GR lecture 3-2

Maxwell equations, some general field theory, simple curved & slanted coordinates

I. MAXWELL EQUATIONS

So far in our discussion of electromagnetism, we've discussed the dynamics of a charge in an external EM field. The dynamics of the field itself can be obtained by adding an appropriate term to the particle action from the previous lecture:

$$S = -m\int\sqrt{-dx_\mu dx^\mu} + q\int A_\mu dx^\mu - \frac{\epsilon_0}{4}\int F_{\mu\nu}F^{\mu\nu}\, d^4x \ . \tag{1}$$

The new term is an integral over spacetime, and represents the action of a free EM field (in the absence of charges). The interaction between the charges and the field is still fully captured by the second term. This is the essence of Newton's 3rd law in the context of the action principle: the effects of dynamical objects (such as charges and the EM field) on each other are always equal and opposite, because both are derived from the same interaction term in the action. ϵ_0 is the electric constant, which is often set to 1. Note that the magnetic constant μ_0 never needs to explicitly appear, since $\mu_0 = 1/(\epsilon_0 c^2)$.

The field equations obtained from varying A_μ in the action (1) read:

$$\partial_\nu F^{\mu\nu} = j^\mu/\epsilon_0 \ . \tag{2}$$

Here, $j^\mu = (\rho, \mathbf{j})$ is the 4-current of electric charge, which in the case of (1) consists of a single point particle, but in general can describe some continuous distribution of charges and currents. Due to the antisymmetry of $F_{\mu\nu}$, eq. (2) automatically implies local charge conservation:

$$\partial_\mu j^\mu = \epsilon_0 \partial_\mu \partial_\nu F^{\nu\mu} = 0 \ . \tag{3}$$

In space and time components, eq. (2) reads:

$$\boldsymbol{\partial}\cdot\mathbf{E} = \rho/\epsilon_0 \ ; \quad \boldsymbol{\partial}\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = \mathbf{j}/\epsilon_0 \ , \tag{4}$$

which is half of Maxwell's equations. The other half of Maxwell's equations reads:

$$\partial_{[\mu}F_{\nu\rho]} = 0 \ , \tag{5}$$

or, in components:

$$\boldsymbol{\partial} \cdot \mathbf{B} = 0 \; ; \quad \boldsymbol{\partial} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \; . \tag{6}$$

Exercise 1. *Consider electromagnetism with no charges, i.e. the action* (1) *with just the 3rd term. Perform the variation of this action with respect to* A_μ*, and derive the charge-free version of* (2)*, i.e.* $\partial_\nu F^{\mu\nu} = 0$*. It may be helpful to solve Exercise 5 first.*

Exercise 2. *Show that the second half* (5) *of Maxwell's equations is automatically satisfied when* $F_{\mu\nu}$ *is constructed from a potential via* $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$*. Remember that partial derivatives commute!*

Exercise 3. *Show that eqs.* (2),(5) *indeed decompose into eqs.* (4),(6).

A key property of electromagnetism is the gauge symmetry:

$$A_\mu \;\to A_\mu - \partial_\mu \theta \; ; \quad F_{\mu\nu} \;\to\; F_{\mu\nu} \; , \tag{7}$$

where $\theta(x^\mu)$ is an arbitrary scalar function on spacetime.

Exercise 4. *Show that the transformation* (7) *of* A_μ *indeed leaves* $F_{\mu\nu}$ *invariant.*

Note that both the EM force law $m\alpha_\mu = qF_{\mu\nu}u^\nu$ and the Maxwell equations (2),(5) can be written purely in terms of the field strength $F_{\mu\nu}$, with no reference to the potential A_μ. In other words, all physical quantities are actually invariant under the gauge transformation (7). This means that the potential A_μ, which is not invariant, is not directly observable. In fact, the only observable quantity is its circulation $\oint A_\mu dx^\mu$ around closed loops. For an infinitesimal loop, this circulation (per unit area) is measured by $F_{\mu\nu}$. Nevertheless, A_μ is necessary for writing the interaction term in the action (1), as well as for deriving the Maxwell equations from an action principle, even in the absence of charges (see Exercise 1).

II. THE STRESS-ENERGY TENSOR OF THE ELECTROMAGNETIC FIELD

The electromagnetic field carries energy and momentum. However, defining its stress-energy tensor is somewhat tricky. One might try and start from first principles, as follows. Consider a field theory of (for simplicity) scalar fields ϕ, with an action of the form:

$$S = \int L(\phi, \partial_\mu \phi) d^4x = \int L(\phi, \boldsymbol{\partial}\phi, \dot{\phi}) \, d^3x \, dt \; . \tag{8}$$

The function L is sometimes called the Lagrangian density, since its d^3x integral gives the usual Lagrangian from analytical mechanics – the one that is integrated over dt to give the action. However, when doing field theory, we often just call L itself the Lagrangian.

Exercise 5. *By varying the action* (8)*, derive the field theory version of the Euler-Lagrange equations of motion:*

$$\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} = 0 \ . \tag{9}$$

In mechanics, the energy E (or the Hamiltonian H) is defined as the Noether charge conjugate to time translations, i.e. the variation $-\partial S/\partial t$ of the action upon varying the time of the trajectory's endpoint. After crunching some integrations by parts, we eventually learn the expression:

$$H = \frac{\partial L}{\partial \dot{q}}\dot{q} - L \ . \tag{10}$$

It isn't too hard to guess the relativistic generalization of this, which 1) passes from overall Lagrangian and energy to their densities, and 2) treats the energy as the time component of a 4-vector:

$$T^\mu{}_\nu = L\delta^\mu_\nu - \frac{\partial L}{\partial(\partial_\mu \phi)}\partial_\nu \phi \ . \tag{11}$$

To make sense of (11) as a generalization of (10), including the sign difference, recall that $T^{tt} = -T^t{}_t$ is the energy density.

Exercise 6. *Prove that the stress-energy tensor* (11) *is conserved:* $\partial_\mu T^\mu{}_\nu = 0$.

Exercise 7. *What is the appropriate generalization of* (11) *for the case of a vector field* A_μ*? Use this to derive the stress-energy tensor* $T^{\mu\nu}$ *from the Lagrangian* $L = -\epsilon_0 F_{\mu\nu}F^{\mu\nu}/4$ *of the electromagnetic field. Is this* $T^{\mu\nu}$ *symmetric in its indices? Is it gauge-invariant?*

The answer to the last exercise suggests that deriving the stress-energy tensor of a field theory can be unexpectedly tricky. There are ways to correct the stress-energy tensor (11) so as to make it both symmetric and gauge-invariant. Later in the course, we will learn a general prescription that replaces (11) entirely, and automatically produces a symmetric and gauge-invariant $T^{\mu\nu}$. For now, we can take a different, informal route to constructing the $T^{\mu\nu}$ of the electromagnetic field. Gauge invariance demands that $T^{\mu\nu}$ be constructed solely out of $F_{\mu\nu}$ and its derivatives. From dimensional analysis and the powers with which A_μ and ∂_μ enter the Lagrangian, the most general symmetric tensor that one can construct is:

$$T^{\mu\nu} = \alpha F^\mu{}_\rho F^{\nu\rho} + \beta F_{\rho\sigma} F^{\rho\sigma} \eta^{\mu\nu} \ . \tag{12}$$

Exercise 8. *Now, fix the unknown coefficients α and β, using the following facts:*

- *$T^{\mu\nu}$ should satisfy a conservation law.*
- *As we know from electrostatics, the energy density of an electric field is $\epsilon_0 \mathbf{E}^2/2$.*

The total energy density of the electromagnetic field works out to be:

$$T^{tt} = \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) \ . \tag{13}$$

The EM field's momentum density is known as the Poynting vector:

$$T^{ti} = \epsilon_0 B^{ij} E_j = \epsilon_0 (\mathbf{E} \times \mathbf{B})^i \tag{14}$$

Exercise 9. *Find the pressure $p = T^i_i/3$ of an electromagnetic field. Show that the electromagnetic stress-energy tensor is traceless $T^\mu_\mu = 0$, just like the stress-energy tensor of photons.*

Curiously, the last statement – that $T^\mu_\mu = 0$ for arbitrary EM fields just like for photons – is only true in 4 dimensions. This has to do with the fact that in 4 dimensions, electromagnetism has conformal symmetry, while in general, it only has scale symmetry.

III. SOME CURVED COORDINATES: POLAR, SPHERICAL, RINDLER

We are now ready to start working with curved coordinates. We begin with a few simple examples. We know and love the polar coordinate system (ρ, ϕ) for the plane, where ρ is the distance from the origin, and ϕ is the angle from the horizontal line. These are related to the Cartesian coordinates (x, y) via:

$$x = \rho \cos\phi \ ; \quad y = \rho \sin\phi \ . \tag{15}$$

Consider now an infinitesimal displacement dx^i, which we may express as either (dx, dy) or $(d\rho, d\phi)$. What is the length-squared of this vector? In Cartesian coordinates, it is

$ds^2 = dx^2 + dy^2$. In polar coordinates, we can notice that the radial direction is orthogonal to the azimuthal one, so we can continue using the Pythagoras theorem. However, while the length of a radial displacement is dr, the length of an angular displacement is not $d\phi$, but $rd\phi$! Thus, the length-squared of an infinitesimal displacement reads:

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 \ . \tag{16}$$

On the other hand, we know that length-squared should take the general quadratic form $ds^2 = g_{ij}dx^i dx^j$. We deduce the metric in polar coordinates as:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix} \quad \Longleftrightarrow \quad g_{\rho\rho} = 1 \ ; \ g_{\phi\phi} = \rho^2 \ ; \ g_{\rho\phi} = 0 \ . \tag{17}$$

If we directly add a third dimension z, we get cylindrical coordinates, with metric:

$$g_{\rho\rho} = g_{zz} = 1 \ ; \quad g_{\phi\phi} = \rho^2 \ , \tag{18}$$

with all other components vanishing.

Another useful coordinate system for 3d space is given by the spherical coordinates (r, θ, ϕ), defined via:

$$x = r\sin\theta\cos\phi \ ; \quad y = r\sin\theta\sin\phi \ ; \quad z = r\cos\theta \ . \tag{19}$$

Again, we can guess the metric. The r, θ and ϕ directions are still orthogonal to each other. The length of a radial displacement is dr, while the lengths of angular displacement are scaled by the radius r of the relevant sphere. In addition, while displacements in θ are along large circles of radius r, displacements in ϕ are along smaller circles, or radius $r\sin\theta$. The metric thus reads:

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \Longleftrightarrow \quad g_{rr} = 1; \ g_{\theta\theta} = r^2; \ g_{\phi\phi} = r^2\sin^2\theta \ , \tag{20}$$

with all other components vanishing.

Exercise 10. *Instead of using geometric intuition, derive the polar and spherical metrics by plugging the coordinate relations* (15),(19) *directly into* $ds^2 = dx^2 + dy^2$ *or* $ds^2 = dx^2 + dy^2 + dz^2$.

Exercise 11. *Write down the components of the inverse metric* g^{ij} *in polar and spherical coordinates.*

We can play the same kind of game in spacetime. Instead of polar coordinates in the xy plane, consider the tx plane. The role of circles $x^2 + y^2 = \rho^2$ is now played by hyperbolas $x^2 - t^2 = \rho^2$. In fact, in the same way that circles are lines of constant curvature in the Euclidean plane, these hyperbolas are lines of constant curvature in the Lorentzian plane. When such lines are timelike, they can be interpreted as worldlines with constant acceleration. By analogy with polar coordinates, we can define an angular coordinate τ (not proper time!) that runs along these hyperbolas:

$$t = \rho \sinh \tau \ ; \quad x = \rho \cosh \tau \ . \tag{21}$$

The coordinates (ρ, τ) are called Rindler coordinates.

Exercise 12. *Write down the metric components $g_{\rho\rho}, g_{\tau\tau}, g_{\rho\tau}$ in Rindler coordinates.*

Exercise 13. *A particle is traveling along a $\rho =$ const worldline. Consider this particle at Rindler time τ. In terms of the inertial (t, x) basis, write down the particle's 4-velocity u^μ and 4-acceleration a^μ. What is the magnitude $|\mathbf{a}|$ of the particle's ordinary acceleration in its rest frame?*

Unlike polar coordinates, Rindler coordinates do not span the full tx plane (at least, not without using complex values). In particular, the ordinary range $0 < \rho < \infty$ for the radial coordinate, together with the full range $-\infty < \tau < \infty$ for the boost angle, covers only the "right-hand wedge" of the plane, enclosed between the $t = \pm x$ lightrays at $x > 0$. This quarter of Minkowski spacetime is sometimes confusingly called "Rindler space", as if it were something different entirely. A better name is "Rindler wedge". The importance of Rindler coordinates is that they provide the simplest, prototypical example, of a spacetime divided by causal horizons. In particular:

- An accelerated observer along a $\rho =$ const worldline can receive causal signals only from below the $t = x$ line, and can send causal signals only above the $t = -x$ line.

- The Rindler wedge is the causal domain of dependence of the $x > 0$ half-space: given an initial state on the $x > 0$ half-axis at $t = 0$, the Rindler wedge is the region within which we can deduce the past and future from causal equations of motion without requiring any knowledge about the degrees of freedom at $x < 0$. Note that in Newtonian mechanics, where there is no lightspeed barrier on causal influences, such a spacetime region does not exist at all!

Exercise 14. *Consider a particle at rest at some positive* x *value* $x = L$. *Describe its trajectory in Rindler coordinates. Where is the particle at* $\tau \to \infty$? *Is this relevant to something you know about falling into a black hole?*

Exercise 15. *Construct an analogue of spherical coordinates for Minkowski spacetime. In other words, express* (t, x, y, z) *in terms of suitable coordinates* (R, χ, θ, ϕ), *where* R *is the radial distance from the origin, and* (χ, θ, ϕ) *are angles. Write down the metric in terms of your coordinates. What is the region of Minkowski spacetime that your coordinates span? For concreteness, suppose that the radius* R *is spacelike.*

IV. SLANTED COORDINATES

In the examples above, we've seen coordinates that weren't orthonormal, but were still orthogonal: the metric didn't develop any non-diagonal components. This is of course not true for general coordinates. Even without introducing curvature, we can construct slanted coordinates by considering a general linear transformation of (t, x, y, z):

$$x'^{\mu} = (M^{-1})_{\nu}{}^{\mu} x^{\nu} \; ; \quad g'_{\mu\nu} = M_{\mu}{}^{\rho} M_{\nu}{}^{\sigma} g_{\rho\sigma} \; . \tag{22}$$

Starting with the orthonormal metric $g_{\mu\nu} = \eta_{\mu\nu}$, the metric $g'_{\mu\nu}$ in the new coordinates can be a general symmetric matrix (up to the constraint on its signature, as in the case of ordinary space). In fact, it need not even be true that one coordinate is timelike, and the others are spacelike. Minkowski spacetime can be spanned e.g. by 4 timelike vectors, or by 4 spacelike ones!

Exercise 16. *Consider the coordinates:*

$$t' = t \; ; \quad x' = t + 0.001x \; ; \quad y' = t + 0.001y \; ; \quad z' = t + 0.001z \; . \tag{23}$$

Find the metric $g'_{\mu\nu}$ *and the inverse metric* $g'^{\mu\nu}$ *in these coordinates. Hint: it may help to invert the relations* (23). *What are the signs of the diagonal elements?*

In general, we can say the following about the signs of the metric's diagonal elements. The diagonal element g_{11} is the length-squared $g_{\mu\nu} v^{\mu} v^{\nu}$ of the vector $v^{\mu} = (0, 1, 0, 0)$, i.e. of a vector pointing along the "x^1 axis" – the line of fixed (x^0, x^2, x^3). Thus, g_{11} is positive/negative/zero when the x^1 axis is spacelike/timelike/null. Now, consider the inverse metric element g^{11}. This is the "length-squared" $g^{\mu\nu}u_\mu u_\nu$ of the covector $u_\mu = (0,1,0,0)$, which "points along" the hypersurface of fixed x^1 (which is spanned by the x^0, x^2 and x^3 axes). Thus, g^{11} is positive/negative/zero when the normal to the constant-x^1 hypersurface is spacelike/timelike/null. Note that g_{11} and g^{11} can have different signs.

Consider now the off-diagonal elements of $g_{\mu\nu}$. These indicate axes that are not orthogonal to each other (while nonzero off-diagonal elements of $g^{\mu\nu}$ indicate hypersurfaces that aren't orthogonal to each other). Of particular physical interest are the mixed space/time components g_{ti} (assuming a "not-too-wild" coordinate system, in which t is timelike, and the other coordinates x^i are spacelike). When these components are non-vanishing, they indicate that the t axis is non-orthogonal to some of the x^i axes, i.e. that it is non-orthogonal to the constant-t hypersurface. The physical meaning of this is that the t axis has nonzero velocity with respect to the normal to the constant-t hypersurface. Another way to think of this is that the t axis – the worldline of a particle at rest in our coordinate system – is being "dragged along" some of the x^i coordinates. This is a good way to describe the spacetime near a rotating black hole.

Exercise 17. *Express the mixed components g^{ti} of the inverse metric in terms of g_{tt}, g_{ti} and g^{ij}. For $g_{\mu\nu}$ that is not too different from $\eta_{\mu\nu}$, is it possible for g^{ti} to vanish while g_{ti} doesn't, or for g_{ti} to vanish while g^{ti} doesn't?*

When coordinatizing a Lorentzian plane, such as the tx plane, instead of using a timelike t axis and a spacelike x axis, it is often useful to use a pair of null axes:

$$u = t + x \; ; \quad v = t - x \; . \tag{24}$$

Unlike the t and x axes, these lightlike axes are unique – there are exactly two lightrays in a given plane!

Exercise 18. *Find the metric components g_{uu}, g_{vv} and g_{uv}. To avoid factors of 2, don't forget that $g_{uv} = g_{vu}$.*

Exercise 19. *Perform a Lorentz boost with rapidity θ on the (t,x) plane. How does it affect the null coordinates (u,v)?*