sub-leading soft graviton theorems, together with global space-time translation symmetries were used to completely fix the leading OPE coefficients of gluons and gravitons respectively in

[*M. Pate, A. M. Raclariu, A. Strominger, E. Y. Yuan; hep-th/1910.07424*]

- Constraints from Poincare symmetries for massless as well massive particles have also been explored in [*Y. T. A. Law, M. Zlotnikov; hep-th/1910.04356, 2004.04309*]

Celestial OPE

- What is the structure of higher order terms in the celestial OPE ?
- Usually in 2-*d* CFTs, the OPE can be organised into representations of the underlying Virasoro symmetry algebra.
- In this talk we will see that for *tree level MHV graviton amplitudes*, the OPE in the dual celestial CFT can be organised into representations of an *infinite dimensional local symmetry algebra*.
- This symmetry algebra will comprise of a current algebra, constructed using the *subleading soft graviton theorem*, and *supertranslations*.

Outline

- Current algebra from subleading soft graviton theorem.
- Celestial OPE of gravitons.
    - Extracting OPE from tree level MHV graviton amplitudes.
    - OPE coefficients from extended symmetry algebra.

Conformal Soft theorems & Current Algebra

Conformal Soft Theorems in Gravity

- Soft theorems play an integral role in the study of scattering amplitudes in QFT. Here we will be interested in soft limits of gravitational amplitudes.

- Tree level gravity amplitudes have been shown to exhibit soft factorisation [*Weinberg, 1965*]

[*F. Cachazo, A. Strominger; hep-th/1404.4091*]

[*see also Ashoke Sen's talk @ RDST, ICTS, 2020*]

$$\mathcal{A}_{n+1}(p_1, \cdots, p_n; q) \xrightarrow{q\to 0} \left(S_{(0)} + S_{(1)} + S_{(2)}\right) \mathcal{A}_n(p_1, \cdots, p_n) + \cdots$$

leading soft factor   *sub-leading soft factor*   *sub-sub-leading soft factor*

- In the conformal basis, analog of soft limit is

[*L. Donnay, A. Puhm, A. Strominger; hep-th/1810.05219*]

[*W. Fan, A. Fotopoulos, T. R. Taylor, hep-th/1903.01676*]

[*D. Nandan, A. Schreiber, A. Volovich, M. Zlotnikov; hep-th/1904.10940*]

[*T. Adamo, L. Mason, A. Sharma; hep-th/1905.09224*]

[*A. Puhm; hep-th/1905.09799*]

[*A. Guevara; hep-th/1906.07810*]

$$\Delta \to 1, 0, -1, \ldots$$

leading conformal soft   *sub-leading conformal soft*   *sub-sub-leading conformal soft*

Subleading Conformal Soft Limit

Consider

$$\left\langle G_{\Delta}^{+}(z,\bar{z}) \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle$$

positive helicity graviton primary

matter conformal primaries

$$h = \frac{\Delta + \sigma}{2}$$

$$\bar{h} = \frac{\Delta - \sigma}{2}$$

- In the subleading conformal soft limit

$$\lim_{\Delta \to 0} \Delta \left\langle G_{\Delta}^{+}(z,\bar{z}) \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle = \left\langle S_1^{+}(z,\bar{z}) \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle$$

where

$$\left\langle S_1^{+}(z,\bar{z}) \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle = \sum_{k=1}^{n} \frac{(\bar{z}-\bar{z}_k)^2}{z-z_k} \left[\frac{2\bar{h}_k}{\bar{z}-\bar{z}_k} - \partial_{\bar{z}_k} \right] \left\langle \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle$$

Subleading conformal soft limit

- Treat z and $\bar{z}$ as independent variables and expand the subleading soft factor in powers of $\bar{z}$

$$S_1^+(z,\bar{z}) = -J^1(z) + 2\,\bar{z}\,J^0(z) - \bar{z}^2\,J^{-1}(z)$$

where

$$\left\langle J^1(z)\prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = \sum_{k=1}^n \frac{\bar{L}_1(k)}{z-z_k}\left\langle \prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle \qquad \bar{L}_1(k) = \bar{z}_k^2\,\partial_{\bar{z}_k} + 2\,\bar{h}_k\,\bar{z}_k$$

$$\left\langle J^0(z)\prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = \sum_{k=1}^n \frac{\bar{L}_0(k)}{z-z_k}\left\langle \prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle \qquad \bar{L}_0(k) = \bar{h}_k + \bar{z}_k\,\partial_{\bar{z}_k}$$

$$\left\langle J^{-1}(z)\prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = \sum_{k=1}^n \frac{\bar{L}_{-1}(k)}{z-z_k}\left\langle \prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle \qquad \bar{L}_{-1}(k) = \partial_{\bar{z}_k}$$

Current Algebra

- We can define modes of these currents. Their commutation relations are

$$[J^a_m, J^b_n] = (a-b) J^{a+b}_{m+n}, \quad a,b = 0, \pm 1, \quad m,n \in \mathbb{Z}$$

  Level zero $\overline{SL(2,\mathbb{C})}$ current algebra.

- The zero modes $\{J^1_0, J^0_0, J^{-1}_0\} = \{\bar{L}_1, \bar{L}_0, \bar{L}_{-1}\}$ are generators of $\overline{SL(2,\mathbb{C})}$

  : antiholomorphic Lorentz transformations.

- Commutators with primary field

$$[J^1_n, \phi_{h,\bar{h}}(z,\bar{z})] = z^n \left(\bar{z}^2 \partial_{\bar{z}} + 2\bar{h}\,\bar{z}\right) \phi_{h,\bar{h}}(z,\bar{z}) \qquad [J^0_n, \phi_{h,\bar{h}}(z,\bar{z})] = z^n \left(\bar{z}\,\partial_{\bar{z}} + \bar{h}\right) \phi_{h,\bar{h}}(z,\bar{z})$$

$$[J^{-1}_n, \phi_{h,\bar{h}}(z,\bar{z})] = z^n\, \partial_{\bar{z}} \phi_{h,\bar{h}}(z,\bar{z})$$

- Commutators with $SL(2,\mathbb{C})$ generators

$$[L_m, J^a_n] = -n\, J^a_{m+n}, \quad a = 0, \pm 1, \quad m = 0, \pm 1$$

Interpretation as diffeomorphism

- Consider infinitesimal diffeomorphisms of the form

$$z \to z, \quad \bar{z} \to \bar{z} + A(z) + B(z)\bar{z} + C(z)\bar{z}^2$$

A, B, C *: analytic functions*

- Mode expanding these functions we can define a basis of vector fields

$$J_n^{-1} = -z^n \frac{d}{d\bar{z}}, \quad J_n^0 = -z^n \bar{z} \frac{d}{d\bar{z}}, \quad J_n^1 = -z^n \bar{z}^2 \frac{d}{d\bar{z}}$$

- Their commutators are identical to that of a level zero $\overline{SL(2,\mathbb{C})}$ current algebra

$$[J_m^a, J_n^b] = (a-b)J_{m+n}^{a+b}, \quad a, b = -1, 0, 1$$

- So the current algebra mentioned before can be geometrically interpreted as the subalgebra of the algebra of diffeomorphisms on the plane with analytic singularities.
- This interpretation is potentially useful for relating the current algebra to asymptotic symmetries.

[*M. Campiglia, A. Laddha; hep-th/1408.2228*]
[*L. Donnay, S. Pasterski, A. Puhm; hep-th/2005.08990*]

OPE with subleading soft graviton

- It is useful to consider the OPE between the subleading soft graviton operator and a matter primary. This can be derived by starting from

$$\left\langle S_1^+(z,\bar z)\phi_{h_1,\bar h_1}(z_1,\bar z_1)\prod_{i=2}^{n}\phi_{h_i,\bar h_i}(z_i,\bar z_i)\right\rangle = \sum_{k=1}^{n}\frac{(\bar z-\bar z_k)^2}{z-z_k}\left[\frac{2\bar h_k}{\bar z-\bar z_k}-\partial_{\bar z_k}\right]\left\langle\prod_{i=1}^{n}\phi_{h_i,\bar h_i}(z_i,\bar z_i)\right\rangle$$

  and expanding the R.H.S. in powers of $(z-z_1),\ (\bar z-\bar z_1)$

- The OPE turns out to be

$$\boxed{\begin{aligned} S_1^+(z,\bar z)\phi_{h_1,\bar h_1}(z_1,\bar z_1) = &-\sum_{p=1}^{\infty}(z-z_1)^{p-1}(J^1_{-p}\phi_{h_1,\bar h_1}(z_1,\bar z_1))(z_1,\bar z_1)\\ &+2(\bar z-\bar z_1)\left(\frac{\bar h_1}{z-z_1}\phi_{h_1,\bar h_1}(z_1,\bar z_1)+\sum_{p=1}^{\infty}(z-z_1)^{p-1}(J^0_{-p}\phi_{h_1,\bar h_1})(z_1,\bar z_1)\right)\\ &-(\bar z-\bar z_1)^2\left(\frac{1}{z-z_1}\partial_{\bar z_1}\phi_{h_1,\bar h_1}(z_1,\bar z_1)+\sum_{p=1}^{\infty}(z-z_1)^{p-1}(J^{-1}_{-p}\phi_{h_1,\bar h_1})(z_1,\bar z_1)\right)\end{aligned}}$$

Current algebra descendants

$J^a_{-p}\phi_{h,\bar{h}}(z,\bar{z}), \quad p \geq 1$ are current algebra descendants of the primary operator $\phi_{h,\bar{h}}(z,\bar{z})$

- Correlation functions with other primary operators

$$\left\langle (J^1_{-p}\phi_{h_1,\bar{h}_1})(z_1,\bar{z}_1)\prod_{i=2}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = \mathcal{J}^1_{-p}(z_1,\bar{z}_1)\left\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$
$$= -\sum_{k\neq 1}\frac{(\bar{z}_k-\bar{z}_1)^2\partial_{\bar{z}_k}+2\bar{h}_k(\bar{z}_k-\bar{z}_1)}{(z_k-z_1)^p}\left\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$

$$\left\langle (J^0_{-p}\phi_{h_1,\bar{h}_1})(z_1,\bar{z}_1)\prod_{i=2}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = \mathcal{J}^0_{-p}(z_1,\bar{z}_1)\left\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$
$$= -\sum_{k\neq 1}\frac{\bar{h}_k+(\bar{z}_k-\bar{z}_1)\partial_{\bar{z}_k}}{(z_k-z_1)^p}\left\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$

$$\left\langle (J^{-1}_{-p}\phi_{h_1,\bar{h}_1})(z_1,\bar{z}_1)\prod_{i=2}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = \mathcal{J}^{-1}_{-p}(z_1,\bar{z}_1)\left\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$
$$= -\sum_{k\neq 1}\frac{1}{(z_k-z_1)^p}\partial_{\bar{z}_k}\left\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$

Current algebra primaries

- The OPE with the subleading soft graviton operator implies

$$(J^a_n\,\phi_{h,\bar{h}})(z,\bar{z}) = 0, \quad \forall n > 0, \quad a = 0, \pm 1$$

$$(J^1_0\,\phi_{h,\bar{h}})(z,\bar{z}) = \bar{L}_1(\bar{z})\phi_{h,\bar{h}}(z,\bar{z}) = 0$$

Thus conformal primaries are also primaries of the $\overline{SL(2,\mathbb{C})}$ *current algebra.*

Leading conformal soft limit & supertranslations

- The leading soft graviton theorem is related to supertranslation Ward identities.
- In order to consider this in the conformal basis, it will be useful to introduce the retarded time coordinates.
- In that case we can use the following *modified* version of the Mellin transform

$$\left\langle \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(u_i, z_i, \bar{z}_i) \right\rangle = \int_0^\infty \prod_{i=1}^{n} d\omega_i \, \omega_i^{\Delta_i - 1} \, e^{-i \sum\limits_{i=1}^{n} \varepsilon_i \omega_i u_i} \, \mathcal{A}_n(\omega_i, z_i, \bar{z}_i) \, \delta^{(4)}\left(\sum_{i=1}^{n} \varepsilon_i \omega_i q_i^\mu \right)$$

[*S. Banerjee; hep-th/1801.10171*]

[*S. Banerjee, S.G., P. Pandey, A. P. Saha; hep-th/1909.03075*]

- Now under the leading conformal soft limit

$$\lim_{\Delta \to 1} (\Delta - 1) \left\langle G^+_\Delta(z, \bar{z}) \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(u_i, z_i, \bar{z}_i) \right\rangle = \left\langle S^+_0(z, \bar{z}) \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(u_i, z_i, \bar{z}_i) \right\rangle$$

$$= - \sum_{k=1}^{n} \frac{\bar{z} - \bar{z}_k}{z - z_k} \, i\partial_{u_k} \left\langle \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(u_i, z_i, \bar{z}_i) \right\rangle$$

[*S. Banerjee; hep-th/1804.06646*]

[*S. Banerjee, S.G.,R. Gonzo; hep-th/2002.00975*]

Leading soft graviton theorem and supertranslations

- Expanding the leading soft factor in the R.H.S. in powers of $\bar{z}$ we get

$$S_0^+(z,\bar{z}) = P_0(z) - \bar{z}P_{-1}(z)$$

where

$$\left\langle P_0(z)\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(u_i,z_i,\bar{z}_i)\right\rangle = \sum_{k=1}^{n}\frac{\bar{z}_k}{z-z_k}\,i\partial_{u_k}\left\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(u_i,z_i,\bar{z}_i)\right\rangle$$

$$\left\langle P_{-1}(z)\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(u_i,z_i,\bar{z}_i)\right\rangle = \sum_{k=1}^{n}\frac{1}{z-z_k}\,i\partial_{u_k}\left\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(u_i,z_i,\bar{z}_i)\right\rangle$$

- These are Ward identities for infinitesimal global space-time translations

$$u \rightarrow u + \epsilon\bar{z} \qquad u \rightarrow u + \epsilon$$

Modes of supertranslation currents

- We can consider modes of the supertranslation currents : $P_{n,0},\ P_{n,-1}$

$$[P_{n,0}, P_{m,-1}] = 0$$

- Global space-time translation generators $\{P_{-1,-1}, P_{0,-1}, P_{-1,0}, P_{0,0}\}$

- Commutators with primary fields

$$[P_{n,0}, \phi_{h,\bar{h}}(u, z, \bar{z})] = z^{n+1}\bar{z}\, i\partial_u \phi_{h,\bar{h}}(u, z, \bar{z})$$

$$[P_{n,-1}, \phi_{h,\bar{h}}(u, z, \bar{z})] = z^{n+1}\, i\partial_u \phi_{h,\bar{h}}(u, z, \bar{z})$$

- Note that if we consider conformal primaries which do not depend on the retarded time coordinate then

$$i\partial_u \phi^{\epsilon}_{h,\bar{h}}(u, z, \bar{z}) \to \epsilon\, \phi^{\epsilon}_{h+\frac{1}{2},\bar{h}+\frac{1}{2}}(z, \bar{z})$$

$\epsilon = \pm 1$ *(outgoing/incoming particles)*

OPE with leading soft graviton

- Let us now consider the OPE between a positive helicity leading soft graviton operator and a generic conformal primary. This can be derived following the same procedure as in the subleading soft graviton case.

$$\boxed{\begin{aligned} &S_0^+(z,\bar{z})\phi_{h_1,\bar{h}_1}(u_1,z_1,\bar{z}_1) \\ &= \sum_{a=2}^{\infty}(z-z_1)^{a-2}(P_{-a,0}\phi_{h_1,\bar{h}_1})(u_1,z_1,\bar{z}_1) \\ &-(\bar{z}-\bar{z}_1)\left(\frac{1}{z-z_1}i\partial_{u_1}\phi_{h_1,\bar{h}_1}(z_1,\bar{z}_1)+\sum_{a=2}^{\infty}(z-z_1)^{a-2}(P_{-a,-1}\phi_{h_1,\bar{h}_1})(u_1,z_1,\bar{z}_1)\right)\end{aligned}}$$

- Note that this OPE implies

$$(P_{n,0}\phi_{h,\bar{h}})(u,z,\bar{z})=0, \quad n\geq -1$$

$$(P_{n,-1}\phi_{h,\bar{h}})(u,z,\bar{z})=0, \quad n\geq 0$$

Supertranslation Descendants

$$(P_{-a,0}\phi_{h,\bar{h}})(u,z,\bar{z}),\ (P_{-a,1}\phi_{h,\bar{h}})(u,z,\bar{z}), \quad a\geq 2$$

are supertranslation descendants of $\phi_{h,\bar{h}}(u,z,\bar{z})$

- Correlation functions with other primary operators

$$\left\langle (P_{-a,0}\phi_{h_1,\bar{h}_1})(u_1,z_1,\bar{z}_1)\prod_{i=2}^{n}\phi_{h_i,\bar{h}_i}(u_i,z_i,\bar{z}_i)\right\rangle = \mathcal{P}_{-a,0}(u_1,z_1,\bar{z}_1)\left\langle \prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(u_i,z_i,\bar{z}_i)\right\rangle$$
$$= -\sum_{k\neq 1}\frac{\bar{z}_k-\bar{z}_1}{(z_k-z_1)^{a-1}}\,i\partial_{u_k}\left\langle \prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(u_i,z_i,\bar{z}_i)\right\rangle$$

$$\left\langle (P_{-a,-1}\phi_{h_1,\bar{h}_1})(u_1,z_1,\bar{z}_1)\prod_{i=2}^{n}\phi_{h_i,\bar{h}_i}(u_i,z_i,\bar{z}_i)\right\rangle = \mathcal{P}_{-a,-1}(u_1,z_1,\bar{z}_1)\left\langle \prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(u_i,z_i,\bar{z}_i)\right\rangle$$
$$= -\sum_{k\neq 1}\frac{1}{(z_k-z_1)^{a-1}}\,i\partial_{u_k}\left\langle \prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(u_i,z_i,\bar{z}_i)\right\rangle$$

Current algebra & Supertranslation commutators

- The generators defined using the leading soft graviton theorem (*for positive helicity graviton*) have the following commutation relations with the current algebra generators

$$[J^1_m, P_{n,-1}] = P_{m+n,0}, \quad [J^0_m, P_{n,-1}] = \frac{1}{2}P_{m+n,-1}, \quad [J^{-1}_m, P_{n,-1}] = 0$$

$$[J^1_m, P_{n,0}] = 0, \quad [J^0_m, P_{n,0}] = -\frac{1}{2}P_{m+n,0}, \quad [J^{-1}_m, P_{n,0}] = -P_{m+n,-1}$$

- The global space-time translation generators

$$\{P_{-1,-1}, P_{0,-1}, P_{-1,0}, P_{0,0}\}$$

are part of the extended algebra generated by $\{P_{n,-1}, P_{n,0}, J^a_n\}$

In the rest of the talk we will see how this extended algebra features in the celestial OPE of gravitons.

Summary (so far)

$\overline{SL(2,\mathbb{C})}$ Current Algebra

$$[J^a_m, J^b_n] = (a-b)J^{a+b}_{m+n}, \quad a,b = 0,\pm 1, \quad m,n \in \mathbb{Z}$$

$$J^1_0 = \bar{L}_1, \quad J^0_0 = \bar{L}_0, J^{-1}_0 = \bar{L}_{-1}$$

Supertranslations

$$[P_{m,n}, P_{m',n'}] = 0, \quad n,n' = 0,-1$$

Mixed Commutators

$$[J^1_m, P_{n,-1}] = P_{m+n,0}, \quad [J^0_m, P_{n,-1}] = \frac{1}{2}P_{m+n,-1}, \quad [J^{-1}_m, P_{n,-1}] = 0$$

$$[J^1_m, P_{n,0}] = 0, \quad [J^0_m, P_{n,0}] = -\frac{1}{2}P_{m+n,0}, \quad [J^{-1}_m, P_{n,-1}] = -P_{m+n,-1}$$

$$[L_m, J^a_{m+n}] = -n J^a_{m+n}, \quad m = 0,\pm 1, \quad n \in \mathbb{Z}$$

$$[L_m, P_{a,b}] = \left(\frac{n-1}{2} - a\right) P_{a+n,b}, \quad n = 0,\pm 1, \quad b = 0,-1$$

Definition of primary states under extended algebra

$$(J^a_n \phi_{h,\bar{h}})(z,\bar{z}) = 0, \quad \forall n > 0, \quad a = 0, \pm 1$$

$$(J^1_0 \phi_{h,\bar{h}})(z,\bar{z}) = \bar{L}_1(\bar{z})\phi_{h,\bar{h}}(z,\bar{z}) = 0$$

$$L_1(z)\phi_{h,\bar{h}}(z,\bar{z}) = 0$$

$$(P_{n,0}\phi_{h,\bar{h}})(z,\bar{z}) = 0, \quad n \geq -1$$

$$(P_{n,-1}\phi_{h,\bar{h}})(z,\bar{z}) = 0, \quad n \geq 0$$

$$L_0(z)\phi_{h,\bar{h}}(z,\bar{z}) = h\phi_{h,\bar{h}}(z,\bar{z}), \quad \bar{L}_0(\bar{z})\phi_{h,\bar{h}}(z,\bar{z}) = \bar{h}\phi_{h,\bar{h}}(z,\bar{z})$$

Celestial OPE from MHV Amplitudes

4-point MHV graviton amplitude

- Celestial OPE of positive helicity gravitons was studied using the (*modified*) Mellin transform of 4-pt. tree level MHV graviton amplitude in Einstein gravity in [*S. Banerjee, S.G., R. Gonzo, hep-th/2002.00975*]
- Upto the first subleading order in the OPE limit it was shown that

$$\boxed{\begin{aligned} &G^{+}_{\Delta_1}(z_1,\bar{z}_2)G^{+}_{\Delta_2}(z_2,\bar{z}_2) \\ &= -\frac{\bar{z}_{12}}{z_{12}} B(i\lambda_1, i\lambda_2) P_{-1,-1} G^{+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) \\ &\quad -\bar{z}_{12}\, B(i\lambda_1, i\lambda_2) \left(\frac{i\lambda_2 - i\lambda_1}{i\lambda_1 + i\lambda_2} P_{-2,-1} + \frac{i\lambda_1}{i\lambda_1 + i\lambda_2} L_{-1} P_{-1,-1} \right) G^{+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) + \cdots \end{aligned}}$$

- Let's consider the subleading soft limit $i\lambda_1 \to -1$ in the above,

$$\lim_{i\lambda_1 \to -1} (1 + i\lambda_1) G^{+}_{\Delta_1}(z_1,\bar{z}_1) G^{+}_{\Delta_2}(z_2,\bar{z}_2)$$

$$= \frac{\bar{z}_{12}}{z_{12}} \left[(i\lambda_2 - 1) P_{-1,-1} + z_{12} ((1 + i\lambda_2) P_{-2,-1} - L_{-1} P_{-,-1}) \right] G^{+}_{\Delta_2 - 1}(z_2,\bar{z}_2)$$

Constraints from subleading soft limit

- Now recall that consistency with the subleading conformal soft theorem requires

$$\lim_{i\lambda_1 \to -1} (1 + i\lambda_1) G^+_{\Delta_1}(z_1, \bar{z}_1) G^+_{\Delta_2}(z_2, \bar{z}_2)$$

$$= \frac{\bar{z}_{12}}{z_{12}} \left[(i\lambda_2 - 1) P_{-1,-1} + 2\, z_{12}\, J^0_{-1} P_{-1,-1} \right] G^+_{\Delta_2 - 1}(z_2, \bar{z}_2)$$

- So we must have

$$\left[L_{-1} P_{-1,-1} + 2\, J^0_{-1} P_{-1,-1} - \Delta_2\, P_{-2,-1} \right] G^+_{\Delta_2 - 1}(z_2, \bar{z}_2) = 0$$

Shifting $\Delta_2 \to \Delta_2 + 1$ gives,

$$\left[L_{-1} P_{-1,-1} + 2\, J^0_{-1} P_{-1,-1} - (1 + \Delta_2) P_{-2,-1} \right] G^+_{\Delta_2}(z_2, \bar{z}_2) = 0$$

- Thus

$$\left[\mathcal{L}_{-1} \mathcal{P}_{-1,-1} + 2\, \mathcal{J}^0_{-1} \mathcal{P}_{-1,-1} - (1 + \Delta_2) \mathcal{P}_{-2,-1} \right] \langle G^+_{\Delta_2} G^-_{\Delta_3} G^-_{\Delta_4} \rangle = 0$$

We thus have a first order linear partial differential equation for the 3-pt. amplitude. This can be explicitly verified by computing the 3-pt. Mellin amplitude.

Limitations of 3-pt. function

- In order to see the action of the all the current algebra and supertranslation generators in the OPE from MHV graviton amplitudes we need to start from $n > 4$ point amplitude.

- This is because the generators $P_{-a,0}, a \geq 2, \quad J^1_{-n}, n \geq 1$ annihilate the 3-pt Mellin amplitude due to

$$\langle G^-_{\Delta_1} G^-_{\Delta_2} G^+_{\Delta_3} \rangle \propto \delta(\bar{z}_{13})\delta(\bar{z}_{23})$$

- The delta function is a consequence of energy-momentum conservation.

- Here we will consider n=6 point graviton amplitudes.

Higher point MHV graviton amplitudes

- There are many available representations of n-point tree-level MHV graviton amplitudes. For our purposes the most useful is the one due to Hodges

[A. Hodges; hep-th/1108.2227; 1204.1930]

$$\mathcal{A}_n = \langle 12 \rangle^8 \, \frac{\det(\Phi^{ijk}_{pqr})}{\langle ij\rangle\langle ik\rangle\langle jk\rangle\langle pq\rangle\langle pr\rangle\langle qr\rangle}$$

$\Phi^{ijk}_{pqr} : (n-3)\times(n-3)$ matrix obtained by removing rows $\{i,j,k\}$ and columns $\{p,q,r\}$ from the $n\times n$ matrix Φ

$$\Phi_{ij} = \begin{cases} \frac{[ij]}{\langle ij\rangle}, & i\neq j \\ -\sum_{k\neq i} \frac{[ik]\langle xk\rangle\langle yk\rangle}{\langle ak\rangle\langle xa\rangle\langle ya\rangle}, & i=j \end{cases}$$

- Salient features:
  - manifests S_n permutation symmetry of the amplitude.
  - manifests soft limits.

6-point MHV amplitude

- For $n = 6$ let us choose the rows and columns to be removed to be

$$\{i, j, k\} = \{1, 2, 3\} \quad \text{and} \quad \{p, q, r\} = \{4, 5, 6\}$$

- 6-graviton MHV amplitude is then

$$\mathcal{A}_6 = \frac{\langle 12 \rangle^8 \ \det\left(\Phi^{123}_{456}\right)}{\langle 12 \rangle \langle 13 \rangle \langle 23 \rangle \langle 45 \rangle \langle 46 \rangle \langle 56 \rangle}$$

- In terms of $(\omega_i, z_i, \bar{z}_i)$

$$\langle ij \rangle = -2\varepsilon_i \varepsilon_j \sqrt{\omega_i \omega_j}\, z_{ij} \qquad [ij] = 2\sqrt{\omega_i \omega_j}\, \bar{z}_{ij}$$

$$\begin{aligned}\mathcal{A}_6 = -4 \prod_{i=1}^{6} \varepsilon_i \ \frac{\omega_1^3 \omega_2^3}{\omega_3 \omega_4 \omega_5 \omega_6} \ \frac{z_{12}^8}{z_{12} z_{13} z_{23} z_{45} z_{46} z_{56}} &\left[\frac{\bar{z}_{14}}{z_{14}} \left(\frac{\bar{z}_{25} \bar{z}_{36}}{z_{25} z_{36}} - \frac{\bar{z}_{26} \bar{z}_{35}}{z_{26} z_{35}} \right) \right. \\ &\left. - \frac{\bar{z}_{24}}{z_{24}} \left(\frac{\bar{z}_{15} \bar{z}_{36}}{z_{15} z_{36}} - \frac{\bar{z}_{16} \bar{z}_{35}}{z_{16} z_{35}} \right) + \frac{\bar{z}_{34}}{z_{34}} \left(\frac{\bar{z}_{15} \bar{z}_{26}}{z_{15} z_{26}} - \frac{\bar{z}_{16} \bar{z}_{25}}{z_{16} z_{25}} \right) \right]\end{aligned}$$

5-point MHV amplitude

- For studying the celestial OPE we will also need the 5-point MHV graviton amplitude. Using Hodge's formula this is

$$\mathcal{A}_5 = \frac{\langle 12 \rangle^8 \ \det\left(\Phi^{123}_{345}\right)}{\langle 12\rangle\langle 13\rangle\langle 23\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle}$$

- Transforming to $(\omega_i, z_i, \bar{z}_i)$ variables we get

$$\mathcal{A}_5 = 4\ \varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5\ \frac{\omega_1^3\omega_2^3}{\omega_3^2\omega_4\omega_5}\ \frac{z_{12}^8}{z_{12}z_{13}z_{23}z_{34}z_{35}z_{45}}\left(\frac{\bar{z}_{14}\bar{z}_{25}}{z_{14}z_{25}} - \frac{\bar{z}_{15}\bar{z}_{24}}{z_{15}z_{24}}\right)$$

6-point Mellin amplitude

- Modified Mellin transform of 6-pt. MHV amplitude

$$\mathcal{M}_6 = \int_0^\infty \prod_{i=1}^{6} d\omega_i \, \omega_i^{\Delta_i - 1} \, e^{-i \sum\limits_{i=1}^{6} \varepsilon_i \omega_i u_i} \, \mathcal{A}_6(\omega_i, z_i, \bar{z}_i) \, \delta^{(4)} \left(\sum_{i=1}^{6} \varepsilon_i \omega_i q_i^\mu \right)$$

- Let's take gravitons $(5^+, 6^+)$ to be outgoing. It's also useful to change variables as follows

$$\omega_5 = \omega_P \, t \qquad \omega_6 = \omega_P (1 - t) \qquad t \in [0, 1]$$

- Then the delta function can be represented as

$$\delta\left(\sum_{i=1}^{6} \varepsilon_i \omega_i q_i^\mu \right) = \frac{i}{4} \frac{1}{(r_{12,34} - \bar{r}_{12,34}) z_{13} \bar{z}_{13} z_{24} \bar{z}_{24}} \prod_{i=1}^{4} \delta(\omega_i - \omega_i^*)$$

$$\omega_i^* = \varepsilon_i \, \omega_P \left(\sigma_{i,1} + z_{56} \, t \, \sigma_{i,2} + \bar{z}_{56} \, t \, \sigma_{i,3} + z_{56} \bar{z}_{56} \, t \, \sigma_{i,4} \right), \quad i \in 1, 2, 3, 4.$$

$\sigma_{i,j}$: *explicit functions of* $z_{ij}, \bar{z}_{ij}, \quad (i,j) \in (1, 2, 3, 4, 6)$

6-point Mellin amplitude

- The delta function localises integral over $\omega_i, \quad i \in (1,2,3,4)$

  Doing the integral over ω_P gives

$$\boxed{\mathcal{M}_6 = \mathcal{N}\,\mathcal{F}(z_i,\bar{z}_i)\int_0^1 dt\; t^{i\lambda_5-1}(1-t)^{i\lambda_6-1}\prod_{i=1}^{4}\Theta\left(\varepsilon_i\left(\sigma_{i,1}+z_{56}\,t\,\sigma_{i,2}+\bar{z}_{56}\,t\,\sigma_{i,3}+z_{56}\bar{z}_{56}\,t\,\sigma_{i,4}\right)\right)\mathcal{I}(t)}$$

where

$$\mathcal{I}(t)=\prod_{i=1}^{2}\left(1+z_{56}\,t\,\frac{\sigma_{i,2}}{\sigma_{i,1}}+\bar{z}_{56}\,t\,\frac{\sigma_{i,3}}{\sigma_{i,1}}+z_{56}\bar{z}_{56}\,t\,\frac{\sigma_{i,4}}{\sigma_{i,1}}\right)^{3+i\lambda_i}\prod_{i=3}^{4}\left(1+z_{56}\,t\,\frac{\sigma_{i,2}}{\sigma_{i,1}}+\bar{z}_{56}\,t\,\frac{\sigma_{i,3}}{\sigma_{i,1}}+z_{56}\bar{z}_{56}\,t\,\frac{\sigma_{i,4}}{\sigma_{i,1}}\right)^{i\lambda_i-1}\times$$
$$\left[1+\frac{t}{\mathcal{U}_1}\left(z_{56}\mathcal{U}_1+\bar{z}_{56}\mathcal{U}_2+z_{56}\bar{z}_{56}\mathcal{U}_3+u_{56}\right)\right]^{-4-i\Lambda}$$

$$\mathcal{N}=-i\prod_{i=1}^{4}\varepsilon_i\prod_{j=1}^{2}(\varepsilon_j\sigma_{j,1})^{3+i\lambda_j}\prod_{k=3}^{4}(\varepsilon_k\sigma_{k,1})^{i\lambda_k-1}\,\frac{z_{12}^8}{z_{12}z_{13}z_{14}z_{16}z_{23}z_{24}z_{26}z_{34}z_{36}z_{46}}\,\frac{\Gamma(4+i\Lambda)}{(i\mathcal{U}_1)^{4+i\Lambda}}\qquad \Lambda=\sum_{i=1}^{6}\lambda_i$$

$$\mathcal{F}(z_i,\bar{z}_i)=\frac{1}{z_{46}z_{56}}\left(1-\frac{z_{56}}{z_{46}}\right)^{-1}\sum_{i=1}^{3}z_{i4}\bar{z}_{i6}\,\sigma_{i,1}\left(1-\frac{\bar{z}_{56}}{\bar{z}_{i6}}\right)\left(1-\frac{z_{56}}{z_{i6}}\right)^{-1}$$

$$\boxed{\begin{array}{ll}\mathcal{U}_1=\sum_{i=1}^{4}\sigma_{i,1}u_{i6} & \mathcal{U}_2=\sum_{i=1}^{4}\sigma_{i,2}u_{i6}\\ \mathcal{U}_3=\sum_{i=1}^{4}\sigma_{i,3}u_{i6} & \mathcal{U}_4=\sum_{i=1}^{4}\sigma_{i,4}u_{i6}\end{array}}$$

Extracting the celestial OPE: General strategy

- In order to extract the celestial OPE of gravitons $(5^+, 6^+)$ we expand the Mellin amplitude around
$$z_{56} = 0, \bar{z}_{56} = 0, u_{56} = 0$$
- The remaining $(z_{ij}, \bar{z}_{ij}, u_{ij}),\ (i,j) \in (1,2,3,4,6)$ are kept fixed and non-zero in this OPE limit.
- While expanding the Mellin amplitude in the OPE limit we can drop the terms that come from differentiating the theta functions w.r.t. $z_{56}, \bar{z}_{56}$. Such terms are proportional to delta functions whose arguments are functions of $(z_{ij}, \bar{z}_{ij}),\ (i,j) \in (1,2,3,4,6)$ and so don't contribute in the OPE limit.

OPE decomposition: Leading term

- Leading term from amplitude

$$\mathcal{M}_6 = -\frac{\bar{z}_{56}}{z_{56}}\,\mathcal{N}\int_0^1 dt\; t^{i\lambda_5-1}(1-t)^{i\lambda_6-1}\prod_{i=1}^{4}\Theta\left(\varepsilon_i\sigma_{i,1}\right)+\cdots$$

$$= -\frac{\bar{z}_{56}}{z_{56}}\;B(i\lambda_5, i\lambda_6)\;\mathcal{P}_{-1,-1}\mathcal{M}_5+\cdots$$

where $\mathcal{M}_5$ *: (modified) Mellin transform of 5-pt. MHV amplitude*

$$\mathcal{M}_5 = i\prod_{i=1}^{4}\varepsilon_i\;\prod_{j=1}^{2}(\varepsilon_j\sigma_{j,1})^{3+i\lambda_i}\;\prod_{k=3}^{4}(\varepsilon_k\sigma_{k,1})^{i\lambda_k-1}\;\frac{z_{12}^8}{z_{12}z_{13}z_{14}z_{16}z_{23}z_{24}z_{26}z_{34}z_{36}z_{46}}\;\frac{\Gamma(3+i\Lambda)}{(i\mathcal{U}_1)^{3+i\Lambda}}\;\prod_{l=1}^{4}\Theta\left(\varepsilon_l\sigma_{l,1}\right)$$

and

$$-\mathcal{N}\prod_{i=1}^{4}\Theta\left(\varepsilon_i\sigma_{i,1}\right) = i\,\partial_{u_6}\mathcal{M}_5 = \mathcal{P}_{-1,-1}\mathcal{M}_5 \qquad\qquad \Lambda = \sum_{i=1}^{6}\lambda_i$$

- This implies the leading OPE

$$\boxed{G^+_{\Delta_5}(z_5,\bar{z}_5)G^+_{\Delta_6}(z_6,\bar{z}_6) = -\frac{\bar{z}_{56}}{z_{56}}\;B(i\lambda_5, i\lambda_6)\;P_{-1,-1}G^+_{\Delta_5+\Delta_6-1}(z_6,\bar{z}_6)}$$

OPE decomposition: Subleading terms

$\mathcal{O}(1)$ term

$$\mathcal{M}_6\Big|_{\mathcal{O}(1)} = -B(i\lambda_5, i\lambda_6)\sum_{i=1}^{4}\frac{\bar{z}_{i6}}{z_{i6}}\sigma_{i,1}\ \mathcal{P}_{-1,-1}\mathcal{M}_5 = B(i\lambda_5, i\lambda_6)\ \mathcal{P}_{-2,0}\mathcal{M}_5$$

- Consistency with subleading conformal soft limit requires

$$\lim_{i\lambda_5\to -1}(1+i\lambda_5)\mathcal{M}_6\Big|_{\mathcal{O}(1)} = -\mathcal{J}^1_{-1}\mathcal{P}_{-1,-1}\mathcal{M}^{'}_5$$

where $\mathcal{M}^{'}_5 = \mathcal{M}_5\big|_{i\lambda_5=-1}$

- From the amplitude we get, $\lim_{i\lambda_5\to -1}(1+i\lambda_5)\mathcal{M}_6\Big|_{\mathcal{O}(1)} = -(i\lambda_6-1)\mathcal{P}_{-2,0}\mathcal{M}^{'}_5$

- Thus we must have the relation

$$\mathcal{J}^1_{-1}\mathcal{P}_{-1,-1}\mathcal{M}^{'}_5 = (i\lambda_6-1)\mathcal{P}_{-2,0}\mathcal{M}^{'}_5$$

Replacing $i\lambda_6 \to (i\lambda_6 + i\lambda_5 + 1)$ yields the relation

$$\boxed{\mathcal{J}^1_{-1}\mathcal{P}_{-1,-1}\mathcal{M}_5 = (i\lambda_5 + i\lambda_6)\mathcal{P}_{-2,0}\mathcal{M}_5 = (\Delta_6 - 1)\mathcal{P}_{-2,0}\mathcal{M}_5}$$

OPE decomposition: Subleading terms

$\mathcal{O}(z_{56})$ term from amplitude

$$\mathcal{M}_6\Big|_{\mathcal{O}(z_{56})} = z_{56}\, B(i\lambda_5, i\lambda_6)\left[\frac{i\lambda_6 - i\lambda_5}{i\lambda_5 + i\lambda_6}\,\mathcal{P}_{-3,0} + \frac{i\lambda_5}{i\lambda_5 + i\lambda_6}\,\mathcal{L}_{-1}\mathcal{P}_{-2,0}\right]\mathcal{M}_5$$

- As before, consistency with subleading conformal soft limit requires

$$\boxed{\mathcal{L}_{-1}\mathcal{P}_{-2,0}\mathcal{M}_5 = (2 + i\lambda_5 + i\lambda_6)\mathcal{P}_{-3,0}\mathcal{M}_5 - \mathcal{J}^1_{-2}\mathcal{P}_{-1,-1}\mathcal{M}_5}$$

- Consequently the $\mathcal{O}(z_{56})$ term can also be written as

$$\mathcal{M}_6\Big|_{\mathcal{O}(z_{56})} = z_{56}\, B(i\lambda_5, i\lambda_6)\left[(1 + i\lambda_5)\mathcal{P}_{-3,0} - \frac{i\lambda_5}{i\lambda_5 + i\lambda_6}\,\mathcal{J}^1_{-2}\mathcal{P}_{-1,-1}\right]\mathcal{M}_5$$

OPE decomposition: Subleading terms

$\mathcal{O}(\bar{z}_{56})$ term from amplitude

$$\mathcal{M}_6\Big|_{\mathcal{O}(\bar{z}_{56})} = \bar{z}_{56}\, B(i\lambda_5, i\lambda_6) \left[\frac{i\lambda_5}{i\lambda_5 + i\lambda_6} \left(\bar{\mathcal{L}}_{-1}\mathcal{P}_{-2,0} - \mathcal{L}_{-1}\mathcal{P}_{-1,-1}\right) + \frac{i\lambda_5 - i\lambda_6}{i\lambda_5 + i\lambda_6}\, \mathcal{P}_{-2,-1}\right] \mathcal{M}_5$$

- In this case consistency with subleading conformal soft limit requires

$$\boxed{\left(\bar{\mathcal{L}}_{-1}\mathcal{P}_{-2,0} - \mathcal{L}_{-1}\mathcal{P}_{-1,-1} + (2 + i\lambda_5 + i\lambda_6)\mathcal{P}_{-2,-1}\right) \mathcal{M}_5 = 2\,\mathcal{J}^0_{-1}\mathcal{P}_{-1,-1}\mathcal{M}_5}$$

Note that compared to the 3-pt case we now have the non-trivial action of the generator $P_{-2,0}$

- This allows us to re-write the $\mathcal{O}(\bar{z}_{56})$ term as

$$\mathcal{M}_6\Big|_{\mathcal{O}(\bar{z}_{56})} = \bar{z}_{56}\, B(i\lambda_5, i\lambda_6) \left[\frac{2i\lambda_5}{i\lambda_5 + i\lambda_6}\mathcal{J}^0_{-1}\mathcal{P}_{-1,-1} - (1 + i\lambda_5)\mathcal{P}_{-2,-1}\right] \mathcal{M}_5$$

OPE decomposition: Subleading terms

Finally let us consider the $\mathcal{O}(z_{56}\bar{z}_{56})$ term from amplitude

$$\mathcal{M}_6\Big|_{\mathcal{O}(z_{56}\bar{z}_{56})}$$

$$= z_{56}\bar{z}_{56}\, B(i\lambda_5, i\lambda_6)\Bigg[\frac{i\lambda_5(1+i\lambda_5)}{(i\lambda_5+i\lambda_6)(1+i\lambda_5+i\lambda_6)}\left(\mathcal{L}_{-1}\mathcal{P}_{-2,-1} - \bar{\mathcal{L}}_{-1}\mathcal{P}_{-3,0} + \mathcal{L}_{-1}\bar{\mathcal{L}}_{-1}\mathcal{P}_{-2,0} - \frac{1}{2}\mathcal{L}_{-1}^{2}\mathcal{P}_{-1,-1}\right)$$

$$- \left(1 + \frac{i\lambda_5\, i\lambda_6}{1+i\lambda_5+i\lambda_6}\right)\mathcal{P}_{-3,-1} + \frac{2\, i\lambda_5\, i\lambda_6}{(i\lambda_5+i\lambda_6)(i\lambda_5+i\lambda_6+1)}\,\mathcal{J}^{0}_{-2}\mathcal{P}_{-1,-1}\Bigg]\mathcal{M}_5$$

- This is already in a form that is manifestly consistent with leading and subleading conformal soft limits.

Celestial OPE coefficients from Extended Symmetry Algebra

Primary Descendants

- From the explicit amplitude computations we have seen that certain linear combinations of descendants vanish when acting on MHV amplitudes. In fact such combinations must also be primaries under the extended symmetry algebra. This is analogous to null state relations in ordinary 2d CFTs.

- We can determine these primary descendants purely from symmetry considerations as follows. Consider the linear combination

$$\Psi^\sigma(z,\bar z) = \left(J^1_{-1} P_{-1,-1} + c P_{-2,0}\right) G^\sigma_\Delta(z,\bar z) \qquad\qquad \sigma = \pm 2$$

Both states in the above are individually primaries under Poincare.

Demanding this to be a primary under the current algebra implies: $J^a_n \Psi^\sigma = 0, \quad n \geq 1$

In particular applying J_1^{-1} gives, $\boxed{c = -(2\bar h + 1)}$ $\longrightarrow$ *precisely agrees with the amplitude calculation*

It can also be easily verified that this state is also annihilated by $P_{a,0}, a \geq 0, \quad P_{n,-1}, n \geq 1$

Consequently Ψ^σ *is a primary descendant of the extended symmetry algebra.*

Primary Descendants

- Let us consider another instance of a primary descendant. For this consider the linear combination

$$\Phi = \left(L_{-1}P_{-1,-1} + c_1 J^0_{-1}P_{-1,-1} + c_2 P_{-3,-1} + c_3 \bar{L}_{-1}P_{-2,0}\right) G^{\sigma}_{\Delta}(z,\bar{z})$$

Demanding that

$$L_1\Phi = \bar{L}_1\Phi = P_{0,-1}\Phi = P_{-1,0}\Phi = 0$$

and using the previous *vanishing condition* we get

$$\boxed{c_1 = 2, \quad c_2 = -(\Delta+1), \quad c_3 = -1, \quad \sigma = 2}$$

The values of these coefficients again precisely match with those found from the amplitude.

Then it can be verified that for the above values Φ is also annihilated by

$$J^a_n, n > 0; \quad P_{n,-1}, n \geq 0; \quad P_{n,0}, n \geq -1$$

Thus Φ is a primary descendant of the extended symmetry algebra.

Differential equations for MHV amplitudes

- These primary descendants when inserted in Mellin transformed MHV graviton amplitudes yield

$$\left\langle \left[J^1_{-1} P_{-1,-1} - (2\bar{h}+1) P_{-2,0}\right] G^{\sigma}_{\Delta}(z,\bar{z}) \prod_i G^{\sigma_i}_{\Delta_i}(z_i,\bar{z}_i) \right\rangle_{MHV} = 0 \qquad \sigma = \pm 2$$

$$\implies \left[\mathcal{J}^1_{-1} \mathcal{P}_{-1,-1} - (2\bar{h}+1) \mathcal{P}_{-2,0}\right] \left\langle G^{\sigma}_{\Delta}(z,\bar{z}) \prod_i G^{\sigma_i}_{\Delta_i}(z_i,\bar{z}_i) \right\rangle_{MHV} = 0$$

$$\left\langle \left[L_{-1} P_{-1,-1} + 2 J^0_{-1} P_{-1,-1} - (\Delta+1) P_{-2,-1} - \bar{L}_{-1} P_{-2,0}\right] G^{+}_{\Delta}(z,\bar{z}) \prod_i G^{\sigma_i}_{\Delta_i}(z_i,\bar{z}_i) \right\rangle_{MHV} = 0$$

$$\implies \left[\mathcal{L}_{-1} \mathcal{P}_{-1,-1} + 2 \mathcal{J}^0_{-1} \mathcal{P}_{-1,-1} - (\Delta+1) \mathcal{P}_{-2,-1} - \bar{\mathcal{L}}_{-1} \mathcal{P}_{-2,0}\right] \left\langle G^{+}_{\Delta}(z,\bar{z}) \prod_i G^{\sigma_i}_{\Delta_i}(z_i,\bar{z}_i) \right\rangle_{MHV} = 0$$

Some comments on the differential equations

- The first equation can be analytically verified using Hodge's representation of *n-pt.* MHV graviton amplitudes. We have analytically checked the second equation for *5*-pt MHV amplitudes and numerically for *6*-points.

- The 2 sets of differential equations are not independent. The second equation can be derived from the first using special conformal transformations.

- The presence of both holomorphic, i.e., z-derivatives and anti-holomorphic, i.e, $\bar{z}$-derivatives reflects the fact that the underlying infinite dimensional symmetry algebra does not admit holomorphic factorisation.

- We will now see that these equations can be used to fully determine the structure of the leading OPE coefficients for gravitons primaries.

- Here will only deal with the case where both gravitons considered in the OPE are outgoing in the S-matrix. Analogous techniques can be used to determine the leading OPE structure for the case where one graviton is incoming and the other outgoing.

Leading OPE coefficient from differential equations

- Let the leading OPE for (*outgoing*) graviton primaries be of the form

$$G^{+}_{\Delta}(z,\bar{z})G^{\sigma_1}_{\Delta_1}(z_1,\bar{z}_1) = C_{pq}(\Delta,\Delta_1,\sigma_1)(z-z_1)^p(\bar{z}-\bar{z}_1)^q G^{\sigma_2}_{\Delta_2}(z_1,\bar{z}_1)$$

- Consider the differential equation involving $\mathcal{J}^1_{-1},\ \mathcal{P}_{-2,0}$

$$\mathcal{J}^1_{-1}\mathcal{P}_{-1,-1}\left\langle G^{\sigma}_{\Delta}(z,\bar{z})\prod_i G^{\sigma_i}_{\Delta_i}(z_i,\bar{z}_i)\right\rangle_{MHV} = (2\bar{h}+1)\mathcal{P}_{-2,0}\left\langle G^{\sigma}_{\Delta}(z,\bar{z})\prod_i G^{\sigma_i}_{\Delta_i}(z_i,\bar{z}_i)\right\rangle_{MHV}$$

- Then take the holomorphic collinear limit $z \to z_1$
- Expand the differential operators and keep the singular terms in this limit.
- Insert the above form of the OPE inside the correlator corresponding to the MHV amplitude.
- Match coefficients of $(z-z_1)^{p-1}(\bar{z}-\bar{z}_1)^{q+1}$ on both sides.

Leading OPE coefficient from differential equations

- This yields the relation

$$(\Delta-1)C_{pq}(\Delta,\Delta_1+1,\sigma_1)=(\Delta_1-\sigma_1+q)C_{pq}(\Delta+1,\Delta_1,\sigma_1)$$

- The same procedure for the other differential equation gives

$$(\Delta-q)C_{pq}(\Delta,\Delta_1+1,\sigma_1)=(\Delta_1-\sigma_1+2q+p)C_{pq}(\Delta+1,\Delta_1,\sigma_1)$$

- These equations have non trivial solutions iff

$$q=1,\quad p=-1$$

- The leading OPE then has the form

$$G^{+}_{\Delta}(z,\bar{z})G^{\sigma_1}_{\Delta_1}(z_1,\bar{z}_1)=\frac{\bar{z}-\bar{z}_1}{z-z_1}C_{-1,1}(\Delta,\Delta_1,\sigma_1)\,G^{\sigma_1}_{\Delta+\Delta_1}(z_1,\bar{z}_1)$$

This is precisely the structure of the leading OPE in a 2-derivative theory of gravity. So, this analysis suggests that the bulk gravitational theory that is dual to the MHV sector of the celestial CFT must be a 2-derivative theory.

Leading OPE coefficient

- From the differential equations we have the relation

$$(\Delta - 1)C_{-1,1}(\Delta, \Delta_1 + 1, \sigma_1) = (\Delta_1 - \sigma_1 + 1)C_{-1,1}(\Delta + 1, \Delta_1, \sigma_1)$$

- Now demanding invariance of the OPE under global u-translations gives

$$C_{-1,1}(\Delta, \Delta_1, \sigma_1) = C_{-1,1}(\Delta + 1, \Delta_1, \sigma_1) + C_{-1,1}(\Delta, \Delta_1 + 1, \sigma_1)$$

- These equations are solved by

$$C_{-1,1}(\Delta, \Delta_1, \sigma_1) = \alpha\, B(\Delta - 1, \Delta_1 - \sigma_1 + 1)$$

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$ *: Euler Beta function*

- The constant can be determined from the leading soft limit and this gives $\alpha = -1$

  So we have

$$\boxed{G^{+}_{\Delta}(z, \bar{z}) G^{\sigma_1}_{\Delta_1}(z_1, \bar{z}_1) = -\frac{\bar{z} - \bar{z}_1}{z - z_1} B(\Delta - 1, \Delta_1 - \sigma_1 + 1)\, G^{\sigma_1}_{\Delta + \Delta_1}(z_1, \bar{z}_1)}$$

Recursion relations for descendant OPE coefficients

- Let us consider the subleading terms in the OPE corresponding to descendants. For positive helicity (*outgoing*) gravitons this has the structure

$$G^+_{\Delta_1}(z_1,\bar z_1)G^+_{\Delta_2}(z_2,\bar z_2) = -\frac{\bar z_{12}}{z_{12}}B(i\lambda_1,i\lambda_2)\,P_{-1,-1}G^+_{\Delta_1+\Delta_2-1}(z_2,\bar z_2)$$

$$+\left(\alpha_1\,P_{-2,0}+\alpha_2 J^1_{-1}P_{-1,-1}\right)G^+_{\Delta_1+\Delta_2-1}(z_2,\bar z_2)$$
$$+z_{12}\left(\alpha_3\,P_{-3,0}+\alpha_4 J^1_{-2}P_{-1,-1}+\alpha_5 L_{-1}P_{-2,0}\right)G^+_{\Delta_1+\Delta_2-1}(z_2,\bar z_2)$$

$$+\bar z_{12}\left(\alpha_6\,P_{-2,-1}+\alpha_7 J^0_{-1}P_{-1,-1}+\alpha_8 J^{-1}_0 P_{-2,0}+\alpha_{10}L_{-1}P_{-1,-1}\right)G^+_{\Delta_1+\Delta_2-1}(z_2,\bar z_2)\,+\cdots$$

- Demanding invariance of the above OPE under the extended symmetry algebra yields recursion relations for the descendant OPE coefficients. Taking into account the vanishing conditions mentioned before, these recursion relations can be solved in terms of the leading OPE coefficient.

- Upto the order shown above, we have checked that the solutions to such recursion relations precisely agree with corresponding results obtained by expanding the Mellin transform of the 6-pt MHV graviton amplitude in the OPE limit.

Conclusions

- We constructed a current algebra using the subleading soft graviton theorem for a *positive helicity outgoing* graviton.

- We showed that *for MHV amplitudes* the celestial OPE of positive helicity gravitons can be organised into representations of an extended symmetry algebra comprising of the $\overline{SL(2,\mathbb{C})}$ current algebra, supertranslations and holomorphic Lorentz generators.

- We showed that there exist *null-state* relations which lead to partial differential equations for celestial correlators dual to MHV graviton amplitudes.

- Our analysis suggests that the celestial CFT that computes MHV graviton amplitudes can be treated as a *decoupled sector* which is entirely governed by the $\overline{SL(2,\mathbb{C})}$ current algebra and supertranslations.

Future Directions

- Understand the structure of celestial CFT correlators dual to NMHV graviton amplitudes. In this case the current algebras corresponding to both positive as well as negative helicity soft gravitons are likely to play an important role.
- Does there exist similar structure in other cases such as Einstein-Yang Mills amplitudes ?
- Implications for double copy structures for celestial correlators.

THANK YOU