# GR lecture 11

## Linearized GR and gravitational waves

### I. CARROLL'S BOOK: SECTIONS 7.1, 7.2, 7.4

### II. IS GR MORE SINGULAR OR LESS SINGULAR THAN OTHER FIELD THEORIES?

In the previous lectures, we learned about Schwarzschild black holes. Some of their features are overly idealized. Astrophysical black holes, which aren't eternal, don't come with a white hole, or with a second external universe. However, the formation of horizons and singularities is a very generic feature: GR predicts it for a wide range of initial conditions for e.g. collapsing stars (future singularity – black hole), or expanding cosmologies (past singularity – Big Bang). This is a consequence of two features of GR. First, gravity (even in the Newtonian version) always attracts, and thus tends to produce collapses. This is in contrast to electromagnetism, where attraction always comes together with repulsion. Second, in GR, gravity can be self-reinforcing: the non-linearity of the Einstein equation means that the gravitational field itself can contribute to the attraction that eventually causes collapse into a singularity. In this sense, when we look at short-distance phenomena, GR is more singular than other field theories: it is more prone to producing infinities, such as infinite density and curvature. The same pattern continues when we try to quantize GR: we find infinities in the Feynman diagrams which can't be renormalized away.

At the same time, there is a different sense in which GR is less singular than electromagnetism. To see this, let's consider the question of the electromagnetic self-interaction of a point charge. On one hand, we must accept that a point charge must interact with its own electromagnetic field: otherwise, we are in conflict with the observed fact that accelerated charges lose energy by emitting light. On the other hand, if we try to calculate the electrostatic energy associated with a ball of charge $Q$ and radius $R$, we get $\sim Q^2/R$, which of course diverges as $R \to 0$. The interpretation/treatment of this infinite energy in electromagnetism is a subtle and non-trivial issue. It is partially alleviated at the quantum level through the existence of antiparticles with opposite charge, but it never completely goes away.

In contrast, consider a "gravitational point charge", i.e. a "point mass", i.e. a black hole. Here, the "gravitational self-interaction energy" is difficult to define, or to separate from the "intrinsic energy" of the collapsed matter. However, the total energy of this system is very clearly defined, and is manifestly finite! Indeed, in gravity, energy, which in the rest frame is the same as mass, is just the "charge" that couples to the gravitational field. Thus, we can find the energy inside a volume by applying Gauss' law, i.e. by calculating the flux of the gravitational field through an enclosing closed surface. We should note that in GR, as opposed to Newtonian gravity, such a flux is difficult to define, unless our enclosing surface is at infinity; but, ultimately, it can be done. Another way to phrase this is that a black hole's total energy can be inferred by probing its gravitational field at a large distance, where (disregarding cosmological complications) the Newtonian limit applies. This was in fact what allowed us to confidently identify the parameter $M$ in the Schwarzschild metric as the black hole's mass!

Let us make two more comments about the way in which GR leads to a finite total energy. First, this is a non-perturbative phenomenon: it cannot be seen in weak-field perturbation theory. Indeed, consider the Newtonian approximation. There, just as for electrostatics, the gravitational self-interaction energy for a small ball will be $\sim GM^2/R$ (and, in this case, it will be negative). Now consider GR corrections to this Newtonian answer. These will be organized in powers of the dimensionless ratio $GM/R$:

$$E_{\text{self-interaction}} = \frac{GM^2}{R}\sum_{n=0}^{\infty} c_n \left(\frac{GM}{R}\right)^n \quad . \tag{1}$$

Each additional term in this expansion diverges even more badly as $R \to 0$. The finite result becomes apparent only in the full, non-perturbative answer! The second comment is that the finite energy of a black hole can be interpreted in terms of the horizon providing a short-distance cutoff. In other words, a gravitational "point charge" is never really pointlike: as far as outside observers can tell, it can never become smaller than the horizon radius $R \sim GM$. Plugging this into the naive energy formula $\sim GM^2/R$, we get an answer $\sim M$, as we should.

Thus, on one hand, GR is more singular than electromagnetism at short distances. But on the other hand, there exists a class of questions (such as a system's total energy), in which electromagnetism is stuck with short-distance singularities, while GR can give sensible finite answers by rephrasing the question from the point of view of distant observers. This observation is at the heart of the holographic approach to quantum gravity.

## III. INFINITESIMAL COORDINATE TRANSFORMATIONS AND LIE DERIVATIVES

So far in this course, we neglected to spell out explicitly the relation between Lie derivatives and infinitesimal coordinate transformations.

Consider a coordinate transformation $x^\mu \to x'^\mu(x)$. Then the components of a tensor $A^\mu{}_\nu$ with e.g. one upper and one lower index transform as:

$$A'^\mu{}_\nu(x') = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}A^\rho{}_\sigma(x) \ , \tag{2}$$

where we took care to notice not just the basis transformation matrices, but also the fact that $A^\mu{}_\nu$ and $A'^\mu{}_\nu$ are evaluated at different values of the coordinates. Now, suppose that our coordinate transformation is infinitesimal, so that $x'^\mu = x^\mu + \xi^\mu(x)$, where $\xi^\mu(x)$ is small. The LHS of the tensor transformation law (2) now reads:

$$A'^\mu{}_\nu(x') = A'^\mu{}_\nu(x) + \xi^\rho\partial_\rho A'^\mu{}_\nu(x) = A'^\mu{}_\nu(x) + \xi^\rho\partial_\rho A^\mu{}_\nu(x) \ , \tag{3}$$

where, in the second term, we interchanged $A'^\mu{}_\nu$ and $A^\mu{}_\nu$: since they're already multiplied by $\xi^\mu$, the difference would only contribute at second order.

Let us now turn to the RHS of (2). The basis transformation matrices read:

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu\xi^\mu \ ; \quad \frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu_\nu - \partial_\nu\xi^\mu \ . \tag{4}$$

To obtain the second matrix, we can simply invert first one, taking into account that $\partial_\nu\xi^\mu$ is infinitesimal. Equivalently, we can derive it from $x^\mu = x'^\mu - \xi^\mu(x)$, taking into account that $\xi^\mu(x)$ and $\xi^\mu(x')$ are interchangeable, since the difference between them would be second-order in $\xi^\mu$. We can now express the RHS of (2) as:

$$(\delta^\mu_\rho + \partial_\rho\xi^\mu)(\delta^\sigma_\nu - \partial_\nu\xi^\sigma)A^\rho{}_\sigma = A^\mu{}_\nu + A^\rho{}_\nu\partial_\rho\xi^\mu - A^\mu{}_\sigma\partial_\nu\xi^\sigma \ . \tag{5}$$

Putting this together with (3), we obtain the change in the components of $A^\mu{}_\nu$ as:

$$\delta A^\mu{}_\nu(x) \equiv A'^\mu{}_\nu(x) - A^\mu{}_\nu(x) = -\xi^\rho\partial_\rho A^\mu{}_\nu(x) + A^\rho{}_\nu\partial_\rho\xi^\mu - A^\mu{}_\sigma\partial_\nu\xi^\sigma \ , \tag{6}$$

which we identify as the Lie derivative:

$$\delta A^\mu{}_\nu(x) = -\mathcal{L}_\xi A^\mu{}_\nu \ . \tag{7}$$

Of course, this derivation works for tensors with any number of upper and lower indices: each index will come with its own term in (5) from the basis transformation matrix, which will match the corresponding term in the Lie derivative. As a special case, we get the transformation law for the metric under infinitesimal coordinate changes:

$$\delta g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu} = -\left(\xi^\rho \nabla_\rho g_{\mu\nu} + g_{\rho\nu}\nabla_\mu \xi^\rho + g_{\mu\rho}\nabla_\nu \xi^\rho\right) = 0 - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu = -2\nabla_{(\mu}\xi_{\nu)} \ , \tag{8}$$

where we wrote out the Lie derivative in terms of covariant derivatives, and then used the covariant derivative's metric compatibility.

**EXERCISES**

**Exercise 1.** *Calculate the Riemann tensor $R_{\mu\nu\rho\sigma}(x)$ to first order in a small metric perturbation $h_{\mu\nu}(x)$.*

**Exercise 2.** *Calculate the first-order Riemann tensor $R_{\mu\nu\rho\sigma}(x)$ for a gravitational wave $h_{\mu\nu}(x) = C_{\mu\nu}e^{ik_\rho x^\rho}$ that satisfies the linearized vacuum Einstein equations.*

**Exercise 3.** *Consider again a gravitational wave $h_{\mu\nu}(x) = C_{\mu\nu}e^{ik_\rho x^\rho}$. Calculate its gauge transformation $\delta h_{\mu\nu}(x)$ under an infinitesimal diffeomorphism $\xi^\mu(x) = \epsilon^\mu e^{ik_\rho x^\rho}$. Show that de Donder gauge $C_{\mu\nu}k^\nu = \frac{1}{2}C^\nu_\nu k_\mu$ is preserved.*