# GR lecture 1-2

Newtonian gravity vs. inertial forces,
non-orthonormal bases, upper and lower indices

## I. INERTIAL FORCES AND NEWTONIAN GRAVITY

We return to our discussion of reference frames. In Newtonian physics, inertial frames are the ones in which the laws are written. A particle's velocity is ill-defined, because it can appear different in different inertial frames. However, acceleration is well-defined: it is the same in all inertial frames, and it enters the force law $F_i = ma_i$. Non-inertial frames are marked by extra, spurious, accelerations. To fit them artificially into the force law, we multiply them by the mass, and package them as "inertial forces". Conversely, if we notice a force that is conspicuously proportional to the particle's mass, and doesn't depend on any other intrinsic property, we should deduce that it is an inertial force.

But wait! When we first learn about the Earth's gravity, we learn it as an acceleration! Only later do we slap on the $m$ to make it into a force $mg_i$. That looks just like an inertial force! Key in our intuition from before: non-inertial frames are curvature of coordinates, gravity is curvature of spacetime itself. Gravity is when all reference frames are non-inertial. It is the inertial force that you can't get rid of by switching frames! Conversely, gravity takes away the notion of well-defined acceleration: there are no longer preferred (inertial) reference frames in which acceleration should be measured. This is one sense of the "extra layer of relativity" that we acquire when switching from SR to GR.

Now let's get more specific. Let there be a bunch of particles in space. They might be interacting with each other via some force laws. We would like to understand two questions:

- Are we in an inertial frame?
- Are the particles subject to a gravitational field?

First, consider the particles' positions. These aren't enough to decide anything: they are arbitrary initial conditions. The same goes for the particles' velocities: even if they look momentarily like the particles are e.g. rotating around an empty point, we cannot deduce that we're in a rotating frame: it could be a weird initial condition. Things get interesting once we come to accelerations. Now we can ask: are the particles' accelerations accounted for by all the (non-gravitational) forces? If not, perhaps there is some force that we missed. But, with enough statistics, we can be smarter, and ask: do the extra accelerations depend in any way on the particles' intrinsic properties? For example, particles of different charges in an electric field will accelerate differently. If that is not the case, if the extra accelerations depend only on the particles' positions and velocities, then we're either dealing with an inertial force or with gravity. Now, how to tell the difference between the two?

Well, we could just "cheat" and use Newton's ~~force~~ acceleration formula:

$$a_i(x) = G \int d^3x' \rho(x') \, \frac{x'_i - x_i}{\left[(x'_j - x_j)(x'_j - x_j)\right]^{3/2}} \; . \tag{1}$$

The acceleration that is given by this formula is gravity, and any additional acceleration must be an inertial force. This is the attitude that everyone took before Einstein. But, fundamentally speaking, something is off with eq. (1). It is non-local: it requires knowing all the masses in the Universe and their positions. Jumping ahead in the story, Special Relativity makes us very suspicious of non-local laws: we like our laws to be local in time, but then SR means that they must also be local in space. So, is there some <u>local</u> measurement, one not requiring information about faraway masses, that would tell gravity from an inertial force?

The mere presence of acceleration is definitely not enough: it can always arise from an accelerated frame. A time-dependent acceleration is not a smoking gun for gravity either: it can always be reproduced by a frame shift of the form $x_i \to x_i + f_i(t)$. So the difference must lie in <u>spatial</u> derivatives of the acceleration: we should be treating acceleration as a field $a_i(x)$. What spatial derivatives of this field can we write? The simplest one is the divergence $\partial_i a_i$. This is actually fixed by the local version of Newton's formula (1):

$$\partial_i a_i = -G\rho(x) \; . \tag{2}$$

In fact, eq. (1) is equivalent to (2) <u>together with the non-local assumption</u> that the acceleration "at infinity", i.e. far from all masses, vanishes. So, the divergence (2) is a promising candidate for a locally measurable quantity that signals the presence of gravity.

**Exercise 1.** *Consider a surface composed of probe particles. The surface initially encloses a volume $V$, within which there is a mass $M$. The probe particles' initial velocity is zero. Find the second time derivative $\ddot{V}$ in the first instant after the particles are released.*

The divergence (2) has one crucial limitation as a probe of gravity: it is only non-vanishing at points where $\rho(x)$ is nonzero, i.e. when we're on top of the gravity's source. Can gravity be measured away from its source? Is there a local observation on the Earth that can detect the gravitational influence of the Moon? There is, but we must look at more detailed spatial derivatives of $a_i$. We may try to consider the curl $\epsilon_{ijk}\partial_j a_k$ (or simply its bivector version $2\partial_{[i}a_{j]} = \partial_i a_j - \partial_j a_i$). However, that always vanishes: otherwise, inertial/gravitational forces would not be conservative, and we could use them to build a perpetual motion machine!

**Exercise 2** (ADVANCED)**.** *Identify the property of the Riemann tensor that is responsible for the vanishing of $\partial_{[i}a_{j]}$.*

This leaves the symmetric traceless ("spin-2") component of the spatial derivative $\partial_i a_j$, i.e. $\partial_{(i}a_{j)} - \partial_k a_k \delta_{ij}/3$. That is indeed non-vanishing for the Newtonian field (1), even away from source masses. This component of $\partial_i a_j$ is responsible for the tides, and it is what kills you when you fall into a black hole.

**Exercise 3.** *Consider a small block of probe particles hanging above the Earth, at radius $R$ from the Earth's center. The Earth's mass is $M$. The block's height is $h$, and its base area is $A$. The particles are initially at rest. Find the second time derivatives $\ddot{h}$ and $\ddot{A}$ in the first instant after the particles are released. Check for consistency with exercise 1!*

Our analysis has one remaining weakness. While the focus on $\partial_i a_j$ rules out inertial forces from uniform acceleration, it does not rule out inertial forces from rotation!

**Exercise 4.** *Consider centrifugal acceleration arising from a rotating frame with angular velocity $\omega_{ij}$. Calculate $\partial_i a_j(x)$. Is there a mass distribution whose gravity would reproduce the same acceleration field? The answer is a bit subtle.*

In fact, in our lives on the surface of the Earth, we experience centrifugal acceleration due to the Earth's rotation together with the acceleration of gravity, and the two are difficult to distinguish. One could, of course, notice that the stars are rotating overhead, but that is not a local observation... However, a rotating frame can, quite famously, be detected by local observation, via the Coriolis ~~force~~ acceleration $2\omega_{ij}v^j$. This goes beyond considering $a_i(x)$ for particles initially at rest, and probes the dependence of the acceleration on the particle's velocity $v_i$. So, we arrive at the following conclusions vis. Newtonian gravity and inertial forces:

- Gravity is not a force, and also not quite an acceleration: acceleration at a single point can be either due to gravity or due to an inertial force. There is no way to tell. This is called the "equivalence principle". In fact, the main difference between a Universe with gravity and one without is that without gravity, acceleration is well-defined (after ruling out inertial forces), but with gravity it's not.

- Having ruled out a rotating frame by measuring the Coriolis force, the gravitational field is locally measured by the gradient of the acceleration $\partial_i a_j$. The antisymmetric part $\partial_{[i}a_{j]}$ always vanishes. The trace part $\partial_i a_i$ measures the local density of the masses which source the gravitational field. The traceless part is capable of "propagating" away from the source masses, into the surrounding space.

**Exercise 5.** *Let's be more systematic about the classification of inertial forces.*

1. *Write down the most general coordinate transformation $x_i \to x'_i(x_j, t)$ such that the $x'_i$ are still Cartesian coordinates at every time $t$. What parameters does your transformation depend on? If the parameters are subject to constraints, specify them.*

2. *Consider a particle with trajectory $x_i(t)$ in the initial frame. Write down its acceleration $\ddot{x}'_i$ in the new frame. Identify terms that reproduce the effects of 1) uniform acceleration, 2) angular acceleration, 3) the centrifugal force, and 4) the Coriolis force. Show that this exhausts all the possible inertial forces.*

P.S. We've seen that if we allow some assumptions about spatial infinity – e.g. that all the masses are clustered in some finite volume, that the "true" gravitational acceleration at infinity vanishes, that there exist "fixed stars" which we can use to notice a rotating frame, etc. – then the "extra layer of relativity" goes away: accelerations can again be well-defined, not just their gradients. This is the basis of the holographic approach to quantum gravity, where we avoid the complications of GR by purposefully focusing on spatial infinity.

## II. ARBITRARY BASES, UPPER AND LOWER INDICES

So far, in our treatment of tensors and indices, we've been assuming an orthonormal basis $(x, y, z)$. Sometimes, it's a good idea to consider non-orthonormal bases for our tensors:

- In condensed matter, there is sometimes a non-orthonormal basis adapted to a crystal lattice.
- In Special Relativity, there isn't quite an orthonormal basis in the usual sense, because $t$ is different from $(x, y, z)$.
- In General Relativity, spacetime is curved, thus we will have no choice but to deal with non-orthonormal coordinate axes.

When discussing non-orthonormal bases, the difference between covariant ("lower") and contravariant ("upper") indices becomes important. We will now gradually begin to introduce these. In addition, we will switch to denoting our axes as $(1, 2, 3)$ rather than $(x, y, z)$, to emphasize that they can be more general. Finally, for the current discussion, it's instructive to combine the old-fashioned vector notation with the modern index notation: a vector $\mathbf{v}$ is defined by its components $v^i$, via:

$$\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3 = v^i\mathbf{e}_i \ , \tag{3}$$

where $\mathbf{e}_i = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ are the basis elements. Suppose for now that this basis is orthonormal: $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Then all the formulas from the last lecture make sense. We can change to a different orthonormal basis by a rotation (and perhaps reflection) of the basis: $\mathbf{e}_i \to M_i{}^j\mathbf{e}_j$. To preserve the orthonormality, we must have $M_i{}^k M_j{}^l \delta_{kl} = \delta_{ij}$, or, in matrix notation, $M^T M = 1$. As expected, the symmetry group that preserved orthonormality is the group $O(3)$ of orthogonal matrices.

Now, let us allow arbitrary linear changes of basis:

$$\mathbf{e}_i \ \to \ M_i{}^j\,\mathbf{e}_j \ . \tag{4}$$

Thus, we "extend our symmetry group" from $O(3)$ to the full group $GL(3)$ of $3\times3$ matrices. Now we no longer have the orthonormality $\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$. Instead, the right-hand side becomes:

$$\mathbf{e}_i \cdot \mathbf{e}_j = M_i{}^k M_j{}^k \equiv g_{ij} \ . \tag{5}$$

This matrix $g_{ij}$ is called the metric, and will be absolutely crucial for us in the future. For an arbitrary basis $\mathbf{e}_i$, the metric is an arbitrary symmetric matrix, up to a small but important caveat: its square-type structure $g = MM^T$ (using matrix notation) serves as a kind of positivity constraint. This constraint actually encodes the existence of an orthonormal basis, in which the basis elements all square to 1. A completely general symmetric matrix $g_{ij}$ describes a geometry in which there is at best a "pseudo-orthonormal" basis, in which the basis elements can square to either 1, $-1$ or 0.

Under the basis change (4), the components $v^i$ of a vector $\mathbf{v}$ must change as well (this is of course true for both orthogonal and non-orthogonal transformations). In particular, for the decomposition (3) to remain true, the components $v^i$ must transform as:

$$v^i \;\rightarrow\; (M^{-1})_j{}^i\, v^j \;, \tag{6}$$

where $(M^{-1})_i{}^j$ is the matrix inverse to $M_i{}^j$. Note that, if we pay attention to the index placement, the matrices acting on $\mathbf{e}_i$ vs. $v^i$ are related through an inverse and transpose operation $M \leftrightarrow (M^{-1})^T$. Since the basis is no longer orthonormal, the scalar product is no longer given by $u^i v^i$. Instead, combining (3) and (5), we have:

$$\mathbf{u} \cdot \mathbf{v} = g_{ij} u^i v^j \;. \tag{7}$$

In particular, the length-squared of a vector is given by $\mathbf{v} \cdot \mathbf{v} = g_{ij} v^i v^j$. More generally, the scalar product (7) contains all the geometric information about lengths and angles. Thus, the metric $g_{ij}$ is all we need to do geometry in the general basis $\mathbf{e}_i$.

Some rules are beginning to emerge here:

- Upper and lower indices transform differently under changes of basis.
- An upper index can be contracted with a lower index.
- To take a scalar product, we cannot contract two upper indices directly. Instead, we must contract them both with the two lower indices of $g_{ij}$.

In principle, at this point we can forget about the basis vectors $\mathbf{e}_i$, and go back to using just vector components $v^i$ (and higher-order tensor components $T^{ijk\dots}$), armed with the metric $g_{ij}$ for whenever we want to contract two upper indices. Or we can even remember the scalar product rule (7) at the back of our heads, revert to using the old notation $u_i v_i$, and forget all about upper vs. lower indices. That is the approach of old books on Special Relativity, including the Feynman Lectures. Instead, the modern approach takes a middle path, which we will now describe.

## III. THE CO-BASIS; COVARIANT VS. CONTRAVARIANT VECTORS

Let's have our cake and eat it whole. Our basis elements $\mathbf{e}_i$ are non-orthonormal, as in (5). But we can introduce a second basis $\mathbf{e}^i$ (note the different index placement!) whose elements are chosen to have nice 0 or 1 scalar products with the elements of the original one:

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j \ . \tag{8}$$

Thus, the direction of e.g. $\mathbf{e}^1$ is chosen orthogonal to the plane of $\mathbf{e}_2$ and $\mathbf{e}_3$; its magnitude is then chosen to give a unit scalar product with $\mathbf{e}_1$. Explicitly, this can be achieved by choosing:

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)} \ , \tag{9}$$

and similarly for $\mathbf{e}^2$ and $\mathbf{e}^3$. If the $\mathbf{e}_i$ are our original basis, then we refer to the $\mathbf{e}^i$ as the co-basis (just like in sine vs. cosine). We can define the components $v_i$ of a vector $\mathbf{v}$ in the co-basis, via:

$$\mathbf{v} = v_i \mathbf{e}^i \ . \tag{10}$$

Under a general change of basis (4), the co-basis must change as well, so as to preserve (8):

$$\mathbf{e}^i \ \rightarrow \ (M^{-1})_j{}^i \, \mathbf{e}^j \ . \tag{11}$$

Then, to preserve (10), the components $v_i$ of a vector must transform as:

$$v_i \ \rightarrow \ M_i{}^j \, v_j \ . \tag{12}$$

Since the components $v_i$ in the co-basis transform just like the basis elements $\mathbf{e}_i$, they are called covariant. Since the components $v^i$ in the original basis transform in the opposite way, they are called contravariant. Yes, this is confusing, and in practice I prefer simply saying "upper-index" or "lower-index".

What is the relationship between the components $v^i$ and $v_i$ of the same vector $\mathbf{v}$? Let us start from $v_i \mathbf{e}^i = v^i \mathbf{e}_i$, and take the scalar product of both sides with $\mathbf{e}_j$. Using (5) and (8), this gives:

$$v_i = g_{ij} v^j \ . \tag{13}$$

Thus, the metric $g_{ij}$ can be used to lower indices.

**Exercise 6.** *Prove that this also works for the basis elements, i.e. that* $\mathbf{e}_i = g_{ij}\mathbf{e}^j$.

It follows that, to raise indices, we can use the inverse of the metric:

$$v^i = (g^{-1})^{ij} v_j \ . \tag{14}$$

From now on, we will refer to the inverse matrix $(g^{-1})^{ij}$ simply as $g^{ij}$. As always, whatever can be done to a vector, can also be done to separate indices of a tensor, e.g.:

$$T_{ijkl} = g_{lm} T_{ijk}{}^m \ . \tag{15}$$

Similarly, under a change of basis, the components of a tensor will transform as:

$$T_{ijk}{}^m \to M_i{}^n M_j{}^p M_k{}^q (M^{-1})_r{}^m \, T_{npq}{}^r \ . \tag{16}$$

**Exercise 7.** *Prove that* $(g^{-1})^{ij} \equiv g^{ij}$ *really is the raised-index version of* $g_{ij}$*, according to the rules* (13)-(14).

**Exercise 8.** *Prove that* $g^{ij}$ *can be defined alternatively as* $\mathbf{e}^i \cdot \mathbf{e}^j$.

The main upshot of this song and dance is that the scalar product of two vectors can now be written simply as:

$$\mathbf{u} \cdot \mathbf{v} = g_{ij} u^i v^j = u_i v^i = u^i v_i \ . \tag{17}$$

Thus, we come back to the simple contraction of repeated indices, as long as we remember that one has to be upper, and the other lower.

The above discussion in terms of basis vs. co-basis should be illuminating for some, and mystifying for others. At the end of the day, we can again throw away the explicit basis vectors, and work just with components $v^i$ or $v_i$. The metric now almost never has to appear in formulas explicitly – we can hide it away most of the time as in (17). The important new rules to remember are:

1. Free indices must be the same on both sides of an equation, including their upper/lower placement.

2. A contracted index pair must always be one upper & one lower.

## IV. INVARIANT TENSORS

When we were working with orthonormal bases, we had the special tensor $\delta_{ij}$, invariant under $O(3)$, and the special tensor $\epsilon_{ijk}$, invariant under $SO(3)$. In the new general framework, the role of $\delta_{ij}$ splits into three different objects. Two of these we've encountered: the metric $g_{ij}$ and the inverse metric $g^{ij}$. Their components are not invariant under general $GL(3)$ changes of basis. However, there is also a version with one index up and the other down, which is again a Kronecker delta $\delta^i_j$, with components 0 for $i \neq j$ and 1 for $i = j$.

**Exercise 9.**

1. *Prove that $\delta^i_j$ is invariant under a general change of basis.*
2. *Prove that $\delta^i_j$ can be regarded as a version of $g_{ij}$ with one index raised, or of $g^{ij}$ with one index lowered.*

For the Levi-Civita tensor $\epsilon_{ijk}$ in the context of general bases, there are two commonly used conventions, each with advantages and disadvantages. In one of them, we maintain $\epsilon_{123} = \epsilon^{123} = 1$; then the Levi-Civita does not depend on the metric, but doesn't quite behave as a tensor. In the other, we set $\epsilon_{123} = \sqrt{g}$ and $\epsilon^{123} = 1/\sqrt{g}$, where $g$ is the determinant of the metric $g_{ij}$; this definition does behave as a tensor, at the cost of depending on the metric. It may be best to avoid this discussion for now - we've had enough $\epsilon$'s for the time being.