

# GR lecture 12

## Gravitational waves: polarizations and production

### I. CARROLL'S BOOK: SECTIONS 7.4-7.5

### II. A WORKED-OUT EXAMPLE OF DE DONDER AND TRANSVERSE-TRACELESS GAUGE

In the previous lecture, we considered small perturbations of the metric around flat spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \quad (1)$$

and established a notation in which indices are raised and lowered using the flat metric  $\eta_{\mu\nu}$ . We found it convenient to define the trace-reversed version of the metric perturbation  $h_{\mu\nu}$ :

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} ; \quad \tilde{h} = -h . \quad (2)$$

In terms of this quantity, we defined de Donder gauge:

$$\partial^\mu \tilde{h}_{\mu\nu} = 0 . \quad (3)$$

In this gauge, to first order in  $h_{\mu\nu}$ , the Einstein tensor becomes simply:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{1}{2}\square\tilde{h}_{\mu\nu} , \quad (4)$$

where  $\square$  is the d'Alembertian:

$$\square \equiv \partial_\mu \partial^\mu = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} . \quad (5)$$

The linearized Einstein equation thus reads:

$$\square\tilde{h}_{\mu\nu} = -16\pi GT_{\mu\nu} . \quad (6)$$

Let's now consider the source-free version:

$$\square\tilde{h}_{\mu\nu} = 0 . \quad (7)$$

The non-trivial solutions to this equation are gravitational waves. By the usual Fourier transform techniques, these solutions are spanned by plane waves of the form:

$$\tilde{h}_{\mu\nu}(x) = \tilde{C}_{\mu\nu} e^{ik_\rho x^\rho} , \quad (8)$$

where the wavevector  $k_\mu$  and the polarization tensor  $\tilde{C}_{\mu\nu}$  are constants. The field equation (7) implies that  $k_\mu$  is lightlike:  $k_\mu k^\mu = 0$ , while de Donder gauge (3) implies that  $\tilde{C}_{\mu\nu}$  is transverse:  $k^\mu \tilde{C}_{\mu\nu} = 0$ . As an example, consider the null wavevector:

$$k^\mu = \omega(1, 1, 0, 0) ; \quad k_\mu = \omega(-1, 1, 0, 0) , \quad (9)$$

which represents a wave with frequency  $\omega$  moving along the  $x$  direction. The most general symmetric polarization tensor  $\tilde{C}_{\mu\nu}$  that satisfies  $k^\mu \tilde{C}_{\mu\nu} = 0$  reads:

$$\tilde{C}_{\mu\nu} = \begin{pmatrix} \alpha & -\alpha & \beta & \gamma \\ -\alpha & \alpha & -\beta & -\gamma \\ \beta & -\beta & \tilde{C}_{yy} & \tilde{C}_{yz} \\ \gamma & -\gamma & \tilde{C}_{yz} & \tilde{C}_{zz} \end{pmatrix} . \quad (10)$$

From  $\tilde{h}_{\mu\nu}$ , we can extract the actual metric perturbation  $h_{\mu\nu}$ , via:

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2} \tilde{h} \eta_{\mu\nu} = C_{\mu\nu} e^{ik_\rho x^\rho} , \quad (11)$$

where the polarization tensor  $C_{\mu\nu}$  in our example (10) reads:

$$C_{\mu\nu} = \tilde{C}_{\mu\nu} - \frac{1}{2} \tilde{C} \eta_{\mu\nu} = \begin{pmatrix} \frac{2\alpha + \tilde{C}_{yy} + \tilde{C}_{zz}}{2} & -\alpha & \beta & \gamma \\ -\alpha & \frac{2\alpha - \tilde{C}_{yy} - \tilde{C}_{zz}}{2} & -\beta & -\gamma \\ \beta & -\beta & \frac{\tilde{C}_{yy} - \tilde{C}_{zz}}{2} & \tilde{C}_{yz} \\ \gamma & -\gamma & \tilde{C}_{yz} & \frac{\tilde{C}_{zz} - \tilde{C}_{yy}}{2} \end{pmatrix} . \quad (12)$$

Now, recall that de Donder gauge still leaves us with some of the gauge symmetry  $h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$ . Specifically, we can still apply gauge transformations that satisfy  $\square \xi_\mu = 0$ . The parameter of this residual gauge symmetry can again be decomposed into plane waves of the form:

$$\xi_\mu(x) = \epsilon_\mu e^{ik_\rho x^\rho} . \quad (13)$$

In our example (12), we can now perform a gauge transformation with  $\epsilon_\mu$  of the form:

$$\epsilon_\mu = \frac{1}{\omega} \left( -\frac{2\alpha + \tilde{C}_{yy} + \tilde{C}_{zz}}{4} , \frac{2\alpha - \tilde{C}_{yy} - \tilde{C}_{zz}}{4} , -\beta , -\gamma \right) , \quad (14)$$

which leaves us with the traceless-transverse polarization tensor:

$$C_{\mu\nu} = \tilde{C}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\tilde{C}_{yy} - \tilde{C}_{zz}}{2} & \tilde{C}_{yz} \\ 0 & 0 & \tilde{C}_{yz} & \frac{\tilde{C}_{zz} - \tilde{C}_{yy}}{2} \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C_{\oplus} & C_{\otimes} \\ 0 & 0 & C_{\otimes} & -C_{\oplus} \end{pmatrix}. \quad (15)$$

Here,  $C_{\oplus}$  and  $C_{\otimes}$  are the amplitudes of the “+” and “×” polarizations of the gravitational wave.