GR lecture 9

Solar system effects of GR; Schwarzschild black holes

I. CARROLL'S BOOK: SECTIONS 5.4-5.7

II. PRECESSION OF MERCURY

The gravitational field of the Sun, outside the Sun itself, is well-described by the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ , \tag{1}$$

which can be derived from the vacuum Einstein equations via a 3+1d version of the calculation we performed in the last section. Since the Sun is much larger than its Schwarzschild radius, we are always in the limit $GM/r \ll 1$. Let us consider the shapes of orbits, i.e. geodesics, in the metric (1). By spherical symmetry, an orbit will always remain in the same "plane", which we can choose as $\theta = \pi/2$. From the translation symmetries in t and ϕ, we get conservation laws for energy and momentum. It's convenient to talk about energy and momentum per unit mass of the moving particle, i.e. of the planet. These read:

$$E = -u_t \ ; \quad L = u_\phi = g_{\phi\phi} u^\phi = r^2 \frac{d\phi}{d\tau} \ , \tag{2}$$

where τ is the planet's proper time, and $u^\mu = dx^\mu/d\tau$ is the 4-velocity. The constraint that u^μ is a unit vector reads:

$$\begin{aligned}
-1 = u_\mu u^\mu &= g^{tt}(u_t)^2 + g_{rr}(u^r)^2 + g^{\phi\phi}(u_\phi)^2 \\
&= -\frac{E^2}{1 - 2GM/r} + \frac{1}{1 - 2GM/r}\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2} \\
&= -\frac{E^2}{1 - 2GM/r} + \frac{L^2}{r^4(1 - 2GM/r)}\left(\frac{dr}{d\phi}\right)^2 + \frac{L^2}{r^2} \ ,
\end{aligned} \tag{3}$$

where in the last line we used $d\phi/d\tau = L^2/r^2$. At this point, it's very convenient to switch variables from r to $u \equiv 1/r$:

$$-1 = -\frac{E^2}{1 - 2GMu} + \frac{L^2}{1 - 2GMu}\left(\frac{du}{d\phi}\right)^2 + L^2 u^2 \ , \tag{4}$$

Rearranging the terms and introducing a factor of 1/2 just for fun, we get:

$$\frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + V(u) = \frac{E^2 - 1}{2L^2} \ ; \quad V(u) \equiv -\frac{GMu}{L^2} + \frac{u^2}{2} - GMu^3 \ . \tag{5}$$

Thus, the spatial shape $u(\phi)$ of the orbit has been reduced to a mechanical problem with "total energy" $(E^2-1)/(2L^2)$ and "potential" $V(u)$. It is worthwhile to compare our result so far to the one in the standard Kepler problem. There, we have:

$$\begin{aligned}
&L = r^2\frac{d\phi}{dt}\ ; \\
&\epsilon = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{r^2}{2}\left(\frac{d\phi}{dt}\right)^2 - \frac{GM}{r} = \frac{L^2}{2r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{L^2}{2r^2} - \frac{GM}{r} \\
&\quad = \frac{L^2}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{L^2u^2}{2} - GMu\ ,
\end{aligned} \tag{6}$$

where we used ϵ for energy this time, and again defined $u = 1/r$. Rearranging terms, this becomes:

$$\frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + V_{\text{non-rel}}(u) = \frac{\epsilon}{L^2}\ ; \quad V_{\text{non-rel}}(u) = -\frac{GMu}{L^2} + \frac{u^2}{2}\ . \tag{7}$$

Comparing (5) with (7), we find two differences. First, $(E^2-1)/2$ is replaced with ϵ. This is not surprising: recall that E is energy per unit mass; thus, in the non-relativistic limit, it will take the form $1+\epsilon$, where 1 is the rest energy, and $\epsilon \ll 1$ is the energy from non-relativistic physics. The second difference is the last term in (5), which, quite remarkably, is the full "relativistic correction" to the problem of the orbit's shape.

We are now ready to discuss the orbit's precession. The fact that Keplerian orbits are closed is encoded in the fact that $u(\phi)$ has a period of precisely 2π, regardless of the amplitude (i.e. of the orbit's eccentricity). This fact in turn is obvious from the fact that $V_{\text{non-rel}}(u)$ is a <u>harmonic</u> potential, with period:

$$\Delta\phi = \frac{2\pi}{\sqrt{d^2V_{\text{non-rel}}/du^2}} = 2\pi\ . \tag{8}$$

In the relativistic problem (5), the potential is no longer harmonic. The easiest case to handle is that of <u>slightly</u> eccentric orbits, in which u performs slight oscillations around the potential's minimum (where the minimum itself corresponds to the circular orbit). Such small oscillations can always be treated as harmonic, governed by the potential's second derivative at the minimum:

$$\frac{d^2V}{du^2} = 1 - 6GMu_0\ , \tag{9}$$

where u_0 is now the minimum of $V(u)$, given by $u_0 = GM/L^2$ in the non-relativistic limit. The period of $u(\phi)$ now becomes:

$$\Delta\phi = \frac{2\pi}{\sqrt{d^2V/du^2}} \approx 2\pi\left(1+3GMu_0\right) = 2\pi + \frac{6\pi GM}{Rc^2} \approx 2\pi + 24\pi^3\left(\frac{R}{cT}\right)^2\ , \tag{10}$$

where we restored the factors of c, denoted the radius of the circular orbit by $R = 1/u_0$, and used Kepler's relation $GM/R^3 = (2\pi/T)^2$ between the orbit's radius R and time period T. The orbit's angular precession per unit time therefore reads:

$$\frac{\Delta\phi - 2\pi}{T} = \frac{24\pi^3 R^2}{c^2 T^3} \ . \tag{11}$$

For the parameters of Mercury's orbit, this evaluates to $41''$/century. The correct GR value for Mercury's precession, which takes into account the orbit's eccentricity, is $43''$/century. Our simple analysis was not so bad!

III. DEFLECTION OF STARLIGHT BY THE SUN

Let's now consider the trajectories of lightrays in the Sun's gravitational field (1). These are somewhat analogous to the hyperbolic orbit or a very energetic object which merely gets slightly deflected by its "collision" with the Sun's field. Thus, the zeroth-order approximation to the trajectory is a straight line, going from $(r, \phi) = (\infty, 0)$ to $(r, \phi) = (\infty, \pi)$, with some minimal distance of approach b from the Sun, which we call the "impact parameter". Light travels along null geodesics, for which we do not have proper time, but we do have an affine parameter λ. Furthermore, we can scale λ so that $p^\mu = dx^\mu/d\lambda$ is the light's 4-momentum. The conserved quantities then look identical to (2):

$$E = -p_t \ ; \quad L = p_\phi = g_{\phi\phi} p^\phi = r^2 \frac{d\phi}{d\lambda} \ . \tag{12}$$

In place of (3), we now have the constraint that p^μ is lightlike:

$$\begin{aligned} 0 = p_\mu p^\mu &= g^{tt}(p_t)^2 + g_{rr}(p^r)^2 + g^{\phi\phi}(p_\phi)^2 \\ &= -\frac{E^2}{1 - 2GM/r} + \frac{L^2}{r^4(1 - 2GM/r)}\left(\frac{dr}{d\phi}\right)^2 + \frac{L^2}{r^2} \ . \end{aligned} \tag{13}$$

Redefining again $u \equiv 1/r$, we arrive at (4) with 0 in place of -1 on the LHS:

$$0 = -\frac{E^2}{1 - 2GMu} + \frac{L^2}{1 - 2GMu}\left(\frac{du}{d\phi}\right)^2 + L^2 u^2 \ , \tag{14}$$

which becomes:

$$\frac{du}{d\phi} = \sqrt{\frac{E^2}{L^2} - u^2(1 - 2GMu)} \ , \tag{15}$$

which can be integrated as:

$$d\phi = \frac{du}{\sqrt{E^2/L^2 - u^2(1-2GMu)}} \ . \tag{16}$$

The total change $\Delta\phi_{\text{total}}$ in the course of the trajectory can be found by integrating (16) from $u = 0$, i.e. $r = \infty$, down to the closest approach radius $b = 1/u_{\text{max}}$ and back again. u_{max} is a crucial input in this integral; it can be found by solving the condition $du/d\phi = 0$. If we neglect the Sun's gravity altogether, i.e. we through away the GM terms, then we get $1/u_{\text{max}} = L/E$, which makes sense: the light's energy E is the same as its linear momentum, and $b = 1/u_{\text{max}}$ is the angular momentum's "arm". At this zeroth-order approximation, the total change in ϕ reads:

$$\Delta\phi_{\text{total}} = 2\int_0^{E/L} \frac{du}{\sqrt{E^2/L^2 - u^2}} = 2\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi \ , \tag{17}$$

as expected. We are interested in the correction to this angle at first order in GM. First, we solve for $u_{\text{max}} = E/L + \epsilon$ to this order:

$$\begin{aligned} 0 = \left(\frac{du}{d\phi}\right)^2\Bigg|_{u=u_{\text{max}}} &= \frac{E^2}{L^2} - u^2 + 2GMu^3 \approx \frac{E^2}{L^2} - \left(\frac{E}{L}+\epsilon\right)^2 + 2GM\left(\frac{E}{L}\right)^3 \\ &= \frac{2E}{L}\left(\frac{GME^2}{L^2} - \epsilon\right) \ . \end{aligned} \tag{18}$$

From which we read off:

$$u_{\text{max}} \approx \frac{E}{L} + \epsilon = \frac{E}{L}\left(1 + \frac{GME}{L}\right) \implies \frac{E}{L} = u_{\text{max}}\left(1 - GMu_{\text{max}}\right) \ . \tag{19}$$

We can now plug this into (16) to get:

$$\frac{d\phi}{du} = \frac{1}{\sqrt{u_{\text{max}}^2(1-2GMu_{\text{max}}) - u^2(1-2GMu)}} \approx \frac{1}{\sqrt{u_{\text{max}}^2 - u^2}} + \frac{GM(u_{\text{max}}^3 - u^3)}{(u_{\text{max}}^2 - u^2)^{3/2}} \ . \tag{20}$$

Crucially, the du integration is now still from 0 to $u_{\text{max}} = 1/b$ and back again. Changing variables as $x = u/u_{\text{max}} = bu$, we thus obtain:

$$\Delta\phi_{\text{total}} = \pi + \frac{2GM}{b}\int_0^1 \frac{(1-x^3)}{(1-x^2)^{3/2}}\, dx \ . \tag{21}$$

One piece of the integral can be evaluated as:

$$\begin{aligned} \int \frac{dx}{(1-x^2)^{3/2}} &= \int dx\left(\frac{1}{\sqrt{1-x^2}} + \frac{x^2}{(1-x^2)^{3/2}}\right) \\ &= \int dx\left(\frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{\sqrt{1-x^2}}\right)'\right) = \frac{x}{\sqrt{1-x^2}} \ . \end{aligned} \tag{22}$$

For the second piece, we change variables to $y = 1 - x^2$ to get:

$$\begin{aligned}\int \frac{x^3 dx}{(1-x^2)^{3/2}} &= \frac{1}{2}\int \frac{(y-1)dy}{y^{3/2}} = \frac{1}{2}\int (y^{-1/2} - y^{-3/2})dy \\ &= y^{1/2} + y^{-1/2} = \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}} = \frac{2-x^2}{\sqrt{1-x^2}} \ .\end{aligned} \tag{23}$$

Putting the pieces together, we get:

$$\int \frac{(1-x^3)}{(1-x^2)^{3/2}}\, dx = \frac{x^2+x-2}{\sqrt{1-x^2}} = \frac{(x-1)(x+2)}{\sqrt{1-x^2}} = -(x+2)\sqrt{\frac{1-x}{1+x}} \ . \tag{24}$$

From which we read off:

$$\Delta\phi_{\text{total}} = \pi - \frac{2GM}{b}\ (x+2)\sqrt{\frac{1-x}{1+x}}\Bigg|_0^1 = \pi + \frac{4GM}{b} \ . \tag{25}$$

We conclude that the deflection angle at first order for a lightray with impact parameter b is $4GM/b$.

IV. STATIONARY OBSERVERS IN THE SCHWARZSCHILD METRIC

In this section, we begin to take the Schwarzschild metric seriously outside the limit $r \gg GM$, i.e. our discussion begins to include Schwarzschild black holes. For now, though, we stay outside the horizon, i.e. we take $r > 2GM$. Consider a so-called “stationary” observer, which stays at the same spatial point (r, θ, ϕ). The 4-velocity u^μ of such an observer has only a time component $u^t = dt/d\tau$, where τ is proper time. We can find the value of u^t from the normalization condition:

$$-1 = u_\mu u^\mu = g_{tt}(u^t)^2 \implies \frac{dt}{d\tau} = u^t = \frac{1}{\sqrt{-g_{tt}}} = \frac{1}{\sqrt{1-2GM/r}} \ . \tag{26}$$

Let’s imagine that this observer emits light signals at some constant intervals $\Delta\tau$ of its proper time, or perhaps a light wave with frequency $1/\Delta\tau$ in the observer’s frame. The signals – or the peaks of the wave – will then propagate along null geodesics of the Schwarzchild metric. Without even solving the geodesic equation, we can use the time invariance of Schwarzchild to make a simple prediction: since the propagation is invariant under shifting $t \to t + \Delta t$, any two signals that start out separated by a time delay Δt will always remain separated by the same time delay! The crucial subtlety is that the time delay is constant when measured via coordinate time dt, as opposed to proper time $d\tau = \sqrt{-g_{tt}}dt$. Thus, if an observer at radius r emits signals with period $\Delta\tau$, an observer at radius $R > r$ will receive them with a larger period:

$$\Delta\tau' = \sqrt{-g_{tt}(R)}\Delta t = \sqrt{\frac{g_{tt}(R)}{g_{tt}(r)}}\,\Delta\tau = \sqrt{\frac{1-2GM/R}{1-2GM/r}}\,\Delta\tau\ . \tag{27}$$

This is one way to derive gravitational redshift, which we discussed already in Lecture 4-2, and can be observed in e.g. the solar absorption spectrum as viewed from Earth. In a more extreme case, we can consider r that is very close to the event horizon $r = 2GM$. Then we find that the redshift becomes infinite, $\Delta\tau' \to \infty$. A similar result is true for an observer that isn't stationary, but is falling into the horizon: to an external observer, her signals will appear more and more stretched in time, and in particular the external observer will never quite see the infalling one cross the horizon.

Another key insight about a stationary observer at $r = \text{const}$ is that such an observer is accelerated: it must use e.g. some rocket engines to avoid falling (note that this is true already for Newtonian gravity, and has nothing to do with black holes). We can find the 4-acceleration as:

$$\alpha^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}u^\nu u^\rho = 0 + \Gamma^\mu_{tt}(u^t)^2 = \frac{1}{1-2GM/r}\,\Gamma^\mu_{tt}\ , \tag{28}$$

where we used $du^\mu/d\tau = 0$, since u^μ is constant along the stationary trajectory. The only nonzero component of Γ^μ_{tt} in the Schwarzschild metric is:

$$\Gamma^r_{tt} = -\frac{1}{2}g^{rr}\partial_r g_{tt} = \frac{1}{2}\left(1-\frac{2GM}{r}\right)\partial_r\left(1-\frac{2GM}{r}\right) = \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)\ . \tag{29}$$

Thus, the nonzero component of the 4-acceleration is:

$$\alpha^r = \frac{GM}{r^2}\ , \tag{30}$$

which precisely agrees with the Newtonian acceleration! However, we must be careful. The observer's proper acceleration – the acceleration as actually measured in her reference frame – is given by the length $\sqrt{\alpha_\mu\alpha^\mu}$ of the 4-acceleration:

$$a = \sqrt{\alpha_\mu\alpha^\mu} = \sqrt{g_{rr}}\alpha^r = \frac{GM/r^2}{\sqrt{1-2GM/r}}\ . \tag{31}$$

This acceleration behaves as the Newtonian GM/r^2 only at $r \gg GM$. As we approach the horizon $r = 2GM$, the acceleration diverges: at the horizon, it takes infinite effort to avoid falling in. If we consider an observer very close to the horizon, i.e. $r = 2GM + \epsilon$, the acceleration (31) simplifies into:

$$a \approx \frac{1/(2GM)}{\sqrt{1 - 2GM/(2GM + \epsilon)}} \approx \frac{1/(2GM)}{\sqrt{1 - (1 - \epsilon/(2GM))}} = \frac{1}{\sqrt{2GM\epsilon}} \ . \tag{32}$$

V. SCHWARZSCHILD-LIKE COORDINATES FOR FLAT SPACETIME

We already know some commonalities between a Rindler horizon in flat spacetime and the Schwarzschild horizon. Let's define Rindler coordinates (ρ, t), where t will play a similar role to the Schwarzschild t:

$$ds^2 = -dT^2 + X^2 = d\rho^2 - \rho^2 dt \ ; \quad (T, X) = (\rho \sinh t, \rho \cosh t) \ . \tag{33}$$

Both in Rindler coordinates and in the $r > 2GM$ region of the Schwarzschild metric, stationary observers are accelerated, and the acceleration diverges as we approach the horizon. Second, an observer or light signal approaching the horizon does not reach it until $t = \infty$ in both cases (see Lecture 3-2 for the Rindler case and see Carroll for the Schwarzschild case). It is possible to intensify the similarity if we switch coordinates in Rindler space from ρ to $r = \rho^2$. This brings the metric to the form:

$$dr = 2\rho d\rho \ \Rightarrow \ d\rho = \frac{dr}{2\sqrt{r}} \ \Rightarrow \ ds^2 = \frac{dr^2}{4r} - r dt^2 \ . \tag{34}$$

The Rindler horizon $\rho = 0$ is now $r = 0$, and its similarities with the Schwarzschild horizon intensify. In both cases, at the horizon, g_{tt} diverges and g_{rr} vanishes. In both cases, "outside the horizon", i.e. $r > 2GM$ in Schwarzschild and $r > 0$ in (34), r is spacelike and t is timelike; "inside the horizon", i.e. at $r < 2GM$ and $r < 0$ respectively, those roles flip. Note that the $r < 0$ region was not visible in the original Rindler coordinates, where ρ^2 was always positive. While the usual Rindler coordinates are associated with the "right-hand quarter" of Minkowski space, the $r < 0$ can be associated with the "top quarter", if we identify:

$$(T, X) = \begin{cases} (\sqrt{r} \sinh t, \sqrt{r} \cosh t) & r > 0 \\ (\sqrt{-r} \cosh t, \sqrt{-r} \sinh t) & r < 0 \end{cases} \ . \tag{35}$$

Note that, while both the $r > 0$ and $r < 0$ patches correspond to legitimate regions of Minkowski space, the patching between them is not smooth. For example, a line of changing r at constant t is a spacelike straight line at $r > 0$ and a timelike straight line at $r < 0$, with a "90-degree kink" in between. The same statements turn out to be true for the $r > 2GM$ and $r < 2GM$ patches of the Schwarzschild spacetime.

VI. THE NEAR-HORIZON LIMIT VS. THE RINDLER METRIC; GUESSING THE KRUSKAL COORDINATES

Let's continue to analyze the near-horizon limit $r = 2GM + \epsilon$ of Schwarzschild. The spacetime metric in this limit reads:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{2GM+\epsilon}\right) dt^2 + \frac{d\epsilon^2}{1 - 2GM/(2GM+\epsilon)} + (2GM+\epsilon)^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\ &= -\frac{\epsilon}{2GM}\, dt^2 + \frac{2GM}{\epsilon}\, d\epsilon^2 + (2GM)^2(d\theta^2 + \sin^2\theta\, d\phi^2)\ . \end{aligned} \tag{36}$$

In the limit, the angular coordinates form a sphere of the approximately constant radius $2GM$. Let us focus on the (t, r) plane, i.e. the (t, ϵ) plane, where more interesting things happen. In particular, up to some rescalings of the coordinates, we recognize it as the Rindler-like metric (34):

$$ds^2 = \frac{d\tilde{\epsilon}^2}{4\tilde{\epsilon}} - \tilde{\epsilon}\, d\tilde{t}^2\ ; \quad \tilde{\epsilon} = 8GM\epsilon\ ; \quad \tilde{t} = \frac{t}{4GM}\ . \tag{37}$$

We conclude that the near-horizon limit of Schwarzschild looks like the Rindler wedge of flat spacetime, with $\tilde{t} = t/(4GM)$ acting as the boost angle and $\sqrt{\tilde{\epsilon}} = \sqrt{8GM(r - 2GM)}$ acting as the radius. In particular, near the horizon, the time translation symmetry of full Schwarzschild looks like a boost symmetry!

With this lesson guiding our intuition, we can now extend the full Schwarzschild spacetime beyond the horizon, using coordinates that remain regular on it. For Rindler coordinates, this is accomplished by passing to the ordinary inertial coordinates (T, X), which see all of Minkowski space, and are regular on the horizons $X = \pm T$. We will similarly try to replace (t, r) with global coordinates (T, R) subject to the flat metric $-dT^2 + dR^2$, for which $t/(4GM)$ behaves as a boost angle, even outside the near-horizon limit. To take into account the changes in the metric as we go far from the horizon, we will allow for r-dependent warping both in the coordinates and in the metric. Thus, we construct our coordinate transformation as:

$$T = f(r) \sinh \frac{t}{4GM}\ ; \quad R = f(r) \cosh \frac{t}{4GM}\ , \tag{38}$$

and we'd like the metric to become:

$$ds^2 = g(r)(-dT^2 + dR^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\ , \tag{39}$$

where r is now no longer one of the coordinates, but is a function of T and R which we can defined implicitly from (38) as $R^2 - T^2 = f(r)^2$. It remains to fix the warping functions $f(r)$ and $g(r)$. To do that, we plug the coordinate transformation (38) into the metric (39):

$$ds^2 = g(r)\left(-\frac{f(r)^2}{(4GM)^2}\,dt^2 + f'(r)^2 dr^2\right) + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\ , \tag{40}$$

and compare to the Schwarzschild metric. From the ratio of g_{tt} and g_{rr}, we can find an equation involving just $f(r)$:

$$\left(1 - \frac{2GM}{r}\right)^2 = -\frac{g_{tt}}{g_{rr}} = \left(\frac{f(r)}{4GMf'(r)}\right)^2\ . \tag{41}$$

This can be easily solved via:

$$\begin{aligned}
&(\ln f(r))' = \frac{f(r)'}{f(r)} = \frac{1}{4GM}\frac{1}{1-2GM/r} = \frac{1}{4GM}\frac{r}{r-2GM} = \frac{1}{4GM} + \frac{1}{2(r-2GM)} \\
&\implies \quad \ln f(r) = \text{const} + \frac{r}{4GM} + \frac{1}{2}\ln(r-2GM) \\
&\implies \quad f(r) = \text{const}\times e^{r/(4GM)}\sqrt{r-2GM}\ .
\end{aligned} \tag{42}$$

The constant prefactor can be swallowed into $g(r)$, so it's arbitrary. The conventional choice is $1/\sqrt{2GM}$, so that:

$$f(r) = e^{r/(4GM)}\sqrt{\frac{r}{2GM} - 1}\ . \tag{43}$$

We can now find $g(r)$ via:

$$\begin{aligned}
&1 - \frac{2GM}{r} = -g_{tt} = g(r)\left(\frac{f(r)}{4GM}\right)^2 = \frac{g(r)}{(4GM)^2}\,e^{r/(2GM)}\left(\frac{r}{2GM} - 1\right) \\
&\implies \quad g(r) = \frac{32G^3M^3}{r}\,e^{-r/(2GM)}\ .
\end{aligned} \tag{44}$$

To sum up, the transformation into Kruskal coordinates and the metric in them read:

$$(T,R) = e^{r/(4GM)}\sqrt{\frac{r}{2GM} - 1}\left(\sinh\frac{t}{4GM}\ ,\ \cosh\frac{t}{4GM}\right)\ ; \tag{45}$$

$$R^2 - T^2 = e^{r/(2GM)}\left(\frac{r}{2GM} - 1\right)\ ; \tag{46}$$

$$ds^2 = \frac{32G^3M^3}{r}\,e^{-r/(2GM)}(-dT^2 + dR^2) + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\ . \tag{47}$$

EXERCISES

Exercise 1. *Calculate the distance along a line of changing r with (t,θ,ϕ) fixed between a point $r = r_0 > 2GM$ and the horizon $r = 2GM$. Take the limit $r_0 \to 2GM$, and compare* *with the near-horizon Rindler radius $\sqrt{\tilde{\epsilon}}$ from section VI. Hint: The following integral may be useful:*

$$\int dx \sqrt{\frac{x}{x-1}} = \sqrt{x(x-1)} + \ln\left(\sqrt{x} + \sqrt{x-1}\right) \ . \tag{48}$$

Exercise 2. *Consider a photon falling directly into a Schwarzschild black hole along a radial lightray. When the photon is at $r \gg M$, its frequency is ω. How many oscillations does the photon make on its way from $r = 4GM$ to $r = 2GM$, and from $r = 2GM$ to $r = 0$? Hint: the oscillation phase is an affine parameter along the lightray.*

Exercise 3. *Consider a massive particle on a circular orbit around a Schwarzschild black hole. Find the orbit's period as a function of its radius r, both in terms of coordinate time t and in terms of proper time τ. Compare with the Newtonian answer. Hint: use the vanishing of the 4-acceleration's r component.*

What is the smallest radius at which a circular orbit is possible? Hint: remember that the worldline must be timelike.