# How SYM domain walls look like

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# Main goal - to reveal the shape of domain walls in pure $\mathcal{N}$=1, D=4 $SU(N)$ SYM

- $\mathcal{N}$=1, D=4 SYM was constructed in 1974

  ( *Wess & Zumino, Ferrara & Zumino, Salam & Strathdee*)
- Studied intensively over 45 years
- Since early 80s it is known that pure $SU(N)$ SYM has $N$ degenerate susy vacua distinguished by different vevs of the gluino condensate (related by $Z_N$ R-symmetry transformations)

$$< \mathrm{Tr}\, \lambda^\alpha \lambda_\alpha >= \Lambda^3 e^{2\pi i \frac{n}{N}}, \quad n = 0, 1, \ldots N-1$$

- There should exist BPS domain walls interpolating between different vacua and having the following tension (*Dvali & Shifman '97*)

$$T_{\mathrm{DW}} = \frac{N}{8\pi^2} \,| < \lambda\lambda >_n - < \lambda\lambda >_l |$$

# Since the 90s, domain walls in $\mathcal{N}$=1, D=4 SYM and SQCD have been intensively studied with the use of different approaches

- For a recent review and latest developments see

V. Bashmakov, F. Benini, S. Benvenuti, and M. Bertolini,
*Living on the walls of super-QCD*, arXiv:1812.04645

D. Delmastro and J. Gomis, Domain Walls in 4d N=1 Supersymmetric Yang-Mills, arXiv:2004.11395

Two complimentary (dual) ways of describing domain walls:

- as effective 3D field theories
- as solitonic solutions of the 4D field theory

# Solitonic solutions of low-energy EFT describing the *pure* SYM domain walls have not been found until recently

- **Reason:** SYM domain walls are not smooth solitonic field configurations. Their existence requires the presence of a source which has been lacking in pure $\mathcal{N}$=1, D=4 SYM
  - dynamical membranes
- **Aim of this talk** to show
  - how to couple the membrane to $\mathcal{N}$=1 SYM and its Veneziano-Yankielowicz effective formulation
  - how the membrane creates BPS domain walls and what is their shape

# Review of pure $\mathcal{N}$=1 SU(N) SYM

- Field content - adjoint vector supermultiplet

$$A_m^I,\ \lambda_\alpha^I,\ \bar\lambda_{\dot\alpha}^I,\ D^I \qquad I \in\ adj(SU(N))$$

- Building block of the SYM action – chiral spinor superfield

$$\mathcal{W}_\alpha(x,\theta) = -i\lambda_\alpha + \theta_\alpha D - \frac{i}{2} F_{mn} \sigma^{mn}{}_\alpha{}^\beta \theta_\beta + \theta^2 \sigma^m_{\alpha\dot\beta} \nabla_m \bar\lambda^{\dot\beta}, \quad \bar D_{\dot\alpha} \mathcal{W}_\beta = 0.$$

$$\mathcal{L}_{\rm SYM} = \frac{1}{4g^2} \int d^2\theta \, {\rm Tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha \ + \ {\rm c.c.}$$

$$= -\,{\rm Tr}\ \left( \frac{1}{4} F_{mn} F^{mn} + i\lambda\sigma^m \nabla_m \bar\lambda - \frac{D^2}{2} \right)$$

# Review of pure $\mathcal{N}$=1 SU(N) SYM

- Special chiral scalar superfield

$$S = \mathrm{Tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha = -\mathrm{Tr}\lambda^\alpha \lambda_\alpha + \sqrt{2}\theta^\alpha \chi_\alpha + \theta^2 F$$

$$\chi = \sqrt{2}\,\mathrm{Tr}\,\left(\frac{1}{2}F_{mn}\sigma^{mn}\lambda - i\lambda D\right)$$

$$F = \mathrm{Tr}\,\left(-2i\lambda\sigma^m \nabla_m \bar{\lambda} - \frac{1}{2}F_{mn}F^{mn} + D^2 - \frac{i}{4}\varepsilon_{mnpl}F^{mn}F^{pl}\right)$$

- Important notice

$$\begin{aligned}\mathrm{Im}\, F = {}^*F_4 \quad &= \quad -^*\left(\mathrm{Tr}\, F_2 \wedge F_2\right) - \partial_m \mathrm{Tr}\, \lambda\sigma^m \bar{\lambda} \\ &= \quad -^*d\,\mathrm{Tr}\,\left(AdA + \frac{2i}{3}A^3 + \frac{1}{3!}dx^k dx^n dx^m \epsilon_{mnkl}\mathrm{Tr}\,\lambda\sigma^l\bar{\lambda}\right) \equiv {}^*dC_3\end{aligned}$$

$$S = -\frac{i}{4}\bar{D}^2\, U, \qquad U(x,\theta,\bar{\theta}) \;-\; \text{real salar superfield} \quad (\textit{Gates '81})$$

# R-symmetry anomaly and N-degeneracy of SYM vacua

- Classical *U*(1) R-symmetry: $\lambda_\alpha \to e^{i\beta}\lambda_\alpha, \quad \bar{\lambda}_{\dot{\alpha}} \to e^{-i\beta}\lambda_\alpha$

- Anomaly: $\partial_m J^m := \partial_m(\lambda\sigma^m\bar{\lambda}) = \frac{2N}{32\pi^2}\varepsilon^{mnpq}F_{mn}F_{pq}$

$$U(1) \quad \to \quad \mathbb{Z}_{2N}$$

- Gluino condensate and further spontaneous breaking of $\mathbb{Z}_{2N} \to \mathbb{Z}_2$

(*Witten '82, Veneziano & Yankielowicz '82,…, Novikov & Shifman '88, … )*

$$< \mathrm{Tr}\,\lambda^\alpha\lambda_\alpha >= \Lambda^3 e^{2\pi i\frac{n}{N}}, \quad n = 0, 1, \ldots N-1$$

# Veneziano-Yankielowicz ’82 Lagrangian revisited

- Provides effective description of colorless bound states of the SYM multiplet (glueballs, gluinoballs and their fermionic superpartner), gluino condensate and the N-degeneracy of the SYM vacuum

$$S = s(x) + \sqrt{2}\theta^\alpha \chi_\alpha(x) + \theta^2(\hat{D}(x) + \mathtt{i}\, {}^*dC_3(x))$$

$${}^*dC_3 = \partial_m C^m, \quad \text{where} \quad C_1 = {}^*C_3$$

- The form of the VY Lagrangian is (almost) fixed by anomalous superconformal Ward identities of the SU(N) SYM - WZ model

$$\mathcal{L}_{VY} = \int d^2\theta d^2\bar{\theta} K(S\bar{S}) + \int d^2\theta W(S) + c.c\,,$$

$$K(S,\bar{S}) = \frac{1}{\rho}(S\bar{S})^{\frac{1}{3}}, \qquad W(S) = \frac{N}{16\pi^2} S\left(\ln\frac{S}{\Lambda^3} - 1\right)$$

# Veneziano-Yankielowicz ’82 Lagrangian revisited

- The superpotential and the Lagrangian are not single valued

$$S \to S\, e^{2\pi\, i} \qquad \mathcal{L}_{VY} \to \mathcal{L}_{VY} + \frac{N}{4\pi}\,\partial_m C^m$$

- The special form of the chiral superfield $S = -\frac{i}{4}\bar{D}^2\, U$ requires the variation of the VY Lagrangian with respect to independent real superfield $U$.
- The variation principle is well-defined only with the addition of the boundary (total derivative) term (*Bandos, Lanza, D.S. ’19*)

$$\mathcal{L}_{\rm bd} = -\frac{1}{8}\left(\int d^2\theta \bar{D}^2 - \int d^2\bar{\theta} D^2\right)\left[\left(\frac{1}{12\rho}\bar{D}^2\frac{\bar{S}^{\frac{1}{3}}}{S^{\frac{2}{3}}} + \frac{1}{16\pi^2}\ln\frac{\Lambda^{3N}}{S^N}\right)U\right] + {\rm c.c.}$$

# Veneziano-Yankielowicz '82 Lagrangian revisited

- Bosonic part of the Lagrangian

$$\mathcal{L}_{\mathrm{VY}}^{\mathrm{bos}} = K_{s\bar{s}}\left(-\partial_m s \partial^m \bar{s} + (\partial_m C^m)^2 + \hat{D}^2\right) + \left(W_s\left(\hat{D} + i\partial_m C^m\right) + \mathrm{c.c.}\right) + \mathcal{L}_{\mathrm{bd}}^{\mathrm{bos}}$$

- boundary term

$$\mathcal{L}_{\mathrm{bd}}^{\mathrm{bos}} = -2\partial_m\left[C^m\left(K_{s\bar{s}}\partial_n C^n - \mathrm{Im}\, W_s\right)\right], \quad K_{s\bar{s}} \equiv \partial_s\partial_{\bar{s}}K(s,\bar{s}), \quad W_s \equiv \partial_s W(s)$$

- auxiliary field equations of motion

$$K_{s\bar{s}}\hat{D} + \mathrm{Re}\, W_s = 0 \quad \rightarrow \quad \hat{D} = -\frac{\mathrm{Re}\, W_s}{K_{s\bar{s}}}$$

$$\partial_m(K_{s\bar{s}}\partial_n C^n - \mathrm{Im}\, W_s) = 0 \quad \rightarrow \quad \partial_m C^m = \frac{\mathrm{Im}\, W_s - \frac{n}{8\pi}}{K_{s\bar{s}}}$$ <- integration constant

$$F \equiv \hat{D} + \mathrm{i}\partial_m C^m = -\frac{\overline{W}_{\bar{s}} + \mathrm{i}\,\frac{n}{8\pi}}{K_{s\bar{s}}}$$

# Veneziano-Yankielowicz '82 Lagrangian revisited

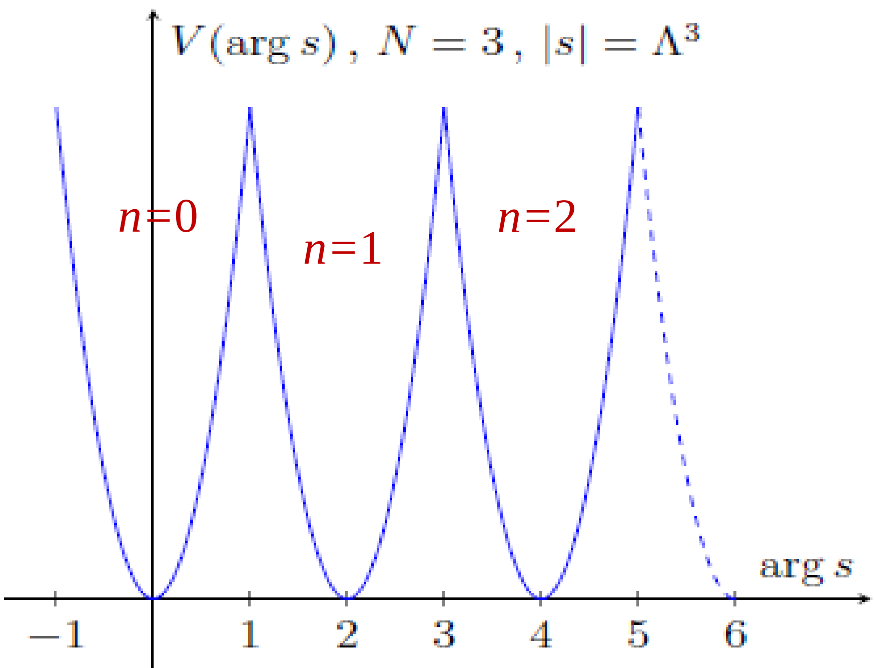
- Scalar field potential (*Kovner & Shifman '97*)

$$V(s,\bar{s}) = \frac{9\rho N}{16\pi^2}|s|^{\frac{4}{3}}\left(\ln^2\frac{|s|}{\Lambda^3} + (\arg s - 2\pi\frac{n}{N})^2\right)\,, \quad n = 0,1,2,\ldots,N-1$$

Potential is single-valued, multi-branched, has cusps at $\arg s = \frac{\pi n}{N}$

and susy minima at $\langle s\rangle = \Lambda^3 e^{2\pi \mathrm{i}\frac{n}{N}} \quad - \quad SU(N)$ SYM vacua

What causes the cusps of the potential?



# Coupling membrane to SYM

- Supersymmetric and kappa-symmetric membrane action
  (*I. Bandos, S. Lanza, D.S. '19*)

$$\mathcal{S}_{M2+SYM} = -\frac{|k|}{4\pi}\int d^3\xi\sqrt{-\det h_{ij}}\,|S| - \frac{k}{4\pi}\int \mathcal{C}_3 \qquad (k = 0, \pm 1, \pm 2, \dots)$$

Nambu-Goto Wess-Zumino

$$S(x,\theta) = \mathrm{Tr}\,\mathcal{W}^\alpha \mathcal{W}_\alpha = -\frac{i}{4}\bar{D}^2\,U, \qquad U(x(\xi),\theta(\xi),\bar{\theta}(\xi)) \;-\; \text{real salar superfield}$$

$$E^a = dx^a + i\,\theta\sigma^a\bar{\theta} + c.c.$$

$$\begin{aligned}\mathcal{C}_3 =& \,\mathrm{i}\,E^a \wedge d\theta^\alpha \wedge d\bar{\theta}^{\dot{\alpha}}\sigma_{a\alpha\dot{\alpha}}U \\ &- \frac{1}{4}E^b \wedge E^a \wedge d\theta^\alpha \sigma_{ab\,\alpha}{}^\beta D_\beta U - \frac{1}{4}E^b \wedge E^a \wedge d\bar{\theta}^{\dot{\alpha}}\bar{\sigma}_{ab}{}^{\dot{\beta}}{}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}U \\ &- \frac{1}{48}E^c \wedge E^b \wedge E^a \epsilon_{abcd}\,\bar{\sigma}^{d\dot{\alpha}\alpha}[D_\alpha, \bar{D}_{\dot{\alpha}}]U\,.\end{aligned}$$

$$\mathcal{C}_3|_{\theta=0} = C_3 = \mathrm{Tr}\left(AdA + \frac{2i}{3}A^3\right)$$

# Kappa-symmetry

- Counterpart of local worldvolume supersymmetry

$$\delta\theta^\alpha = \kappa^\alpha(\xi), \qquad \delta x^m = i\kappa\sigma^m\bar{\theta} + c.c.$$

$$\kappa_\alpha = -\mathtt{i}\frac{kS}{|kS|}\Gamma_{\alpha\dot{\alpha}}\bar{\kappa}^{\dot{\alpha}}, \qquad \Gamma^2 = 1, \qquad \Gamma_{\alpha\dot{\alpha}} \equiv \frac{\mathtt{i}\epsilon^{ijk}}{3!\sqrt{-\det h}}\epsilon_{abcd}E_i^b E_j^c E_k^d\,\sigma^a_{\alpha\dot{\alpha}}$$

Gauges away 2 of 4 fermionic modes $\theta^\alpha(\xi), \bar{\theta}^{\dot{\alpha}}(\xi)$ of the membrane

Worldvolume reparametrization gauges away 3 of 4 bosonic modes $x^m(\xi)$

$$\left(x^3(\xi), \psi^\alpha(\xi)\right) \quad \text{Goldstone supermultiplet}$$

associated with ½ broken supersymmetry in the 4D bulk, while another ½ of susy remains unbroken allowing for BPS configurations

# Induced $\mathcal{N}$=1, 3d Chern-Simons theory on the membrane

- Consider a static membrane in the pure $\mathcal{N}$=1 SYM background

$$\xi^i = x^i, \qquad x^3(\xi) = 0, \qquad \theta^\alpha(\xi), \qquad \bar{\theta}^{\dot{\alpha}}(\xi) = 0$$

- The membrane action reduces to that of $\mathcal{N}$=1, d=3 $SU(N)$ level-$k$ CS action

$$S_{M2} = -\frac{\mathrm{i}k}{4\pi}\int_{\mathcal{C}_3} d^3\xi\, \mathrm{Tr}\psi^\alpha\psi_\alpha + \frac{k}{4\pi}\int_{\mathcal{C}_3} \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right)$$

$$\psi_\alpha(\xi)\ \ (\alpha = 1, 2) \quad \Leftarrow \quad \lambda_1 = \frac{1}{2}(\psi_1 + \mathrm{i}\psi_2)$$

real $SL(2,R)$ spinor

# BPS domain wall solutions sourced by the membrane

- Consider a static membrane in the Veneziano-Yankielowicz model

$$\theta^\alpha(\xi) = \bar{\theta}^{\dot{\alpha}}(\xi) = 0, \quad \xi^i = x^i \ (i = 0, 1, 2), \quad x^3 = 0$$

- The presence of the membrane modifies the bulk field equations by source terms, in particular the gauge 3-form eq.

$$\partial_m(K_{s\bar{s}}\partial_n C^n - \mathrm{Im}\, W_s) = -\frac{k}{8\pi}\delta^3_m \delta(x^3)$$

$$\mathrm{Im}\, F = \partial_m C^m = \frac{8\pi \mathrm{Im}\, W_s - (n + k\Theta(x^3))}{8\pi K_{s\bar{s}}}$$

$$\langle s \rangle = \Lambda^3 e^{2\pi \mathrm{i} \frac{n}{N}} \qquad \Bigg| \qquad \langle s \rangle = \Lambda^3 e^{2\pi \mathrm{i} \frac{n+k}{N}}$$

# BPS domain wall solutions sourced by the membrane

- BPS domain-wall equation is dictated by ½ susy conservation

$$\delta\chi_\alpha = \mathtt{i}\sigma^m_{\alpha\dot{\alpha}}\bar{\epsilon}\,\partial_m s + \epsilon_\alpha F = 0$$

$$\frac{\partial s(x^3)}{\partial x^3} \equiv \dot{s} = \mathrm{i}e^{\mathrm{i}\alpha}F = -\mathrm{i}e^{\mathrm{i}\alpha}\frac{\overline{W}_{\bar{s}} + \frac{\mathrm{i}}{8\pi}(n + k\Theta(x^3))\,s}{K_{s\bar{s}}}, \qquad e^{\mathrm{i}\alpha} = \frac{s(0)}{|s(0)|} \quad \text{(on M2)}$$

Substituting the form of $W$ and $K$ of the Veneziano-Yankielowiz (VY) model, we get

$$\dot{s} = 9\mathrm{i}\,\rho N|s|^{\frac{4}{3}}e^{\mathrm{i}\,\alpha}\left(\ln\frac{\Lambda^3}{|s|} + \mathrm{i}\,\arg s - \frac{2\pi\mathrm{i}}{N}(n + k\Theta(x^3))\right)$$

BPS value of the on-shell action for the VY model + membrane

$$S_{\rm BPS} = S_{\rm VY} + S_{\rm membr} = -\,2\int d^3\xi\,|W_{+\infty} - W_{-\infty}|$$

$$T_{\rm DW} = T_s + T_{\rm membr} = 2\,|W_{+\infty} - W_{-\infty}| = \frac{N}{8\pi^2}\,\Lambda^3\left|e^{2\pi\,i\,\frac{n+k}{N}} - e^{2\pi\,i\,\frac{n}{N}}\right|$$

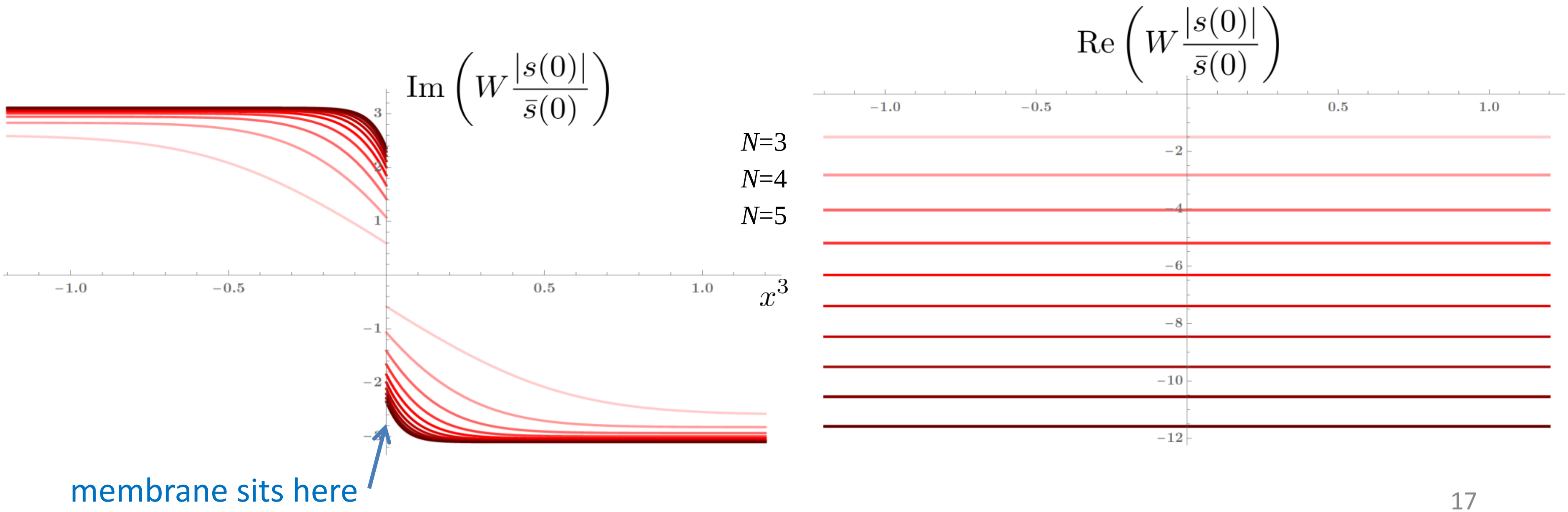
# Shape of BPS domain walls

- $s(x)$-continuous domain wall solutions of the BPS equations exist for the membrane charge having the following values

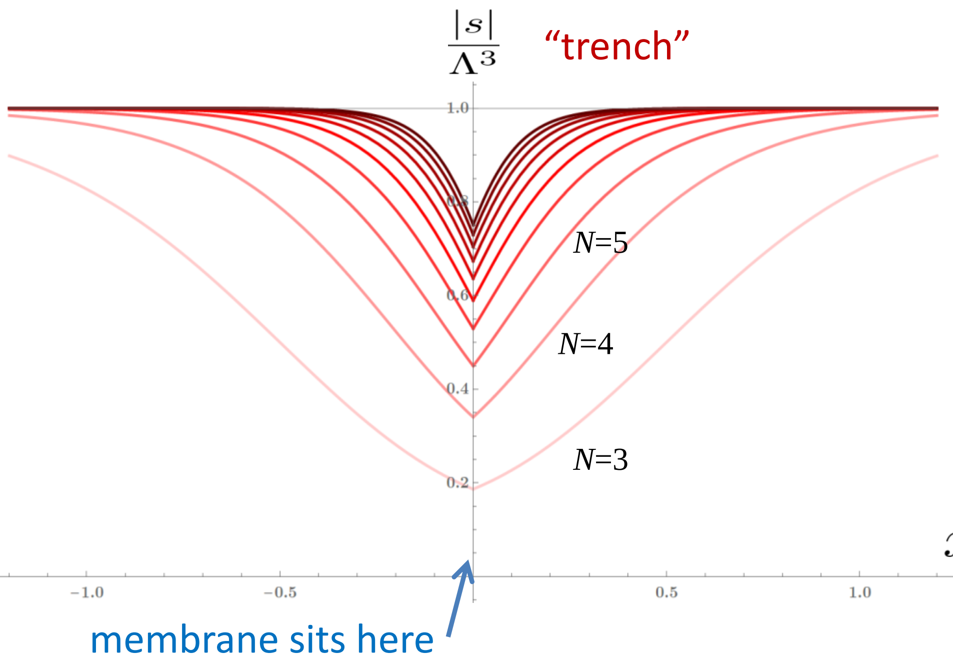
$$|k| \leq \frac{N}{3}$$

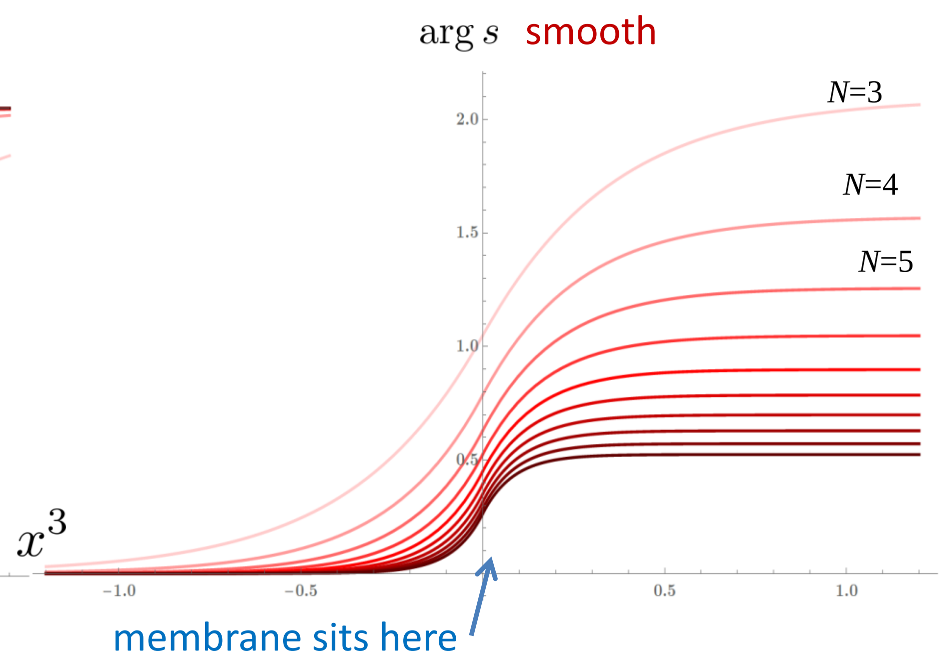
Examples $k = 1, \quad N = 3, 4, 5, 6, 7, \ldots$

Form of the superpotential



# Shape $s(x)$ of BPS domain walls with $|k| \leq \frac{N}{3}$




Solution breaks down for N=2, k=1, i.e. for $|k| > \frac{N}{3}$

# Conclusion

- We have constructed the supersymmetric and kappa-invariant action describing the coupling of a membrane to $\mathcal{N}$=1, D=4 SYM and its Veneziano-Yankielowicz effective sigma-model

- The membrane of charge $k$ separates two SYM vacua with different phases of the gluino condensate

$$\langle s \rangle = \Lambda^3 e^{2\pi \mathrm{i} \frac{n}{N}} \qquad \Big| \qquad \langle s \rangle = \Lambda^3 e^{2\pi \mathrm{i} \frac{n+k}{N}}$$

- and creates BPS domain walls interpolating between these vacua with tension

$$T_{\mathrm{DW}} = \frac{N\Lambda^3}{8\pi^2} \left| e^{2\pi\, i\, \frac{n+k}{N}} - e^{2\pi\, i\, \frac{n}{N}} \right|$$

- Explicit domain wall configurations have been found for $|k| \leq \dfrac{N}{3}$

# Outlook

To relate our construction to the deiscription of SYM domain walls from the perspective of M-theory /type IIA string

Witten ’97: M5-brane wrapping a 3-cycle of $G_2$ manifold

Acharya-Vafa ‘01 description in type IIA:

- Uses a stack of ***k*** D4-branes wrapped on an internal 2-cycle with N RR fluxes

- The resulting 3d worldvolume EFT is an $\mathcal{N}=1$ SYM+CS theory with a gauge group $U(k)_N$ and Chern-Simons level $N$

in IR $U(k)_{N-k,N}$ topological CS

- The 3d worldvolume theory on our membrane is $SU(N)_k$ CS

for *k*=1 it is level-rank dual to the Acharya-Vafa construction

relation in the case $k>1$ should still be understood