Cluster integrable systems and supersymmetric gauge theories

$4d/5d$

$4+1$

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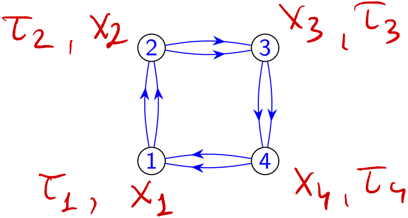
OIST seminar

Zoom, April 2021

Plan, pictures & equations

- Cluster varieties, quiver mutations

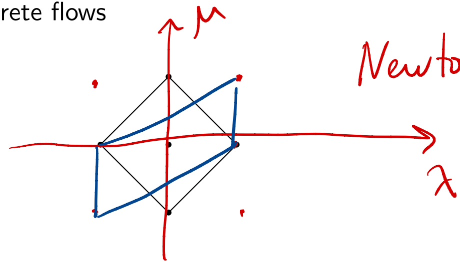
X- cluster variety



$x_1x_2x_3x_4 = q$

- Integrable systems, discrete flows

$\frac{d\lambda}{\lambda}$ $\frac{d\mu}{\mu}$

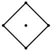


Newton polygon

Plan, pictures & equations

- Deautonomization $q \neq 1$, q-difference Painlevé-Hirota equations

$$\tau_j(qz)\,\tau_j\left(q^{-1}z\right) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}(z)\,\tau_{j-1}(z)$$

- Solutions: supersymmetric gauge theories
  - $SU(N)$ 5d SUSY gauge theory with coupling z and VEV u
  -  – $SU(2)$ Seiberg-Witten curve,
  - $\tau_j(z)$ – dual Nekrasov function.

- Perspectives ...

V. Fock

M. Bershtein
P. Gavrylenko
M. Semenyakin
I. Motorin

Integrability

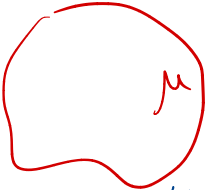
Classical Hamiltonian mechanics (Liouville-Arnold):

- Symplectic $(\mathcal{M}, \varpi)$, $d\varpi = 0$, $\mathcal{M} \subset \mathcal{X}$ Poisson manifold

$$\bar{\omega} = \sum_{i=1}^{\frac{1}{2}\dim \mathcal{M}} dp_i \wedge dq_i \qquad d\bar{\omega} = 0$$

$$\exists\ \bar{\omega}^{-1}$$

$$\{q_i, p_j\} = \delta_{ij} \qquad \{H_i, H_j\} = 0$$



$i_1 = 1 \ldots \frac{1}{2}\dim \mathcal{M}$

- Functions in Poisson involution $H_k \in \mathrm{Fun}(\mathcal{X})$, $\{H_i, H_k\} = 0$

$\mathcal{M}_R \subset X$ $\qquad \mathbb{R}^3$ $\qquad \{x, y\} = z$ ... cycling

$\{x_i, x_j\} = \varepsilon_{ijk} x_k$

$\underset{\text{Casimir}}{R^2} = x^2 + y^2 + z^2 = \text{const}$ $\qquad \{R, x_k\} = 0 \ \forall k$

- Counting: $\dim \mathcal{X} = \tilde{B} + 2g$

$\tilde{B}$ = # Casimir functions

g = # integrals of motion

Cluster integrable system

- Defined by a convex NP $\Delta \subset \mathbb{Z}^2 \subset \mathbb{R}^2$: a curve $\Sigma \subset \mathbb{C}^\times \times \mathbb{C}^\times$

g = genus

$$f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0. \quad (1)$$

GK; F, AM

- Realized on a Poisson X-cluster variety $\mathcal{X}$: Poisson structure

$$\{x_i, x_j\} = \epsilon_{ij} x_i x_j, \qquad \{x_i\} \in \left(\mathbb{C}^\times\right)^{2\mathrm{Area}(\Delta)}. \quad (2)$$

determined by $\mathcal{Q}$, with $\epsilon_{ij} = \#\mathrm{arrows}(i \to j)$.

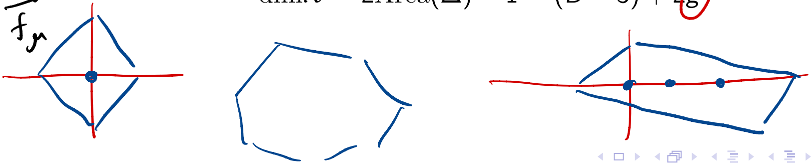
- Integrability: Pick's formula

$\frac{\lambda^i \mu^j d\lambda}{f_\gamma}$

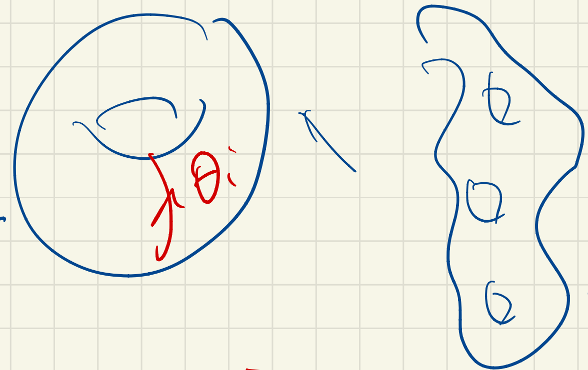
boundary points

$$\dim\mathcal{X} = 2\mathrm{Area}(\Delta) - 1 = (B-3) + 2g \quad (3)$$

q $/\!/$ − # internal points



Jacobian Σ

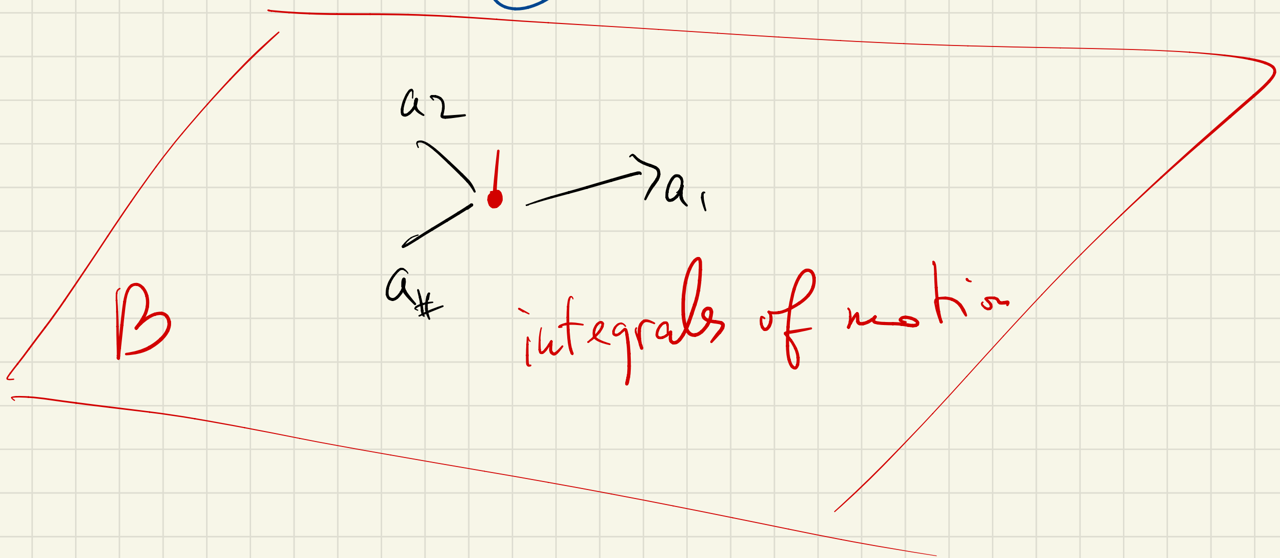


complex curve

$g = \dim B$

$\dim B = g$

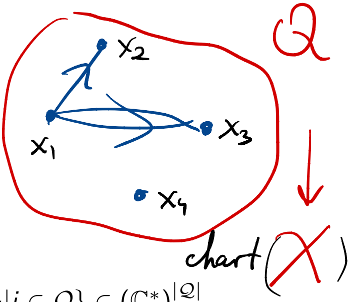
$$\bar{\omega} = \sum_{i=1} da_i \wedge d\theta_i$$



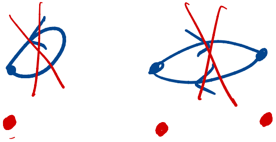
Cluster varieties

Algebraic: Bi-rational transf.

- *Quiver* $\mathcal{Q}$ with $|\mathcal{Q}|$ vertices, oriented edges;



- Forbidden



- Variables $\{x_i | i \in \mathcal{Q}\} \in (\mathbb{C}^*)^{|\mathcal{Q}|}$, alternatively $\{\tau_i | i \in \mathcal{Q}\} \in (\mathbb{C}^*)^{|\mathcal{Q}|}$.

$\{x_1, x_2\} = x_1 x_2 = -\{x_2, x_1\}$

Cluster Poisson variety: logarithmically constant bracket

$\{x_1, x_3\} = 2x_1 x_3$

$$\{x_i, x_j\} = \epsilon_{ij} x_i x_j, \qquad i, j = 1, \ldots, |\mathcal{Q}| \tag{4}$$

(no sum!) with skew-symmetric $\{\log x_i, \log x_j\} = \epsilon_{ij}$ $\{x_4, x_j\} = 0$

$$\epsilon_{ij} = \#\text{arrows}\ (i \to j) = -\epsilon_{ji} \tag{5}$$

Examples

- $x = e^Q$, $y = e^P$, then $\{x, y\} = xy$ with the quiver $\underset{x}{\bullet}\!\to\!\underset{y}{\bullet}$.

$x, y \in \mathbb{C}^*$ $\qquad (\mathbb{C}^*)^{|Q|}$

$SL(2)/\mathrm{Ad}\,H$

$2 =$ Cartan matrix of $SL(2)$

$$\begin{tikzcd} \underset{x}{\bullet} \arrow[r, bend left] \arrow[r, bend right] & \underset{y}{\bullet} \end{tikzcd}$$

- Poisson submanifolds (symplectic leaves) in Lie groups ($SL(2)$ or $PGL(2)$)

Fock - Goncharov map

$$\underset{x}{\circ}\!\to\!\underset{y}{\circ} = \begin{pmatrix} x^{1/2} & 0 \\ 0 & x^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} = g(x, y)$$

$$\underset{x}{\bullet} \xrightarrow[\exp(e)]{} \underset{y}{\bullet}$$

$$\underset{x}{\bullet}\to\underset{y}{\bullet}\leftarrow\underset{\cancel{x}}{\bullet}$$

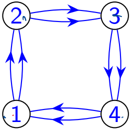
$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\{g \overset{\otimes}{,} g\} = -\frac{1}{2}[r, g \otimes g]$$

Examples

- Mutation class of quiver $\mathcal{Q}$:

$x_1 = x_2$ $\quad$ $x_3 = y_0$ $\quad$ $\hat{C} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$



$0 = \{x_1, x_3\} = \{x_2, x_4\}$

$\dim X = 4 - 2$

$x_4 = x_0$

$x \sim e^Q$

$y \sim e^P$

defines the bracket $\quad y_1 = x_1$

$$\{x_i, x_{i+1}\} = 2x_i x_{i+1}, \qquad i = 1, \dots, 4 \tag{6}$$

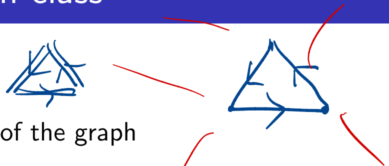
$\{y_i, x_j\} = \hat{C}_{ij} y_i x_j$

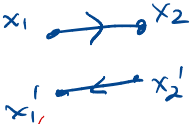
$\{y_1, x_0\} = -2 y_1 x_0$

- **Exercise**: check Jacobi identity!
- **Remarks**:
  - Poisson submanifold in $\widehat{SL(2)}^\natural / \mathrm{Ad} H$
  - $q = x_1 x_2 x_3 x_4$ and $z = x_1 x_3$ are in the center of Poisson algebra.
  - straightforward quantization $\hat{x}_i \hat{x}_j = p^{-2\epsilon_{ij}} \hat{x}_j \hat{x}_i$ (q and p – *two* parameters).

$\{q, x_i\} = 0$ $\qquad$ $H(x_1, x_2, x_3, x_4)$ $\qquad$ $p \sim e^{\hbar}$

Mutation class





Mutations of the graph

$$\mu_j:\ \epsilon_{ik}\mapsto -\epsilon_{ik},\ \text{if } i=j \text{ or } k=j,\qquad \epsilon_{ik}\mapsto \underbrace{\epsilon_{ik}+\frac{\epsilon_{ij}|\epsilon_{jk}|+\epsilon_{jk}|\epsilon_{ij}|}{2}}_{\epsilon'_{ij}}\quad \text{otherwise,}$$

the x-variables (Poisson map) $\{x',x'\}=\epsilon' x' x'$

$\mu_i^2=1$

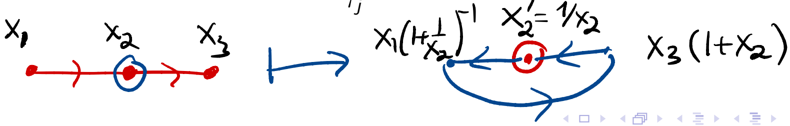
$$\mu_j:\quad x_j\to\frac{1}{x_j},\qquad x_i\to \underbrace{x_i\left(1+x_j^{\mathrm{sgn}(\epsilon_{ij})}\right)^{\epsilon_{ij}}}_{x_i'},\ i\neq j \tag{7}$$

or the τ-variables (symplectic map)

$\mu_i\circ\mu_j=\mu_j\circ\mu_i$

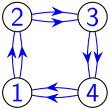
$\epsilon_{ij}=0$

$$\mu_j:\quad \tau_j\mapsto\frac{\prod_{\epsilon_{ij}>0}\tau_i^{\epsilon_{ij}}+\prod_{\epsilon_{ij}<0}\tau_i^{-\epsilon_{ij}}}{\tau_j},\qquad \tau_i\mapsto\tau_i,\qquad i\neq j \tag{8}$$



Mutations as Poisson maps

Mutations of our "basic" quiver:



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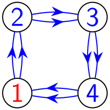

Mutation μ_1

Mutations as Poisson maps

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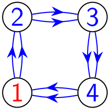
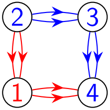


Mutation μ_1

$\longrightarrow$



Reverse all arrows connected to vertex 1

Mutations as Poisson maps

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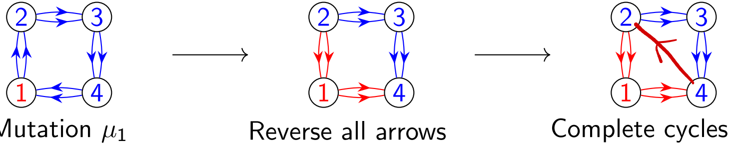

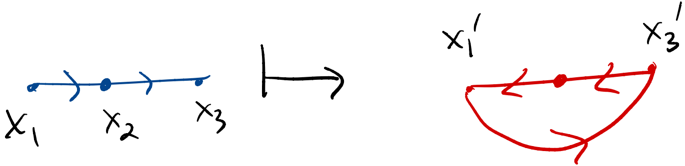

$\{x_1, x_3\} = 0$

Mutations as Poisson maps

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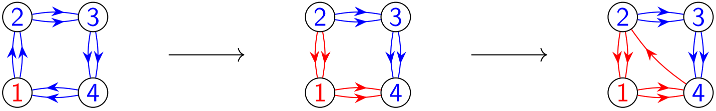
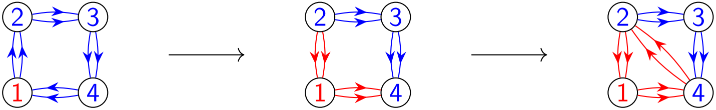


Mutation μ_1 Reverse all arrows Complete cycles

Mutations as Poisson maps

Mutations of our "basic" quiver:



Mutation μ_1 — Reverse all arrows — Complete cycles

Mutations as Poisson maps

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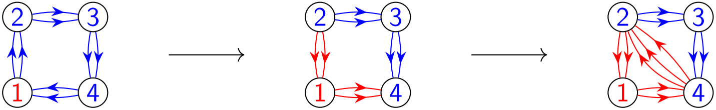

Mutation μ_1 Reverse all arrows Complete cycles

Mutations as Poisson maps

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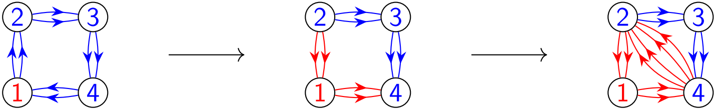
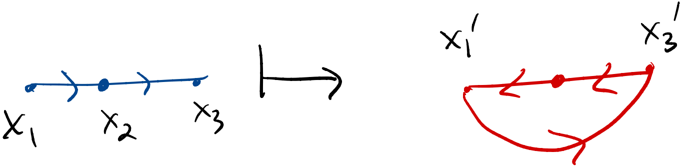


Mutation μ_1 Reverse all arrows Complete cycles



$$\{x_1, x_3\} = 0$$

Mutations as Poisson maps

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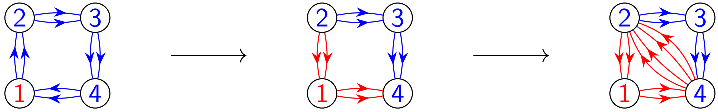


Mutation μ_1 Reverse all arrows Complete cycles

$$\mu_1 : x'_1 = 1/x_1, \quad x'_2 = x_2/\left(1 + 1/x_1\right)^2, \quad x'_4 = x_4\left(1 + x_1\right)^2, \quad x'_3 = x_3$$

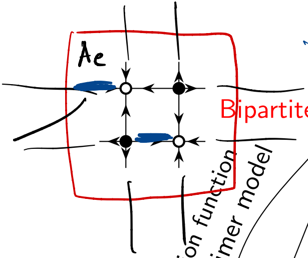
Poisson map: $\{x'_i, x'_k\} = \epsilon'_{ik} x'_i x'_k$

A particular case of generic (bi-rational) transformations:

$$\mu_j : x_j \mapsto x_j^{-1}, \qquad x_i \mapsto x_i \left(1 + x_j^{\mathrm{sgn}\epsilon_{ij}}\right)^{\epsilon_{ij}}, \quad i \neq j$$

8 dimer config

$$\sum_{\{D\}} \prod_{e \in D} A_e = W$$

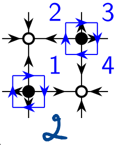


Bipartite graph on torus

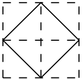
Partition function of the dimer model

Inverse algorithm

- Partition function of dimer model gives set of integrals o motion + casimir funct

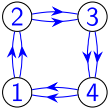


- face variables $F = dA$



Family of spectral curves = Newton polygon

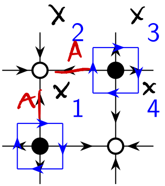
Quiver



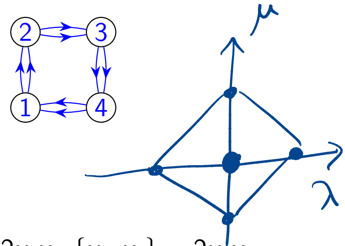
dual surface

Σ

$g \neq 1$



$A \to dA = \underline{\underline{F}} = \{x; \lambda, \mu\}$



Poisson bracket (dual surface $\Sigma \simeq$ spectral curve)

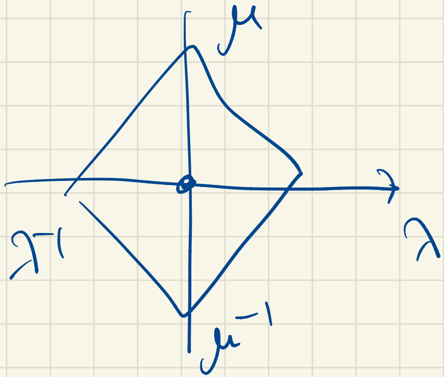
$$\{x_1, x_2\} = 2x_1x_2, \{x_2, x_3\} = 2x_2x_3, \{x_3, x_4\} = 2x_3x_4, \{x_4, x_1\} = 2x_4x_1$$

Dimer partition function $\mathcal{Z}(\lambda, \mu) \mapsto t_{\mathcal{Z}} \cdot \mathcal{Z}(t_\lambda \cdot \lambda, t_\mu \cdot \mu) = f_\Delta(\lambda, \mu)$, with

$$f_\Delta(\lambda, \mu) = \lambda + \lambda^{-1} + \mu + z\mu^{-1} + H \overset{(x)}{=} 0$$

$$q = x_1x_2x_3x_4 = 1, \quad z = x_1x_3, \quad H = \sqrt{x_1x_2} + \frac{1}{\sqrt{x_1x_2}} + \sqrt{\frac{x_1}{x_2}} + z\sqrt{\frac{x_2}{x_1}}$$

Cluster integrable system: $\frac{5d}{SW}$ $(4+1)$ susy (8 scharges)



$$f(\lambda,\mu) = \lambda + \lambda^{-1} + \mu + \frac{z}{\mu} + u = 0$$

z – Casimir function = gauge coupling

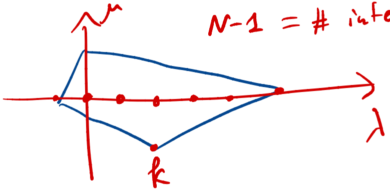
u = vacuum condensate

$g = 1$ $\qquad$ $SU(2)$ gauge group

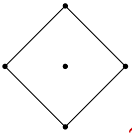
$$a = \oint_A \log\lambda \, \frac{d\mu}{\mu}$$

$$a_D = \oint_B \log\lambda \, \frac{d\mu}{\mu}$$

Newton polygon



N-1 = # internal points (rank of gauge group)



$SU(N)_k$

k = CS level

B = # boundary points = # boundary segments

Newton Polygon (up to $SA(2,\mathbb{Z})$-tranform) of the curve:

$$f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a\mu^b f_{a,b} = \lambda + \frac{1}{\lambda} + \mu + \frac{z}{\mu} + H = 0 \qquad (9)$$

- Renormalizations of λ, μ and f_Δ fix 3 of coefficients $\{f_{a,b}\}$ in the equation;
- Counting: $2\mathrm{Area}(\Delta) - 1 = (B-3) + 2g$ works as $2\cdot 2 - 1 = (4-3) + 2\cdot 1$;
- Toda family

g

Casimir functions, except for g

5d "relativistic" Toda

$H(q,p) = f(e^p, e^q)$

Integrable system

$\{g \otimes g\} = -\frac{1}{2}[r, g \otimes g]$

$H_j = \mathrm{Tr}_{\mu_j} g$

Complete integrability:

- Liouville-Arnold theorem: $\{H_i, H_j\}=0$, $i, j = 1, \ldots, g$;
- $\{\vec{H}\}$ are (properly normalized!) coefficients of dimer partition function, corresponding to *internal* points of Δ.

relativistic Toda

mutation transforms

Discrete integrability:

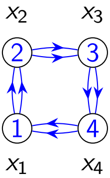
- $\{\vec{H}\}$ are invariant wrt discrete flows $\mathcal{G}_\Delta \subset \mathcal{G}_\mathcal{Q}$;
- $\mathcal{G}_\Delta$ is generated by sequences of quiver mutations (and permutations of vertices) or *spider moves* of the GK bipartite graph.

$$\underline{\underline{T}} : H \mapsto H, \qquad T \in \mathcal{G}_\Delta \tag{10}$$

de-autonomization
gauge theories

Cluster automorphisms

Abelian subgroup $\mathcal{G}_\mathcal{Q} \supset \mathcal{G}_\Delta \supset T$ (discrete flows):



Cluster automorphisms

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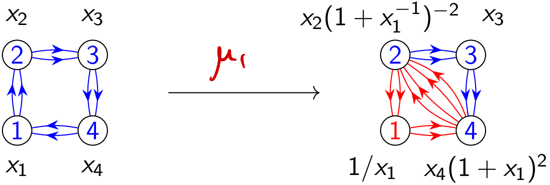

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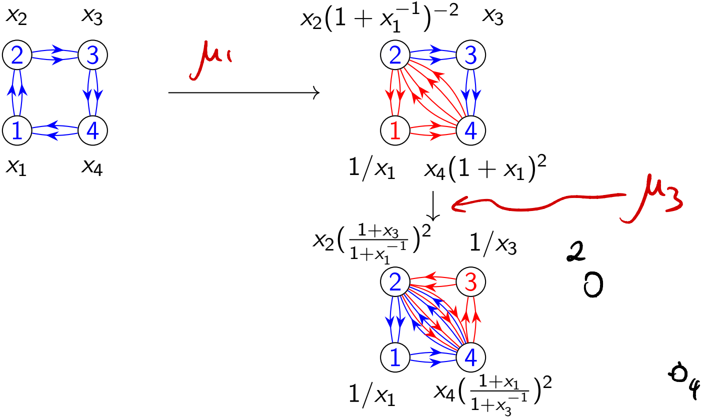

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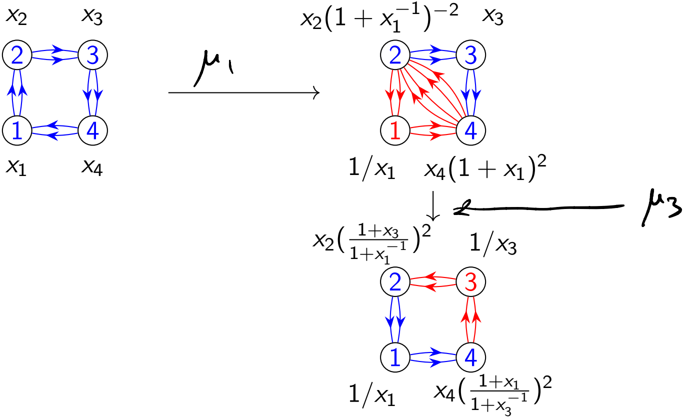
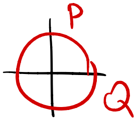
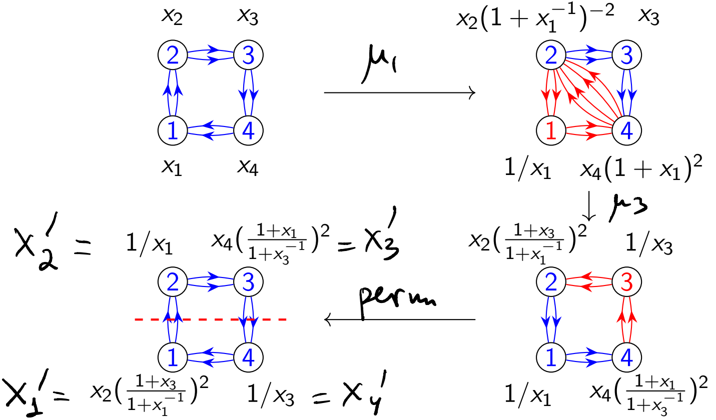


Cluster automorphisms

Abelian subgroup $\mathcal{G}_\mathcal{Q} \supset \mathcal{G}_\Delta \supset T$ (discrete flows):





$$\{X_i', X_j'\} = \{X_i, X_j\}$$

algebraic, rational transformation

Discrete flow

For $q = 1$ the flow T

x_1' x_2' x_3' x_4'

$$T : (x_1, x_2, x_3, x_4) \mapsto \left(x_2 \left(\frac{1 + x_3}{1 + x_1^{-1}} \right)^2, x_1^{-1}, x_4 \left(\frac{1 + x_1}{1 + x_3^{-1}} \right)^2, x_3^{-1} \right)$$

or

$$T : (x_1, x_2, z, q) \mapsto \left(x_2 \left(\frac{x_1 + z}{x_1 + 1} \right)^2, x_1^{-1}, qz, q \right) \underset{q=1}{=} \left(x_2 \left(\frac{x_1 + z}{x_1 + 1} \right)^2, x_1^{-1}, z, q \right)$$

preserves the Hamiltonian $H = \sqrt{x_1 x_2} + \frac{1}{\sqrt{x_1 x_2}} + \sqrt{\frac{x_1}{x_2}} + z\sqrt{\frac{x_2}{x_1}}$.

Discrete flow preserves int. of motion at $q=1$

Deautonomization

Let $x_1x_2x_3x_4 = q \neq 1$

$$T : (x_1, x_2, z, q) \mapsto \left(x_2 \left(\frac{x_1 + z}{x_1 + 1} \right)^2, x_1^{-1}, qz, q \right)$$

Consider z as "time" $x_1 = x(z)$, $x_2 = x^{-1}(q^{-1}z)$, $T : x(z) \mapsto x(qz)$, satisfying

$$x(qz)x(q^{-1}z) = \left(\frac{x(z) + z}{x(z) + 1} \right)^2$$

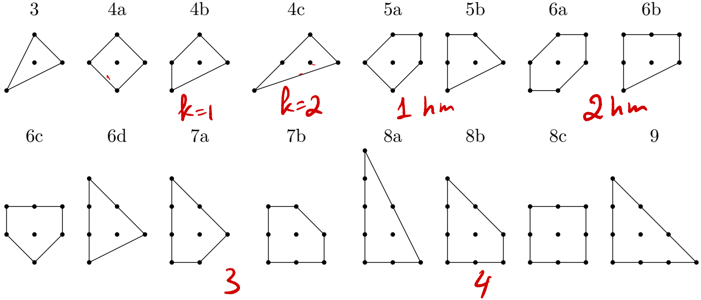
isomonodromic deformations

or q-Painlevé III_3 equation $P(A_7^{(1)'})$.

Just by $q \neq 1$ $\Longrightarrow$ non-autonomous system of Painlevé type

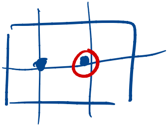
Painlevé NP

with a single internal point and $3 \leq B \leq 9$ boundary points: $\beta = 2N_c - N_f \geq 0$



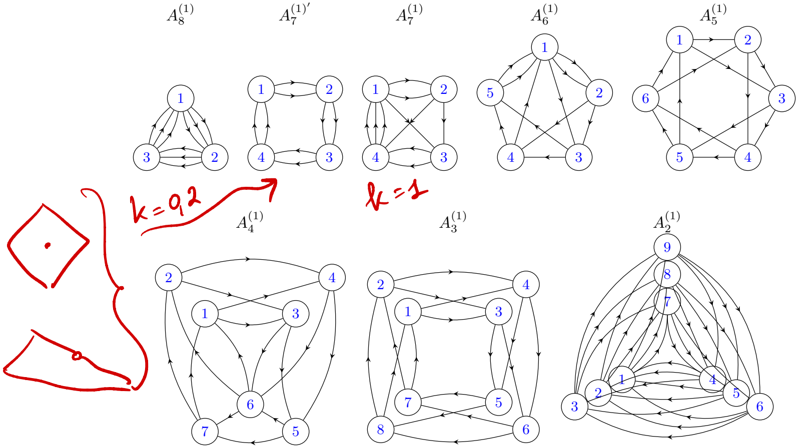
Here Σ: $f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0$ is always a torus $g = 1$.

- 5d SW $SU(2)$ gauge group
- Exceptions



Painlevé quivers

lead to the Painlevé quivers (Sakai classification):



Tau-functions

For the tau-functions $x(z) = z^{1/2}\frac{\tau_1(z)^2}{\tau_0(z)^2}$ one gets bilinear (non-autonomous!) *Hirota equations*

$$\tau_0(qz)\tau_0(q^{-1}z) = \tau_0(z)^2 + z^{1/2}\tau_1(z)^2$$
$$\tau_1(qz)\tau_1(q^{-1}z) = \tau_1(z)^2 + z^{1/2}\tau_0(z)^2$$

"Generic phenomenon": for the $SU(N)_k$-Toda family ($Y^{N,k}$-geometry)

$$\tau_j\left(qz\right)\tau_j\left(q^{-1}z\right) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right)$$
$$j \in \mathbb{Z}/N\mathbb{Z}$$

Origin: mutation of tau-variables ...

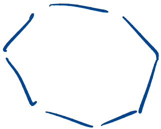
Solutions

Autonomous case: for arbitrary Δ, $j = 1, \dots, B$

$q \to 1$

$$\mathcal{T}_{\vec{n}} \sim \Theta\Big(Z + \sum_j n_j A(P_j)\Big) \prod_{i<j} E(P_i, P_j)^{n_i n_j}$$

$(z_i - z_j)^{n_i n_j}$



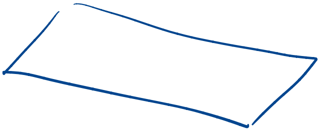
$q \neq 1$: for the (N,k)-theory

$$\tau_j^{N,k}(\vec{u}, \vec{s}; q|z) = \sum_{\vec{\Lambda} \in Q_{N-1}+\omega_j} s^{\Lambda} \mathcal{Z}_{N,k}(\vec{u} q^{\vec{\Lambda}}; q^{-1}, q|z) \tag{11}$$

$q \neq 1$

with sum over A_{N-1} root lattice, $\{\omega_j\}$ are fundamental weights, but $\mathcal{Z}_{N,k} = \mathcal{Z}_{\mathrm{cl}}^{N,k} \cdot \mathcal{Z}_{\mathrm{1loop}}^{N} \cdot \mathcal{Z}_{\mathrm{inst}}^{N,k}$ are 5d Nekrasov functions.

$\mathbb{Z}^N$ $q = \prod x_j$ z, $\vec{u}$



SU(N)

$q_1 = q^{-1}$, $q_2 = q$

Nekrasov functions: 5d SYM

Here:

$$\mathcal{Z}_{\text{cl}}^{N,k} = \exp\left(\log z \frac{\sum (\log u_i)^2}{-2\log q_1 \log q_2} + k \frac{\sum (\log u_i)^3}{-6\log q_1 \log q_2}\right)$$

$T \sim \log \sinh a$

$$\mathcal{Z}_{\text{1loop}}^{N} = \prod_{1\le i\ne j\le N} (u_i/u_j; q_1, q_2)_\infty, \quad \mathcal{Z}_{\text{inst}}^{N,k} = \sum_{\vec{\lambda}} \frac{z^{|\vec{\lambda}|}\prod_{i=1}^N \mathsf{T}_{\lambda^{(i)}}(u; q_1, q_2)^k}{\prod_{i,j=1}^N \mathsf{N}_{\lambda^{(i)},\lambda^{(j)}}(u_i/u_j; q_1, q_2)}$$

with

$$\mathsf{N}_{\lambda,\mu}(u, q_1, q_2) = \prod_{s\in\lambda}(1 - uq_2^{-a_\mu(s)-1} q_1^{\ell_\lambda(s)}) \prod_{s\in\mu}(1 - uq_2^{a_\lambda(s)} q_1^{-\ell_\mu(s)-1})$$

$$\mathsf{T}_\lambda(u; q_1, q_2) = u^{|\lambda|} q_1^{\frac{1}{2}(\|\lambda^t\| - |\lambda^t|)} q_2^{\frac{1}{2}(\|\lambda\| - |\lambda|)} = \prod_{(i,j)\in\lambda} u q_1^{i-1} q_2^{j-1},$$

and $\vec{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(N)})$, $|\vec{\lambda}| = \sum |\lambda^{(i)}|$, $|\lambda| = \sum \lambda_j$, $\|\lambda\| = \sum \lambda_j^2$.

q_1, q_2 2 Nekrasov parameters of Ω background

Nekrasov functions: remarks

$\int A \wedge F$ in 3d

Remarks:

$SU(N)$

- 5d theory, $a = \log u$, $\epsilon_{1,2} = \log q_{1,2}$, $-z \sim \frac{1}{g_{YM}^2}$;
- CS term: $S = S_{\rm YM} + k{\rm Tr}\int A \wedge F \wedge F$, $\underline{|k| \leq N}$

$$\mathcal{Z}_{\rm cl}^{N,k} = \exp\left(\log z \frac{\sum (\log u_i)^2}{-2\log q_1 \log q_2} + k\frac{\sum (\log u_i)^3}{-6\log q_1 \log q_2}\right) \tag{12}$$

- Vertical multiplet: $a_{ij} = \log u_i/u_j$

$a_i - a_j$ masses in vector multiplet

$$\mathcal{Z}_{\rm inst}^{N,k} = \sum_{\vec{\lambda}} \frac{z^{|\vec{\lambda}|}\prod_{i=1}^N \mathsf{T}_{\lambda^{(i)}}(u;q_1,q_2)^k}{\prod_{i,j=1}^N \mathsf{N}_{\lambda^{(i)},\lambda^{(j)}}(u_i/u_j;q_1,q_2)} \tag{13}$$

- Solution for $q_1 q_2 = 1$ (Kiev formula), arbitrary q_1 and q_2 (refined case or two parameters q and $p \sim e^{\hbar}$): *quantization* of cluster integrable system!

Prepotential and Nekrasov function

- SYM 4d/5d: holomorphic prepotential $T_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}$ (action $\mathrm{Im} \int d^4\theta \mathcal{F}(\Phi)$).
- SW theory: Σ of genus=rank, with differentials or dS:

$$\delta dS \simeq \text{holomorphic} \tag{14}$$

  or by *an integrable system*.
- Lattice of charges $\Leftrightarrow$ $H_1(\Sigma)$ with symplectic $\langle , \rangle$,

$$a_i = \oint_{A_i} dS, \qquad a_i^D = \oint_{B_i} dS = \frac{\partial \mathcal{F}}{\partial a_i} \tag{15}$$

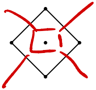
  consistent by symmetricity of $\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} = T_{ij}(a)$.

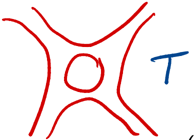
Prepotential from Nekrasov function

$$\mathcal{F}(a) = \lim_{\epsilon_{1,2} \to 0} \epsilon_1 \epsilon_2 Z(a; \epsilon_{1,2}) \tag{16}$$

obtained by instanton calculus (or dual – here q-deformed – CFT).

SW theory: 5d

Our "initial" curve 



$$\lambda + \frac{1}{\lambda} + \mu + \frac{z}{\mu} = u \qquad (17)$$

at $z \to 0$ for $u = 2\cosh a$, $\lambda = 2\cosh\zeta$

$a =$ W-boson mass

$$\mu = 4\sinh\frac{\zeta - a}{2}\sinh\frac{\zeta + a}{2} \qquad (18)$$

The period "matrix" $=$ complexified coupling $T \sim \frac{\vartheta}{2\pi} + i\frac{4\pi^2}{g^2_{\rm YM}}$

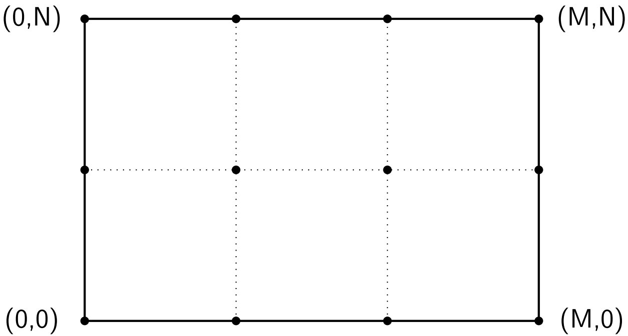
4d: $T \sim \log a$

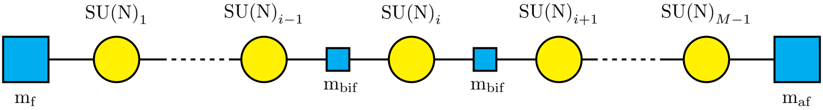
$$T \sim \int_{-a}^{a} d\log\sinh\frac{\zeta - a}{2} \sim \underline{\log\sinh a} = \log\prod(a+n) \qquad (19)$$

$a \to aR$

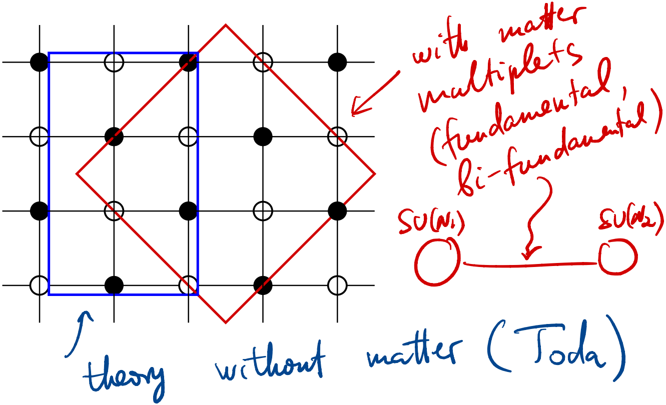
collects contributions from 5d KK modes from $\mathbb{R}^4 \times \mathbb{S}^1_R$.

Quiver gauge theories and spin chains





Poisson quivers and GK graphs



- Toda: $2 \times N$ fundamental domain of square lattice;
- XXZ-type spin chain: $N \times M$ 'fence-net' domain of the same square lattice.

Perspectives

- Cluster integrable system for other $D, C, B, \ldots$ - series;

- Solutions: beyond known cases, non-Lagrangian SUSY gauge theories (S-duality class), no Nekrasov functions (topological strings?);

- Dualities - "deep study" of GK systems: mutations of dual surface/spectral curve and Gaiotto transform ...

- Algebraic aspects: represenations of double-loop~~m~~ algebras, tetrahedron equations etc