GR lecture 7

Decomposition of the Riemann tensor; Geodesic deviation; Einstein's equations

I. CARROLL'S BOOK: SECTIONS 3.7, 3.10, 4.1, 4.2, 4.4, 4.5

II. GEODESIC DEVIATION AS AN ACCELERATION GRADIENT

In class and in Carroll's book, we derived the geodesic deviation equation, which gives the relative 4-acceleration α^μ of a pair of free-falling particles, each with 4-velocity u^μ, separated by a vector s^μ:

$$\alpha^\mu \equiv \frac{D^2 s^\mu}{dt^2} \equiv (u^\nu \nabla_\nu)^2 s^\mu = R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho s^\sigma \ . \tag{1}$$

In the non-relativistic limit, where $u^\mu \approx (1,0,0,0)$, this becomes an ordinary acceleration, given by:

$$a^i = R^i{}_{ttj} s^j \ . \tag{2}$$

This allows us to interpret R_{ittj} as the gradient of the Newtonian acceleration field:

$$R_{ittj} = \partial_j a_i \ . \tag{3}$$

The fact that Newtonian gravity is conservative, i.e. $\partial_{[i} a_{j]} = 0$, is ensured automatically by the symmetries of the Riemann tensor:

$$\partial_i a_j = R_{jtti} = R_{tijt} = R_{ittj} = \partial_j a_i \ , \tag{4}$$

where we used successively the symmetries $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$ and $R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]}$.

Finally, the divergence of the acceleration field is given by:

$$\partial_i a^i = R^i{}_{tti} = -R^i{}_{tit} = -R^\mu{}_{t\mu t} = -R_{tt} \ , \tag{5}$$

where we used the fact that $R^t{}_{ttt} = 0$ (from the index antisymmetries) to convert the 3d trace into a 4d trace. On the other hand, in Newtonian gravity, we know that $\partial_i a^i$ is given by the mass density ρ via:

$$\partial_i a^i = -4\pi G\rho = -4\pi G T_{tt} \ . \tag{6}$$

Here, we used the fact that, in the non-relativistic limit, the energy density T^{tt} consists almost entirely of rest energy, i.e. is equal to the mass density ρ. Also, we assumed that the metric is approximately flat, i.e. gravity is weak, so that $T^{tt} = T_{tt}$. The comparison of eqs. (5) and (6) served as one of our motivations for the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{7}$$

III. GENERAL COVARIANCE / DIFFEOMORPHISM INVARIANCE / BACKGROUND INDEPENDENCE

There is a profound difference between full GR, subject to Einstein's equations, and merely physics on a given curved geometry $g_{\mu\nu}(x)$. Once the metric becomes another dynamical field, with its own field equations that must be solved along with the others, we no longer have fields and particles propagating on top of a given geometry. Instead, everything, including the geometry, is now made of dynamical fields! It's fields propagating on top of each other!

A related important insight is that, if the metric isn't given in advance, then coordinates have no geometric content whatsoever: they are merely labels for distinguishing different points. Coordinate differences get associated with actual distances only after we solve the Einstein equation for the metric. Before we look at some particular solution in some particular coordinate system, it is meaningless to ask for e.g. the value of a field at given coordinates (t, x, y, z), or for the coordinates at which two particles collide with each other. We can only ask questions such as "how much distance did the particles traverse before colliding", which necessarily involve some extra fields – in this case, the metric. Physics is now entirely about relations between different fields. Coordinates merely serve as a convenient mathematical intermediary to tell us when we are talking about two fields at the same point, or at a pair of infinitesimally nearby points.

IV. COUNTING DEGREES OF FREEDOM IN EINSTEIN'S EQUATIONS

A. Number of fields vs. number of equations

This week, we learned Einstein's equations (7). These are the field equations governing the metric $g_{\mu\nu}(x)$, which is now a full-fledged dynamical field. In this section, we count degrees of freedom to ensure that there is indeed one field equation for every dynamical variable. We begin with a simpler example – the Maxwell equations, viewed as equations for the electromagnetic potential $A_\mu(x)$:

$$\partial_\nu F^{\mu\nu} = j^\mu \ , \tag{8}$$

where we're considering flat spacetime for simplicity. Eq. (8) is a vector equation, and has 4 components – the same number as the field $A_\mu(x)$ itself. However, we must be careful. A_μ is subject to the gauge transformations:

$$A_\mu \ \rightarrow \ A_\mu - \partial_\mu \theta \ , \tag{9}$$

which do not change its physical content, and to which the gauge-invariant Maxwell equations (8) are completely blind. This gauge freedom is parameterized by 1 function $\theta(x^\mu)$ on spacetime, leaving $A_\mu(x)$ with $4 - 1 = 3$ truly "physical" components. Luckily (and not by coincidence), a dual statement holds for the field equations. The divergence $\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\mu j^\mu$ of eq. (8) is identically true – the LHS is identically zero, and the RHS vanishes by charge conservation. Therefore, this particular piece of the Maxwell equations doesn't convey any information, and the number of non-trivial equations is actually $4 - 1 = 3$, in agreement with our count of meaningful components of $A_\mu(x)$.

A very similar story holds for GR. Naively, eq. (7) provides us with 10 equations, since the LHS and RHS are both symmetric matrices. This makes one equation for each of the components of the metric field $g_{\mu\nu}(x)$. However, the metric is also subject to a "gauge freedom" of coordinate transformations, also known as diffeomorphisms:

$$g_{\mu\nu} \ \rightarrow \ \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x^\nu} g_{\rho\sigma} \ , \tag{10}$$

which do not change its physical content, and to which the coordinate-independent Einstein equations (7) are completely blind. This diffeomorphism freedom is parameterized by 4 functions $x'^\mu(x^\nu)$ on spacetime, leaving $g_{\mu\nu}(x)$ with $10 - 4 = 6$ truly "physical" components.

To balance this, 4 components of the Einstein equations are actually identities: indeed, the divergence $\nabla^\mu(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = 8\pi G\nabla^\mu T_{\mu\nu}$ is identically true, with the LHS resulting from Bianchi identities, and the RHS from energy-momentum conservation. Thus, the number of non-trivial equations in (7) is actually $10-4=6$, in agreement with our count of meaningful components of $g_{\mu\nu}(x)$.

B. Initial data, or physical degrees of freedom

The result of the previous section – that the number of fields is matched by the number of field equations – tells us that solutions are determined by initial (or, more generally, boundary) data. The next question is: how much initial data do we need? This is what we normally mean by the number of "physical degrees of freedom" in a field – how much freedom is left in choosing the field solution after the field equations (with given sources) are imposed. For the electromagnetic field, this should correspond to the 2 polarization of electromagnetic waves. Similarly, for GR, we will find the 2 polarizations of gravitational waves.

Let us now see how this counting works, starting with electromagnetism. We start with the electromagnetic potential $A_\mu(x^\nu)$, which has 4 components. As discussed earlier, 1 of these components can be gauged away. In particular, we can set $A_t = 0$ everywhere by choosing $\partial_t\theta = A_t$, i.e. $\theta = tA_t(t,\mathbf{x}) + \alpha(\mathbf{x})$. Thus, the initial data on a $t=0$ time slice is now given by $\mathbf{A}(0,\mathbf{x})$, i.e. by 3 functions on the 3d hypersurface. But now we get to use the gauge freedom again, this time just on the 3d hypersurface! Indeed, in the 4d gauge transformation that we used to kill A_t, there is still an unspecified function $\alpha(\mathbf{x})$. We can use this 3d gauge freedom to shift the initial data at $t=0$ as $\mathbf{A} \to \mathbf{A} - \boldsymbol{\partial}\alpha$, removing one more degree of freedom. Thus, after taking gauge invariance into account twice – once in 4d and once more for the 3d initial data – we are left with $4-1-1=2$ degrees of freedom. These are the 2 polarizations of an electromagnetic wave, which can exist independently of any charges or currents.

Let us now repeat the story for GR, where we'll encounter an important extra subtlety. We start again with the 10 components $g_{\mu\nu}(x)$. We can use the 4d diffeomorphism freedom (10) to set $g_{tt} = -1$ and $g_{ti} = 0$ everywhere. This leaves us with just $g_{ij}(t,x^k)$ with $10-4=6$ components as the dynamical field throughout spacetime, and in particular on the initial time slice $t = 0$. Now, as before, we can use coordinate freedom again, this time in 3d, i.e. $x^i \to x'^i(x^j)$, to transform the initial data $g_{ij}(0, x^k)$. This means that 3 of the remaining 6 degrees of freedom are "just gauge". So far, we're left with $10-4-3 = 3$ degrees of freedom. However, we must now take into account one final "gauge freedom": in GR, setting $t = 0$ does not actually specify any particular hypersurface! The coordinates are just labels for points in spacetime, with no geometric content on their own! Thus, in a single solution of GR, we are free to change what we choose to call the $t = 0$ hypersurface, leading to seemingly different, but actually equivalent, initial data. This freedom of repositioning the $t = 0$ hypersurface consists of 1 degree of freedom for each hypersurface point x^i: indeed, the position of a new "$t = 0$" hypersurface can be given as the "graph" of a function $t(x^i)$. Taking this last gauge freedom into account, we are left with $10 - 4 - 3 - 1 = 10 - 4 - 4 = 2$ physical degrees of freedom for the metric. In the end, just as for electromagnetism, the 4d diffeomorphisms get "used twice" to subtract degrees of freedom: once in 4d, and once again on the initial 3d hypersurface. The final count of degrees of freedom is also the same as for electromagnetism: gravitational waves, like light waves, have 2 polarizations.

EXERCISES

Exercise 1. *Find the Ricci tensor and the Ricci scalar for the metrics in Exercises 3-4 from the previous week (you can still assume $t = 0$ for the metric in Exercise 4).*

Exercise 2. *Find the coefficients in the decomposition of the Riemann tensor into its Weyl and Ricci pieces:*

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \beta\left(R_{\mu\rho}g_{\nu\sigma} - R_{\nu\rho}g_{\mu\sigma} - R_{\mu\sigma}g_{\nu\rho} + R_{\nu\sigma}g_{\mu\nu}\right) + \gamma R\left(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right) \ . \quad (11)$$

Express β and γ as functions of the spacetime dimension d. Hint: take traces.

Exercise 3. *In the Newtonian limit of GR, express the traceless part of the "tidal force" $\partial_i a_j$ (where a_i is the gravitational acceleration) in terms of the Weyl tensor $C_{\mu\nu\rho\sigma}$.*

Exercise 4. *Repeat the degrees-of-freedom counting from section IVB for GR in 3d spacetime instead of 4d.*