GR lecture 8

Einstein-Hilbert action; Conical singularities; The Schwarzchild solution

I. CARROLL'S BOOK: SECTIONS 4.3, 5.1-5.4

II. $T^{\mu\nu}$ AS NOETHER CHARGE VS. $T^{\mu\nu}$ AS VARIATION WITH RESPECT TO THE METRIC

When deriving the Einstein equations from an action principle, we found ourselves identifying the stress-energy tensor as:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}L)}{\partial g_{\mu\nu}} \ , \tag{1}$$

or, equivalently:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}L)}{\partial g^{\mu\nu}} = Lg_{\mu\nu} - 2\frac{\partial L}{\partial g^{\mu\nu}} \ , \tag{2}$$

where L is the matter Lagrangian. As we've seen for the electromagnetic field, this definition actually doesn't directly coincide with the one derived by considering $T^{\mu\nu}$ as the 4-current of 4-momentum, which is in turn the Noether charge associated with translations. Nevertheless, the claim is that (1) defines something very much like the Noether current for translations, such that e.g. the integrated total 4-momentum calculated from both definitions is the same (at least in flat spacetime, where such an integrated quantity makes sense). Once we believe that (1) defines something like the 4-current of 4-momentum, then it is clearly the superior definition, since it's automatically symmetric and gauge-invariant. However, why should we believe that? In this section, we attempt to answer that question.

Recall that a symmetric matrix such as $T_{\mu\nu}$ is fully determined by its products $T_{\mu\nu}u^\mu u^\nu$ with arbitrary timelike unit vectors. Thus, to understand $T_{\mu\nu}$, it's enough to consider T_{tt} in arbitrary Lorentz frames. From the Noether point of view, T_{tt} should be the energy density. To make this concrete, let's consider the action in flat spacetime with initial conditions somewhere and final conditions at $t = t_f$. Then the action's variation upon putting the same final conditions but at a slightly later time $t_f + \delta t$ reads:

$$\delta S = -\delta t \int_{t=t_f} d^3x\, T_{tt} \ . \tag{3}$$

Now, let us notice that there's another way to change the time duration of the spacetime region associated with S: instead of changing the final time coordinate t_f, we can just stretch the metric near $t = t_f$! In particular, to obtain the same shift δt_f of proper time, we can stretch a short time interval $(t_f - \Delta t, t_f)$ by a factor of $1 + \delta t/\Delta t$, where we take Δt small but much longer than δt. Thus, we must stretch $\sqrt{-g_{tt}}$ by a factor of $1 + \delta t/\Delta t$, which is equivalent to:

$$\delta g_{tt} = -2\sqrt{-g_{tt}}\,\delta(\sqrt{-g_{tt}}) = -2\frac{\delta t}{\Delta t} \ , \tag{4}$$

where we used the flat value $g_{tt} = -1$ before the variation. The resulting variation of the action $S = \int d^4x\sqrt{-g}\,L$ reads:

$$\delta S = \Delta t \int d^3x\, \frac{\delta(\sqrt{-g}L)}{\delta g_{tt}}\,\delta g_{tt} = -2\delta t \int d^3x\, \frac{\delta(\sqrt{-g}L)}{\delta g_{tt}} \ . \tag{5}$$

Comparing with (3), we conclude that it indeed makes sense to identify (1) as the stress-energy tensor.

III. GAUGE INVARIANCE VS. CONSERVATION

Another comment is that (1) has a close analog in electromagnetism. Indeed, the electric 4-current of a charged field can be defined by varying the action with respect to the electromagnetic potential:

$$j^\mu = \frac{\delta L}{\delta A_\mu} \ . \tag{6}$$

Charge conservation can then be beautifully derived as a consequence of gauge invariance. We simply consider a variation of the particular form $\delta A_\mu = \partial_\mu\theta$, which is a gauge transformation, and must leave the action invariant:

$$0 = \delta S = \int d^4x\, \frac{\delta L}{\delta A_\mu}\delta A_\mu = \int d^4x\, j^\mu \partial_\mu\theta = -\int d^4x\,\theta\,\partial_\mu j^\mu \ , \tag{7}$$

where we integrated by parts and disposed of the boundary term by choosing $\theta(x)$ that vanishes on the boundary. Since (7) must be true for otherwise arbitrary $\theta(x)$, we conclude that charge is locally conserved: $\partial_\mu j^\mu = 0$.

To construct the analogous argument in gravity, recall that an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ acts on various fields through the Lie derivative $\mathcal{L}_\xi$. Indeed, in adapted coordinates for which $\xi^\mu = (\epsilon, 0, 0, 0)$, the coordinate transformation acts simply as the partial derivative $\epsilon(\partial/\partial x^0)$; the Lie derivative $\mathcal{L}_\xi$ is the coordinate-independent formulation of the same geometric concept. As we've seen in Lecture 5, the Lie derivative of the metric can be written in terms of covariant derivatives as:

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 2\nabla_{(\mu}\xi_{\nu)} \ . \tag{8}$$

Since the action should be invariant under this coordinate transformation, we conclude:

$$0 = \delta S = \int d^4x \, \frac{\delta(\sqrt{-g}L)}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = \int d^4x \sqrt{-g}\, T^{\mu\nu} \nabla_\mu \xi_\nu = -\int d^4x \, \xi_\nu \nabla_\mu T^{\mu\nu} \ . \tag{9}$$

Again, for this to be true for arbitrary infinitesimal $\xi^\mu(x)$, we must have the conservation law $\nabla_\mu T^{\mu\nu} = 0$.

IV. THE CONICAL SINGULARITY SOLUTION IN 2+1D GR

As a warmup towards the Schwarzschild solution in 3+1d, let's consider time-independent, rotationally symmetric, non-rotating vacuum solutions in 2+1d. In other words, let's find the gravitational field of a stationary point mass in 2+1d GR. We begin by writing the following ansatz for the metric:

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\phi^2 \ . \tag{10}$$

This is the most general metric that satisfies the following assumptions:

- Rotationally invariant, i.e. invariant under $\phi \to \phi + \text{const}$. This implies $\partial_\phi g_{\mu\nu}$.
- Treats the clockwise and anticlockwise directions equally, i.e. invariant under $\phi \to -\phi$. This implies $g_{t\phi} = g_{r\phi} = 0$.
- Static, i.e. $\partial_t g_{\mu\nu} = 0$ and $g_{tr} = g_{t\phi} = 0$.

As we will see in an exercise, the last assumption isn't actually necessary. Note that we don't need to consider a more general $g_{\phi\phi}(r)$ in (10), since we can always use the tangential length element $\sqrt{g_{\phi\phi}}\, d\phi \equiv r d\phi$ as a definition of the r coordinate. The nonzero elements of

$g_{\mu\nu}$, $g^{\mu\nu}$ and $\partial_\mu g_{\nu\rho}$ read:

$$g_{tt}\ ; \quad g_{rr}\ ; \quad g_{\phi\phi} = r^2\ ; \tag{11}$$

$$g^{tt} = \frac{1}{g_{tt}}\ ; \quad g^{rr} = \frac{1}{g_{rr}}\ ; \quad g^{\phi\phi} = \frac{1}{r^2}\ ; \tag{12}$$

$$\partial_r g_{tt} \equiv g'_{tt}\ ; \quad \partial_r g_{rr} \equiv g'_{rr}\ ; \quad \partial_r g_{\phi\phi} = 2r\ . \tag{13}$$

The Christoffel symbols then read:

$$\Gamma^r_{tt} = -\frac{g'_{tt}}{2g_{rr}}\ ; \quad \Gamma^t_{tr} = \frac{g'_{tt}}{2g_{tt}}\ ; \quad \Gamma^r_{rr} = \frac{g'_{rr}}{2g_{rr}}\ ; \quad \Gamma^r_{\phi\phi} = -\frac{r}{g_{rr}}\ ; \quad \Gamma^\phi_{\phi r} = \frac{1}{r}\ , \tag{14}$$

where all other components are either related to the above by symmetries (e.g. $\Gamma^t_{rt} = \Gamma^t_{tr}$) or vanishing. We see that a lot of Christoffel components have a form similar to $g'_{rr}/(2g_{rr})$. This is not a coincidence: the Christoffel is really about curvature <u>angles</u>, which are related not to the absolute size of the metric, but to its <u>relative</u> rate of change; finally, the factor of 1/2 in the Christoffel's definition tells us that it's directly sensitive not to the metric – which gives lengths squared – but to lengths themselves. Thus, it's a better idea to reparameterize the original metric (10) as:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\phi^2\ , \tag{15}$$

which gives us:

$$\Gamma^r_{tt} = \alpha' e^{2(\alpha-\beta)}\ ; \quad \Gamma^t_{tr} = \alpha'\ ; \quad \Gamma^r_{rr} = \beta'\ ; \quad \Gamma^r_{\phi\phi} = -re^{-2\beta}\ ; \quad \Gamma^\phi_{\phi r} = \frac{1}{r}\ . \tag{16}$$

We can now directly compute the Ricci tensor as:

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\mu \Gamma^\rho_{\nu\rho} + \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho}\ , \tag{17}$$

which yields:

$$\begin{aligned} &R_{tt} = e^{2(\alpha-\beta)}\left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{\alpha'}{r}\right)\ ; \quad R_{rr} = -\alpha'' - \alpha'^2 + \alpha'\beta' + \frac{\beta'}{r}\ ; \\ &R_{\phi\phi} = re^{-2\beta}(\beta' - \alpha')\ . \end{aligned} \tag{18}$$

Let us now apply the vacuum Einstein equations $R_{\mu\nu} = 0$. From $e^{2(\beta-\alpha)}R_{tt} + R_{rr}$, we find $\alpha' + \beta' = 0$. On the other hand, from $R_{\phi\phi}$, we find $\beta' - \alpha' = 0$. It follows that α' and β' both vanish, i.e. that α and β are both constants! We can get rid of these constants by rescaling the coordinates as:

$$t \to e^\alpha t\ ; \quad r \to e^\beta r\ ; \quad \phi \to e^{-\beta}\phi\ , \tag{19}$$

which brings the metric to the flat form:

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 \ . \tag{20}$$

Note that rescaling ϕ as in (19) may affect its 2π periodicity, which so far we've been taking for granted. As we'll now see, the non-trivial part of the geometry (20) is precisely encoded in this periodicity.

First, let's recall that the flat answer (20) should have been expected: we know that in 3d spacetime, $R_{\mu\nu} = 0$ implies that the entire Riemann curvature vanishes. However, now we must be careful. For a point mass, $T_{\mu\nu}$ and thus $R_{\mu\nu}$ vanishes everywhere except at $r = 0$. Thus, we may have some curvature that's concentrated, like a delta function, just at the origin. To see what this curvature should look like, let's "zoom in" on the point mass so it isn't look pointlike anymore. Recall the form of $T_{\mu\nu}$ for a mass density at rest, in locally inertial coordinates:

$$T_{\mu\nu} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \tag{21}$$

By the 3d Einstein equation, the Ricci tensor then takes the form:

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - T g_{\mu\nu}) = 8\pi G \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \ . \tag{22}$$

Thus, we expect a purely spatial 2d curvature $R_{xx} = R_{yy}$. Recall that in 2d, the Riemann tensor has just one independent component. The same curvature can be expressed equivalently as:

$$R_{xyxy} = R_{xx} = R_{yy} = \frac{1}{2} R \ . \tag{23}$$

Thus, returning to the pointlike mass case, we are dealing with a distributional curvature of the form:

$$R_{xyxy} = 8\pi G M \delta^2(\mathbf{x}) \ , \tag{24}$$

where M is the mass at the origin, and $\delta^2(\mathbf{x})$ is a spatial delta function that integrates to 1. What do we call a 2d flat manifold with distributional curvature at the origin? We call this a cone. Indeed, a 2d cone is constructed by simply "cutting out" some angle χ from a flat plane, and gluing the two sides of the cut together. The geometry throughout the cone is the same as that of the plane, i.e. flat, with the exception of the apex. To see that there is curvature at the apex, we recall our definition of the Riemann in terms of parallel transport along a closed loop. If we parallel-transport a vector around the apex of the cone, it ends up at angle χ to its original orientation. Taking χ to be small for simplicity and taking care with the signs (better to make a drawing for this purpose), we find that the rotation matrix upon traversing a counterclockwise loop is:

$$M_i{}^j = \begin{pmatrix} 1 & -\chi \\ \chi & 1 \end{pmatrix} . \tag{25}$$

Recalling that the Riemann tensor element $R^x{}_{yxy}$ is $M_y{}^x$ per unit area of a counterclockwise loop, we read off:

$$R_{xyxy} = \chi\delta^2(\mathbf{x}) . \tag{26}$$

Comparing with (24), we see that in 2+1d GR, the geometry around a (small) mass M is conical, with deficit angle $\chi = 8\pi GM$. Returning to the polar coordinates (20), we note that the deficit angle can be encoded by simply changing the period of ϕ from 2π to $2\pi - \chi$, without any change to the ds^2 formula.

V. PRECESSION OF MERCURY

The gravitational field of the Sun, outside the Sun itself, is well-described by the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) , \tag{27}$$

which can be derived from the vacuum Einstein equations via a 3+1d version of the calculation we performed in the last section. Since the Sun is much larger than its Schwarzschild radius, we are always in the limit $GM/r \ll 1$. Let us consider the shapes of orbits, i.e. geodesics, in the metric (27). By spherical symmetry, an orbit will always remain in the same "plane", which we can choose as $\theta = \pi/2$. From the translation symmetries in t and ϕ, we get conservation laws for energy and momentum. It's convenient to talk about energy and momentum per unit mass of the moving particle, i.e. of the planet. These read:

$$E = -u_t \ ; \quad L = u_\phi = g_{\phi\phi} u^\phi = r^2 \frac{d\phi}{d\tau} \ , \tag{28}$$

where τ is the planet's proper time, and $u^\mu = dx^\mu/d\tau$ is the 4-velocity. The constraint that u^μ is a unit vector reads:

$$\begin{aligned} -1 = u_\mu u^\mu &= g^{tt}(u_t)^2 + g_{rr}(u^r)^2 + g^{\phi\phi}(u_\phi)^2 \\ &= -\frac{E^2}{1-2GM/r} + \frac{1}{1-2GM/r}\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2} \\ &= -\frac{E^2}{1-2GM/r} + \frac{L^2}{r^4(1-2GM/r)}\left(\frac{dr}{d\phi}\right)^2 + \frac{L^2}{r^2} \ , \end{aligned} \tag{29}$$

where in the last line we used $d\phi/d\tau = L^2/r^2$. At this point, it's very convenient to switch variables from r to $u \equiv 1/r$:

$$-1 = -\frac{E^2}{1-2GMu} + \frac{L^2}{1-2GMu}\left(\frac{du}{d\phi}\right)^2 + L^2 u^2 \ , \tag{30}$$

Rearranging the terms and introducing a factor of 1/2 just for fun, we get:

$$\frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + V(u) = \frac{E^2-1}{2L^2} \ ; \quad V(u) \equiv -\frac{GMu}{L^2} + \frac{u^2}{2} - GMu^3 \ . \tag{31}$$

Thus, the spatial shape $u(\phi)$ of the orbit has been reduced to a mechanical problem with "total energy" $(E^2-1)/(2L^2)$ and "potential" $V(u)$. It is worthwhile to compare our result so far to the one in the standard Kepler problem. There, we have:

$$\begin{aligned} L &= r^2 \frac{d\phi}{dt} \ ; \\ \epsilon &= \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{r^2}{2}\left(\frac{d\phi}{dt}\right)^2 - \frac{GM}{r} = \frac{L^2}{2r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{L^2}{2r^2} - \frac{GM}{r} \\ &= \frac{L^2}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{L^2u^2}{2} - GMu \ , \end{aligned} \tag{32}$$

where we used ϵ for energy this time, and again defined $u = 1/r$. Rearranging terms, this becomes:

$$\frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + V_{\text{non-rel}}(u) = \frac{\epsilon}{L^2} \ ; \quad V_{\text{non-rel}}(u) = -\frac{GMu}{L^2} + \frac{u^2}{2} \ . \tag{33}$$

Comparing (31) with (33), we find two differences. First, $(E^2-1)/2$ is replaced with ϵ. This is not surprising: recall that E is energy per unit mass; thus, in the non-relativistic limit, it will take the form $1+\epsilon$, where 1 is the rest energy, and $\epsilon \ll 1$ is the energy from non-relativistic physics. The second difference is the last term in (31), which, quite remarkably, is the full "relativistic correction" to the problem of the orbit's shape.

We are now ready to discuss the orbit's precession. The fact that Keplerian orbits are closed is encoded in the fact that $u(\phi)$ has a period of precisely 2π, regardless of the amplitude (i.e. of the orbit's eccentricity). This fact in turn is obvious from the fact that $V_{\text{non-rel}}(u)$ is a harmonic potential, with period:

$$\Delta\phi = \frac{2\pi}{\sqrt{d^2V_{\text{non-rel}}/du^2}} = 2\pi \ . \tag{34}$$

In the relativistic problem (31), the potential is no longer harmonic. The easiest case to handle is that of slightly eccentric orbits, in which u performs slight oscillations around the potential's minimum (where the minimum itself corresponds to the circular orbit). Such small oscillations can always be treated as harmonic, governed by the potential's second derivative at the minimum:

$$\frac{d^2V}{du^2} = 1 - 6GMu_0 \ , \tag{35}$$

where u_0 is now the minimum of $V(u)$, given by $u_0 = GM/L^2$ in the non-relativistic limit. The period of $u(\phi)$ now becomes:

$$\Delta\phi = \frac{2\pi}{\sqrt{d^2V/du^2}} \approx 2\pi\left(1+3GMu_0\right) = 2\pi + \frac{6\pi GM}{Rc^2} \approx 2\pi + 24\pi^3\left(\frac{R}{cT}\right)^2 \ , \tag{36}$$

where we restored the factors of c, denoted the radius of the circular orbit by $R = 1/u_0$, and used Kepler's relation $GM/R^3 = (2\pi/T)^2$ between the orbit's radius R and time period T. The orbit's angular precession per unit time therefore reads:

$$\frac{\Delta\phi - 2\pi}{T} = \frac{24\pi^3R^2}{c^2T^3} \ . \tag{37}$$

For the parameters of Mercury's orbit, this evaluates to $41''$/century. The correct GR value for Mercury's precession, which takes into account the orbit's eccentricity, is $43''$/century. Our simple analysis was not so bad!

VI. DEFLECTION OF STARLIGHT BY THE SUN

Let's now consider the trajectories of lightrays in the Sun's gravitational field (27). These are somewhat analogous to the hyperbolic orbit or a very energetic object which merely gets slightly deflected by its "collision" with the Sun's field. Thus, the zeroth-order approximation to the trajectory is a straight line, going from $(r, \phi) = (\infty, 0)$ to $(r, \phi) = (\infty, \pi)$, with some minimal distance of approach b from the Sun, which we call the "impact parameter". Light travels along null geodesics, for which we do not have proper time, but we do have an affine parameter λ. Furthermore, we can scale λ so that $p^\mu = dx^\mu/d\lambda$ is the light's 4-momentum. The conserved quantities then look identical to (28):

$$E = -p_t \; ; \quad L = p_\phi = g_{\phi\phi} p^\phi = r^2 \frac{d\phi}{d\lambda} \; . \tag{38}$$

In place of (29), we now have the constraint that p^μ is lightlike:

$$\begin{aligned} 0 = p_\mu p^\mu &= g^{tt}(p_t)^2 + g_{rr}(p^r)^2 + g^{\phi\phi}(p_\phi)^2 \\ &= -\frac{E^2}{1 - 2GM/r} + \frac{L^2}{r^4(1 - 2GM/r)} \left(\frac{dr}{d\phi}\right)^2 + \frac{L^2}{r^2} \; . \end{aligned} \tag{39}$$

Redefining again $u \equiv 1/r$, we arrive at (30) with 0 in place of -1 on the LHS:

$$0 = -\frac{E^2}{1 - 2GMu} + \frac{L^2}{1 - 2GMu} \left(\frac{du}{d\phi}\right)^2 + L^2 u^2 \; , \tag{40}$$

which becomes:

$$\frac{du}{d\phi} = \sqrt{\frac{E^2}{L^2} - u^2(1 - 2GMu)} \; , \tag{41}$$

which can be integrated as:

$$d\phi = \frac{du}{\sqrt{E^2/L^2 - u^2(1 - 2GMu)}} \; . \tag{42}$$

The total change $\Delta\phi_{\text{total}}$ in the course of the trajectory can be found by integrating (42) from $u = 0$, i.e. $r = \infty$, down to the closest approach radius $b = 1/u_{\text{max}}$ and back again. u_{max} is a crucial input in this integral; it can be found by solving the condition $du/d\phi = 0$. If we neglect the Sun's gravity altogether, i.e. we through away the GM terms, then we get $1/u_{\text{max}} = L/E$, which makes sense: the light's energy E is the same as its linear momentum, and $b = 1/u_{\text{max}}$ is the angular momentum's "arm". At this zeroth-order approximation, the total change in ϕ reads:

$$\Delta\phi_{\text{total}} = 2 \int_0^{E/L} \frac{du}{\sqrt{E^2/L^2 - u^2}} = 2 \int_0^1 \frac{dx}{\sqrt{1 - x^2}} = \pi \; , \tag{43}$$

as expected. We are interested in the correction to this angle at first order in GM. First, we solve for $u_{\rm max} = E/L + \epsilon$ to this order:

$$\begin{aligned} 0 = \left(\frac{du}{d\phi}\right)^2\Bigg|_{u=u_{\rm max}} &= \frac{E^2}{L^2} - u^2 + 2GMu^3 \approx \frac{E^2}{L^2} - \left(\frac{E}{L}+\epsilon\right)^2 + 2GM\left(\frac{E}{L}\right)^3 \\ &= \frac{2E}{L}\left(\frac{GME^2}{L^2} - \epsilon\right) . \end{aligned} \tag{44}$$

From which we read off:

$$u_{\rm max} \approx \frac{E}{L} + \epsilon = \frac{E}{L}\left(1 + \frac{GME}{L}\right) \implies \frac{E}{L} = u_{\rm max}\left(1 - GMu_{\rm max}\right) . \tag{45}$$

We can now plug this into (42) to get:

$$\frac{d\phi}{du} = \frac{1}{\sqrt{u_{\rm max}^2(1-2GMu_{\rm max}) - u^2(1-2GMu)}} \approx \frac{1}{\sqrt{u_{\rm max}^2 - u^2}} + \frac{GM(u_{\rm max}^3 - u^3)}{(u_{\rm max}^2 - u^2)^{3/2}} . \tag{46}$$

Crucially, the du integration is now still from 0 to $u_{\rm max} = 1/b$ and back again. Changing variables as $x = u/u_{\rm max} = bu$, we thus obtain:

$$\Delta\phi_{\rm total} = \pi + \frac{2GM}{b}\int_0^1 \frac{(1-x^3)}{(1-x^2)^{3/2}}\, dx . \tag{47}$$

One piece of the integral can be evaluated as:

$$\begin{aligned} \int \frac{dx}{(1-x^2)^{3/2}} &= \int dx \left(\frac{1}{\sqrt{1-x^2}} + \frac{x^2}{(1-x^2)^{3/2}}\right) \\ &= \int dx \left(\frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{\sqrt{1-x^2}}\right)'\right) = \frac{x}{\sqrt{1-x^2}} . \end{aligned} \tag{48}$$

For the second piece, we change variables to $y = 1 - x^2$ to get:

$$\begin{aligned} \int \frac{x^3 dx}{(1-x^2)^{3/2}} &= \frac{1}{2}\int \frac{(y-1)dy}{y^{3/2}} = \frac{1}{2}\int (y^{-1/2} - y^{-3/2})dy \\ &= y^{1/2} + y^{-1/2} = \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}} = \frac{2-x^2}{\sqrt{1-x^2}} . \end{aligned} \tag{49}$$

Putting the pieces together, we get:

$$\int \frac{(1-x^3)}{(1-x^2)^{3/2}}\, dx = \frac{x^2+x-2}{\sqrt{1-x^2}} = \frac{(x-1)(x+2)}{\sqrt{1-x^2}} = -(x+2)\sqrt{\frac{1-x}{1+x}} . \tag{50}$$

From which we read off:

$$\Delta\phi_{\rm total} = \pi - \frac{2GM}{b}\,(x+2)\sqrt{\frac{1-x}{1+x}}\Bigg|_0^1 = \pi + \frac{4GM}{b} . \tag{51}$$

We conclude that the deflection angle at first order for a lightray with impact parameter b is $4GM/b$.

VII. STATIONARY OBSERVERS IN THE SCHWARZSCHILD METRIC

In this section, we begin to take the Schwarzschild metric seriously outside the limit $r \gg GM$, i.e. our discussion begins to include Schwarzschild black holes. For now, though, we stay outside the horizon, i.e. we take $r > 2GM$. Consider a so-called "stationary" observer, which stays at the same spatial point (r, θ, ϕ). The 4-velocity u^μ of such an observer has only a time component $u^t = dt/d\tau$, where τ is proper time. We can find the value of u^t from the normalization condition:

$$-1 = u_\mu u^\mu = g_{tt}(u^t)^2 \implies \frac{dt}{d\tau} = u^t = \frac{1}{\sqrt{-g_{tt}}} = \frac{1}{\sqrt{1 - 2GM/r}} \ . \tag{52}$$

Let's imagine that this observer emits light signals at some constant intervals $\Delta\tau$ of its proper time, or perhaps a light wave with frequency $1/\Delta\tau$ in the observer's frame. The signals – or the peaks of the wave – will then propagate along null geodesics of the Schwarzchild metric. Without even solving the geodesic equation, we can use the time invariance of Schwarzchild to make a simple prediction: since the propagation is invariant under shifting $t \to t + \Delta t$, any two signals that start out separated by a time delay Δt will always remain separated by the same time delay! The crucial subtlety is that the time delay is constant when measured via coordinate time dt, as opposed to proper time $d\tau = \sqrt{-g_{tt}}dt$. Thus, if an observer at radius r emits signals with period $\Delta\tau$, an observer at radius $R > r$ will receive them with a larger period:

$$\Delta\tau' = \sqrt{-g_{tt}(R)}\Delta t = \sqrt{\frac{g_{tt}(R)}{g_{tt}(r)}}\,\Delta\tau = \sqrt{\frac{1 - 2GM/R}{1 - 2GM/r}}\,\Delta\tau \ . \tag{53}$$

This is one way to derive gravitational redshift, which we discussed already in Lecture 4-2, and can be observed in e.g. the solar absorption spectrum as viewed from Earth. In a more extreme case, we can consider r that is very close to the event horizon $r = 2GM$. Then we find that the redshift becomes infinite, $\Delta\tau' \to \infty$. A similar result is true for an observer that isn't stationary, but is falling into the horizon: to an external observer, her signals will appear more and more stretched in time, and in particular the external observer will never quite see the infalling one cross the horizon.

Another key insight about a stationary observer at $r = \text{const}$ is that such an observer is accelerated: it must use e.g. some rocket engines to avoid falling (note that this is true already for Newtonian gravity, and has nothing to do with black holes). We can find the

4-acceleration as:

$$\alpha^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0 + \Gamma^\mu_{tt}(u^t)^2 = \frac{1}{1-2GM/r}\,\Gamma^\mu_{tt} \ , \tag{54}$$

where we used $du^\mu/d\tau = 0$, since u^μ is constant along the stationary trajectory. The only nonzero component of Γ^μ_{tt} in the Schwarzschild metric is:

$$\Gamma^r_{tt} = -\frac{1}{2} g^{rr}\partial_r g_{tt} = \frac{1}{2}\left(1-\frac{2GM}{r}\right)\partial_r\left(1-\frac{2GM}{r}\right) = \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right) \ . \tag{55}$$

Thus, the nonzero component of the 4-acceleration is:

$$\alpha^r = \frac{GM}{r^2} \ , \tag{56}$$

which precisely agrees with the Newtonian acceleration! However, we must be careful. The observer's proper acceleration – the acceleration as actually measured in her reference frame – is given by the length $\sqrt{\alpha_\mu \alpha^\mu}$ of the 4-acceleration:

$$a = \sqrt{\alpha_\mu \alpha^\mu} = \sqrt{g_{rr}}\,\alpha^r = \frac{GM/r^2}{\sqrt{1-2GM/r}} \ . \tag{57}$$

This acceleration behaves as the Newtonian GM/r^2 only at $r \gg GM$. As we approach the horizon $r = 2GM$, the acceleration diverges: at the horizon, it takes infinite effort to avoid falling in. If we consider an observer very close to the horizon, i.e. $r = 2GM + \epsilon$, the acceleration (57) simplifies into:

$$a \approx \frac{1/(2GM)}{\sqrt{1-2GM/(2GM+\epsilon)}} \approx \frac{1/(2GM)}{\sqrt{1-(1-\epsilon/(2GM))}} = \frac{1}{\sqrt{2GM\epsilon}} \ . \tag{58}$$

VIII. SCHWARZSCHILD-LIKE COORDINATES FOR FLAT SPACETIME

We already know some commonalities between a Rindler horizon in flat spacetime and the Schwarzschild horizon. Let's define Rindler coordinates (ρ, t), where t will play a similar role to the Schwarzschild t:

$$ds^2 = -dT^2 + X^2 = d\rho^2 - \rho^2 dt \ ; \quad (T, X) = (\rho\sinh t, \rho\cosh t) \ . \tag{59}$$

Both in Rindler coordinates and in the $r > 2GM$ region of the Schwarzschild metric, stationary observers are accelerated, and the acceleration diverges as we approach the horizon. Second, an observer or light signal approaching the horizon does not reach it until $t = \infty$ in both cases (see Lecture 3-2 for the Rindler case and see Carroll for the Schwarzschild case). It is possible to intensify the similarity if we switch coordinates in Rindler space from ρ to $r = \rho^2$. This brings the metric to the form:

$$dr = 2\rho d\rho \;\Rightarrow\; d\rho = \frac{dr}{2\sqrt{r}} \;\Rightarrow\; ds^2 = \frac{dr^2}{4r} - r dt^2 \; . \tag{60}$$

The Rindler horizon $\rho = 0$ is now $r = 0$, and its similarities with the Schwarzschild horizon intensify. In both cases, at the horizon, g_{tt} diverges and g_{rr} vanishes. In both cases, "outside the horizon", i.e. $r > 2GM$ in Schwarzschild and $r > 0$ in (60), r is spacelike and t is timelike; "inside the horizon", i.e. at $r < 2GM$ and $r < 0$ respectively, those roles flip. Note that the $r < 0$ region was not visible in the original Rindler coordinates, where ρ^2 was always positive. While the usual Rindler coordinates are associated with the "right-hand quarter" of Minkowski space, the $r < 0$ can be associated with the "top quarter", if we identify:

$$(T, X) = \begin{cases} (\sqrt{r}\sinh t, \sqrt{r}\cosh t) & r > 0 \\ (\sqrt{-r}\cosh t, \sqrt{-r}\sinh t) & r < 0 \end{cases} \; . \tag{61}$$

Note that, while both the $r > 0$ and $r < 0$ patches correspond to legitimate regions of Minkowski space, the patching between them is not smooth. For example, a line of changing r at constant t is a spacelike straight line at $r > 0$ and a timelike straight line at $r < 0$, with a "90-degree kink" in between. The same statements turn out to be true for the $r > 2GM$ and $r < 2GM$ patches of the Schwarzschild spacetime.

IX. THE NEAR-HORIZON LIMIT VS. THE RINDLER METRIC; GUESSING THE KRUSKAL COORDINATES

Let's continue to analyze the near-horizon limit $r = 2GM + \epsilon$ of Schwarzschild. The spacetime metric in this limit reads:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{2GM+\epsilon}\right) dt^2 + \frac{d\epsilon^2}{1 - 2GM/(2GM+\epsilon)} + (2GM+\epsilon)^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\ &= -\frac{\epsilon}{2GM}\, dt^2 + \frac{2GM}{\epsilon}\, d\epsilon^2 + (2GM)^2(d\theta^2 + \sin^2\theta\, d\phi^2) \; . \end{aligned} \tag{62}$$

In the limit, the angular coordinates form a sphere of the approximately constant radius $2GM$. Let us focus on the (t, r) plane, i.e. the (t, ϵ) plane, where more interesting things happen. In particular, up to some rescalings of the coordinates, we recognize it as the Rindler-like metric (60):

$$ds^2 = \frac{d\tilde{\epsilon}^2}{4\tilde{\epsilon}} - \tilde{\epsilon}\, d\tilde{t}^2 \ ; \quad \tilde{\epsilon} = 8GM\epsilon \ ; \quad \tilde{t} = \frac{t}{4GM} \ . \tag{63}$$

We conclude that the near-horizon limit of Schwarzschild looks like the Rindler wedge of flat spacetime, with $\tilde{t} = t/(4GM)$ acting as the boost angle and $\sqrt{\tilde{\epsilon}} = \sqrt{8GM(r-2GM)}$ acting as the radius. In particular, near the horizon, the time translation symmetry of full Schwarzschild looks like a boost symmetry!

With this lesson guiding our intuition, we can now extend the full Schwarzschild spacetime beyond the horizon, using coordinates that remain regular on it. For Rindler coordinates, this is accomplished by passing to the ordinary inertial coordinates (T, X), which see all of Minkowski space, and are regular on the horizons $X = \pm T$. We will similarly try to replace (t, r) with global coordinates (T, R) subject to the flat metric $-dT^2 + dR^2$, for which $t/(4GM)$ behaves as a boost angle, even outside the near-horizon limit. To take into account the changes in the metric as we go far from the horizon, we will allow for r-dependent warping both in the coordinates and in the metric. Thus, we construct our coordinate transformation as:

$$T = f(r) \sinh \frac{t}{4GM} \ ; \quad R = f(r) \cosh \frac{t}{4GM} \ , \tag{64}$$

and we'd like the metric to become:

$$ds^2 = g(r)(-dT^2 + dR^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ , \tag{65}$$

where r is now no longer one of the coordinates, but is a function of T and R which we can defined implicitly from (64) as $R^2 - T^2 = f(r)^2$. It remains to fix the warping functions $f(r)$ and $g(r)$. To do that, we plug the coordinate transformation (64) into the metric (65):

$$ds^2 = g(r)\left(-\frac{f(r)^2}{(4GM)^2}\, dt^2 + f'(r)^2 dr^2\right) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ , \tag{66}$$

and compare to the Schwarzschild metric. From the ratio of g_{tt} and g_{rr}, we can find an equation involving just $f(r)$:

$$\left(1 - \frac{2GM}{r}\right)^2 = -\frac{g_{tt}}{g_{rr}} = \left(\frac{f(r)}{4GM\, f'(r)}\right)^2 \ . \tag{67}$$

This can be easily solved via:

$$
\begin{aligned}
&(\ln f(r))' = \frac{f(r)'}{f(r)} = \frac{1}{4GM}\frac{1}{1-2GM/r} = \frac{1}{4GM}\frac{r}{r-2GM} = \frac{1}{4GM} + \frac{1}{2(r-2GM)} \\
&\quad\Longrightarrow \quad \ln f(r) = \text{const} + \frac{r}{4GM} + \frac{1}{2}\ln(r-2GM) \\
&\quad\Longrightarrow \quad f(r) = \text{const} \times e^{r/(4GM)}\sqrt{r-2GM}\ .
\end{aligned} \tag{68}
$$

The constant prefactor can be swallowed into $g(r)$, so it's arbitrary. The conventional choice is $1/\sqrt{2GM}$, so that:

$$
f(r) = e^{r/(4GM)}\sqrt{\frac{r}{2GM} - 1}\ . \tag{69}
$$

We can now find $g(r)$ via:

$$
\begin{aligned}
&1 - \frac{2GM}{r} = -g_{tt} = g(r)\left(\frac{f(r)}{4GM}\right)^2 = \frac{g(r)}{(4GM)^2}\,e^{r/(2GM)}\left(\frac{r}{2GM} - 1\right) \\
&\quad\Longrightarrow \quad g(r) = \frac{32G^3M^3}{r}\,e^{-r/(2GM)}\ .
\end{aligned} \tag{70}
$$

To sum up, the transformation into Kruskal coordinates and the metric in them read:

$$
(T, R) = e^{r/(4GM)}\sqrt{\frac{r}{2GM} - 1}\left(\sinh\frac{t}{4GM}\ ,\ \cosh\frac{t}{4GM}\right)\ ; \tag{71}
$$

$$
R^2 - T^2 = e^{r/(2GM)}\left(\frac{r}{2GM} - 1\right)\ ; \tag{72}
$$

$$
ds^2 = \frac{32G^3M^3}{r}\,e^{-r/(2GM)}(-dT^2 + dR^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\ . \tag{73}
$$

EXERCISES

Exercise 1. *Prove by direct calculation that the variation of the Ricci tensor is:*

$$
\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho_{\nu\mu} - \nabla_\nu \delta\Gamma^\rho_{\rho\mu}\ . \tag{74}
$$

Exercise 2. *Prove the 2+1d version of Birkhoff's theorem. Starting from an ansatz that doesn't assume time independence:*

$$
ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\phi^2\ , \tag{75}
$$

show that the vacuum Einstein equations $R_{\mu\nu} = 0$ imply $\partial_r\alpha = \partial_r\beta = \partial_t\beta = 0$, which brings the metric to the form:

$$
ds^2 = -e^{2\alpha(t)}dt^2 + e^{2\beta}dr^2 + r^2 d\phi^2\ . \tag{76}
$$

Finally, find a coordinate transformation which brings this metric to the flat form (20)*.*

Exercise 3. *Consider the Schwarzschild metric:*

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \ . \tag{77}$$

Show that this metric satisfies the vacuum Einstein equations $R_{\mu\nu} = 0$*. On the other hand, show that* R_{trtr} *is nonzero, and compare it to the Newtonian prediction at* $r \gg GM$*.*