Conformal higher-spin gauge models in curved backgrounds

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Based on:

S. M. Kuzenko, M. Ponds and E. S. N. Raptakis
New locally (super)conformal gauge models in Bach-flat backgrounds
JHEP (in press) [arXiv:2005.08657]

S. M. Kuzenko and M. Ponds
Generalised conformal higher-spin fields in curved backgrounds
JHEP 2004 (2020) 021 [arXiv:1912.00652]

S. M. Kuzenko and M. Ponds
Conformal geometry and (super)conformal higher-spin gauge theories
JHEP 1905 (2019) 113 [arXiv:1902.08010]

Emmanouil Raptakis

Historical background and motivation for this work

- Old conjecture/expectation regarding consistent propagation of conformal higher spin (CHS) fields on Bach-flat backgrounds, going back to E. Fradkin & A. Tseytlin (1985).
- T. Nutma & M. Taronna, arXiv:1404.7452
- M. Grigoriev & A. Tseytlin, arXiv:1609.09381
- Superconformal higher spin (SCHS) models
  SMK, R. Manvelyan & S. Theisen, arXiv:1701.00682
  include several conformal gauge fields.
- M. Beccaria & A. Tseytlin, arXiv:1702.00222
- R. Manvelyan & G. Poghosyan, arXiv:1804.10779

Historical background

- Free CHS models (Fradkin & Tseytlin, 85) are higher-spin extensions of Maxwell's action and the linearised action for conformal gravity in $\mathbb{M}^4$ (conformal graviton model).
- Maxwell's action is gauge invariant in any curved space.
- Gauge invariance of the conformal graviton can be ensured only on Bach-flat backgrounds,

$$B_{ab} = (\mathcal{D}^c\mathcal{D}^d + \frac{1}{2}R^{cd})C_{cabd} = 0 \ .$$

Indeed, consider the nonlinear action for conformal gravity

$$S_{\rm CG} \propto \int \mathrm{d}^4x \, \sqrt{-g} \, (C_{abcd})^2 \ .$$

Its EoM is $B_{ab} = 0$. Picking a solution and linearising $S_{\rm CG}$ about this background leads to a gauge-invariant action for conformal graviton (spin 2).

Historical background

- Conformal gravitino model (spin 3/2) can also be lifted to Bach-flat backgrounds. (Linearisation of conformal SUGRA)
- Old expectation: Gauge-invariant actions for pure conformal spin-s fields should exist for $s > 2$ on Bach-flat backgrounds.
- Conformal spin-3 field.

  T. Nutma & M. Taronna, arXiv:1404.7452

  R. Manvelyan & G. Poghosyan, arXiv:1804.10779

  Attempts to construct a gauge-invariant action for spin-3 succeeded only to first order in the background curvature.
- Old expectation is ruined.

  M. Grigoriev & A. Tseytlin, arXiv:1609.09381

  M. Beccaria & A. Tseytlin, arXiv:1702.00222

Historical background

- Building on the complete interacting bosonic CHS theory

  A. Segal, hep-th/0207212

  Grigoriev & Tseytlin and later Beccaria & Tseytlin argued that, in Bach flat backgrounds, the pure spin-3 action cannot be made gauge invariant on its own beyond first order in the background curvature. Coupling to a conformal spin-1 field is required.
- Conformal spin-3 story remains incomplete.
- Supersymmetry considerations imply that spin-3 field should couple to spin-1 and spin-2 fields. (To be discussed in this talk.)
- Complete gauge-invariant actions in Bach-flat backgrounds were constructed for the following models: (i) conformal maximal depth fields with spin $s = 5/2$ and $s = 3$; (ii) conformal pseudo-graviton (hook field); (iii) superconformal pseudo-graviton multiplet. The models (i) and (ii) will be discussed in this talk.

SUGRA & higher spin gauge fields

- Massless spin-3/2 field in curved backgrounds $\Longrightarrow$ SUGRA
  Buchdahl (1958) $\Longrightarrow$ Deser & Zumino (1976)
  Ferrara, Freedman & van Nieuwenhuizen (1976)
- SUGRA (1976) $\Longrightarrow$ Higher spin gauge models (1978)
  Fronsdal, Fang & Fronsdal
- Conformal SUGRA $\Longrightarrow$ SCHS multiplets
  Kaku, Townsend & van Nieuwenhuizen (1977)
  Ferrara & Zumino (1978)
  Howe, Stelle & Townsend (1981)
- Conformal SUGRA $\Longrightarrow$ Free CHS gauge models
  Fradkin & Tseytlin (1985)
- Off-shell 4D $\mathcal{N} = 2$ supergravity $\Longrightarrow$ 4D massless higher spin theory
  Fradkin & Vasiliev (1979)
  de Wit & van Holten (1979)
  Fradkin & Vasiliev (1986)
- 3D (p, q) AdS supergravity $\Longrightarrow$ 3D higher spin theory
  Achúcarro & Townsend (1986)
  Blencowe (1988) ....

From conformal SUGRA to CHS gauge (super)fields

- $\mathcal{N} = 1$ conformal SUGRA is described by a real unconstrained prepotential $H_{\alpha\dot{\alpha}} = (\sigma_m)_{\alpha\dot{\alpha}} H^m$.
- Linearised gauge freedom

$$\delta H_{\alpha\dot{\alpha}} = \bar{D}_{\dot{\alpha}} \Lambda_\alpha - D_\alpha \bar{\Lambda}_{\dot{\alpha}} \ ,$$

with the gauge parameter Λ_α being unconstrained.

Ferrara & Zumino (1978)

- Gravitational superfield

$$H^m(\theta, \bar{\theta}) = \cdots + \theta\sigma^a\bar{\theta} e_a{}^m + \bar{\theta}^2\theta^\alpha\psi_\alpha{}^m + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}m} + \theta^2\bar{\theta}^2 A^m$$

Ogievetsky & Sokatchev (1977); Siegel (1977)

$e_a{}^m$	vielbein
$\psi_\alpha{}^m$	gravitino
A^m	R-symmetry gauge field

Conformal gauge superfields: Half-integer superspin

Superspin-$(s+\frac{1}{2})$ $s = 1, 2, \dots$

Conformal prepotential $H_{\alpha(s)\dot\alpha(s)} \equiv H_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s}$ is a real superfield, symmetric in its undotted indices, and in its dotted indices.

Gauge transformation law:

$$\delta H_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} = \bar D_{(\dot\alpha_1}\Lambda_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_s)} - D_{(\alpha_1}\bar\Lambda_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_s} ,$$

with unconstrained gauge parameter $\Lambda_{\alpha(s)\dot\alpha(s-1)}$.

Howe, Stelle & Townsend (1981)

The $s = 1$ case corresponds to linearised conformal supergravity

Ferrara & Zumino (1978)

Weak Wess-Zumino gauge:

$$\begin{aligned} H_{\alpha(s)\dot\alpha(s)}(\theta,\bar\theta) &= \theta^\beta\bar\theta^{\dot\beta}e_{\beta\dot\beta,\alpha(s)\dot\alpha(s)} + \bar\theta^2\theta^\beta\psi_{\beta,\alpha(s)\dot\alpha(s)} \\ &\quad -\theta^2\bar\theta^{\dot\beta}\bar\psi_{\alpha(s)\dot\beta,\dot\alpha(s)} + \theta^2\bar\theta^2 A_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} . \end{aligned}$$

$e_{\beta\dot\beta,\alpha(s)\dot\alpha(s)}$ may be identified with Vasiliev's generalised vielbein.

Conformal gauge superfields: Half-integer superspin

Strong Wess-Zumino gauge:

$$H_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s}(\theta,\bar\theta) = \theta^\beta\bar\theta^{\dot\beta}h_{(\beta\alpha_1\dots\alpha_s)(\dot\beta\dot\alpha_1\dots\dot\alpha_s)} + \bar\theta^2\theta^\beta\psi_{(\beta\alpha_1\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_s} -\theta^2\bar\theta^{\dot\beta}\bar\psi_{\alpha_1\dots\alpha_s(\dot\beta\dot\alpha_1\dots\dot\alpha_s)} + \theta^2\bar\theta^2 h_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} ,$$

bosonic fields $h_{\alpha(s+1)\dot\alpha(s+1)}$ and $h_{\alpha(s)\dot\alpha(s)}$ are real.

Residual gauge freedom: $(\mathcal{H}_0 = \theta\sigma^a\bar\theta\partial_a)$

$$\bar D_{(\dot\alpha_1}\Lambda_{\alpha(s)\dot\alpha_2\dots\dot\alpha_s)} = \mathrm{e}^{\mathrm{i}\mathcal{H}_0}\Big\{ -\frac{\mathrm{i}}{2}\zeta_{\alpha(s)\dot\alpha_1\dots\dot\alpha_s} + \mathrm{i}\bar\theta_{(\dot\alpha_1}\rho_{\alpha(s)\dot\alpha_2\dots\dot\alpha_s)} -\mathrm{i}\theta_{(\alpha_1}\bar\rho_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_s} + \frac{s}{s+1}\theta^\beta\bar\theta_{(\dot\alpha_1}\partial_{(\beta}{}^{\dot\gamma}\zeta_{\alpha_1\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_{s-1}\dot\gamma} -\frac{1}{2}\frac{s^2}{(s+1)^2}\theta_{(\alpha_1}\bar\theta_{(\dot\alpha_1}\partial^{\gamma\dot\gamma}\zeta_{\alpha_2\dots\alpha_s)\gamma\dot\alpha_2\dots\dot\alpha_s)\dot\gamma} - 2\mathrm{i}\,\theta_{(\alpha_1}\bar\theta_{(\dot\alpha_1}\zeta_{\alpha_2\dots\alpha_s)\dot\alpha_2\dots\alpha_s)} -\frac{s}{s+1}\theta^2\bar\theta_{(\dot\alpha_1}\partial_{(\alpha_1}{}^{\dot\gamma}\bar\rho_{\alpha_2\dots\alpha_s)\gamma\dot\alpha_2\dots\dot\alpha_s)} \Big\} ,$$

where the bosonic parameters $\zeta_{\alpha(s)\dot\alpha(s)}$ and $\zeta_{\alpha(s-1)\dot\alpha(s-1)}$ are real.

Conformal gauge superfields: Half-integer superspin

Residual gauge transformations:

$$
\begin{aligned}
\delta h_{\alpha_1 \dots \alpha_{s+1} \dot\alpha_1 \dots \dot\alpha_{s+1}} &= \partial_{(\alpha_1 (\dot\alpha_1} \zeta_{\alpha_2 \dots \alpha_{s+1}) \dot\alpha_2 \dots \dot\alpha_{s+1})} \,, \\
\delta h_{\alpha_1 \dots \alpha_s \dot\alpha_1 \dots \dot\alpha_s} &= \partial_{(\alpha_1 (\dot\alpha_1} \zeta_{\alpha_2 \dots \alpha_s) \dot\alpha_2 \dots \dot\alpha_s)} \,, \\
\delta \psi_{\alpha_1 \dots \alpha_{s+1} \dot\alpha_1 \dots \dot\alpha_s} &= \partial_{(\alpha_1 (\dot\alpha_1} \rho_{\alpha_2 \dots \alpha_{s+1}) \dot\alpha_2 \dots \dot\alpha_s)} \,.
\end{aligned}
$$

Switch to vector notation: $h_{\alpha_1 \dots \alpha_s \dot\alpha_1 \dots \dot\alpha_s} \to h_{a_1 \dots a_s} = h_{(a_1 \dots a_s)} \equiv h_{a(s)}$

$$
h_{a_1 \dots a_s} := \left(-\frac{1}{2}\right)^s (\sigma_{a_1})^{\alpha_1 \dot\alpha_1} \dots (\sigma_{a_s})^{\alpha_s \dot\alpha_s} h_{\alpha_1 \dots \alpha_s \dot\alpha_1 \dots \dot\alpha_s}
$$

and extend the symmetric traceless field $h_{a(s)}$ to a symmetric field $\boldsymbol{h}_{a(s)}$ by introducing compensating degrees of freedom. Gauge symmetry:

$$
\delta \boldsymbol{h}_{a_1 \dots a_s} = \partial_{(a_1} \xi_{a_2 \dots a_s)} + \eta_{(a_1 a_2} \lambda_{a_3 \dots a_s)} \,, \qquad \eta^{bc} \xi_{bca_1 \dots a_{s-3}} = 0 \,,
$$

SCHS multiplets contain the CHS fields of Fradkin & Tseytlin (1985)

Free conformal higher-spin theories in $\mathbb{M}^4$

- Spin-s bosonic CHS field is a real, traceful & totally symmetric tensor:

$$\varphi_{a_1\dots a_s}(x) = \varphi_{(a_1\dots a_s)}(x) \equiv \varphi_{a(s)}(x) , \qquad \varphi^b{}_{ba(s-2)} \neq 0$$

- CHS action originally formulated in terms of projectors

$$S^{(s)}_{\text{CHS}} = \int \mathrm{d}^4x \, \varphi^{a(s)} \Box^s \Pi_{(s)} \varphi_{a(s)} , \qquad \Box := \eta^{ab}\partial_a\partial_b$$

E.S. Fradkin & A.A. Tseytlin (1985)

- Operators $\Pi_{(s)}$ project onto traceless and transverse subspace

$$\eta^{bc}\Pi_{(s)}\varphi_{bca(s-2)} = 0 , \qquad \partial^b\Pi_{(s)}\varphi_{ba(s-1)} = 0$$

R.E. Behrends & C. Fronsdal (1957)

- Action has HS gauge symmetry and 'generalised' algebraic Weyl symmetry

$$\delta_\xi\varphi_{a(s)} = \partial_{(a_1}\xi_{a_2\dots a_s)} , \qquad \delta_\lambda\varphi_{a(s)} = \eta_{(a_1a_2}\lambda_{a_3\dots a_s)}$$

- Action is invariant under conformal transformations of $\mathbb{M}^4$
- Convenient to fix algebraic symmetry by gauging away trace of CHS field

$$\varphi_{a(s)} \equiv h_{a(s)} , \qquad h^b{}_{ba(s-2)} = 0$$

The vielbein formalism

- Spacetime manifold parametrized by local coordinates x^m
- Introduce orthonormal basis e_a in the tangent space at each point

$$e_a = e_a{}^m(x)\partial_m \ , \qquad g_{mn}(x)e_a{}^m(x)e_b{}^n(x) = \eta_{ab} \ .$$

- Convert all 'world tensors' to 'Lorentz tensors' using the vielbein

$$\text{e.g.} \qquad V_m(x) \mapsto V_a(x) = e_a{}^m(x)V_m(x)$$

- Replace regular derivative with Lorentz covariant derivative

$$\mathcal{D}_a = e_a{}^m\partial_m - \frac{1}{2}\omega_a{}^{bc}M_{bc} \ , \qquad \big[\mathcal{D}_a, \mathcal{D}_b\big] = -T_{ab}{}^c\mathcal{D}_c - \frac{1}{2}R_{ab}{}^{cd}M_{cd} \ .$$

- The Lorentz connection ω_{abc} is determined by the vielbein through torsion-free constraint,

$$T_{ab}{}^c = 0 \quad \Longrightarrow \quad \omega_{abc} \equiv \omega_{abc}(e) \ .$$

Linearised conformal gravity (spin-2)

- **Top down approach**: Conformal gravity action

$$S_{\rm CG} = \int {\rm d}^4x\, e\, C^{abcd} C_{abcd} \ , \qquad e = \det(e_m{}^a)$$

- Invariant under Weyl (local scale) transformations of the vielbein

$$\delta_\sigma e_a{}^m = \sigma(x) e_a{}^m \quad \Longrightarrow \quad \begin{cases} \delta_\sigma e = -4\sigma e \\ \delta_\sigma C_{abcd} = 2\sigma C_{abcd} \end{cases} \quad \Longrightarrow \quad \delta_\sigma S_{\rm CG} = 0$$

- Equation of motion: $B_{ab} = 0$ (Bach-flat geometry)

$$e_b{}^m \frac{\delta S_{\rm CG}}{\delta e_m{}^a} \propto B_{ab} = (\mathcal{D}^c \mathcal{D}^d + \frac{1}{2} R^{cd}) C_{cabd} \qquad \text{Bach tensor}$$

- Perturb the vielbein around a (Bach-flat) background geometry

$$e_a{}^m \ \to \ \tilde{e}_a{}^m = e_a{}^m + h_a{}^b e_b{}^m \ , \qquad |h_{ab}| \ll 1$$

- Gauge freedom allows to choose h_{ab} to be symmetric and traceless

$$h_{ab} = h_{ba} \ , \qquad \eta^{ab} h_{ab} = 0$$

Linearised conformal gravity (spin-2)

- Weyl action to quadratic order in h_{ab}

$$S^{(\text{Lin})}_{\text{CG}} = \int \mathrm{d}^4x\, e\, \widetilde{C}^{abcd}\widetilde{C}_{abcd} = \int \mathrm{d}^4x\, e \left\{ {}^{(1)}C_{abcd}\,{}^{(1)}C^{abcd} + 2C_{abcd}\,{}^{(2)}C^{abcd} \right\}$$

- $\widetilde{C}_{abcd} = C_{abcd} + {}^{(1)}C_{abcd} + {}^{(2)}C_{abcd}$

$$\begin{aligned} {}^{(1)}C_{abcd} = &- 2h^f{}_{[a}C_{b]fcd} + 4P_{[a\{c}h_{d\}b]} + 4P^f{}_{[a}\eta_{b]\{c}h_{d\}f} + \frac{4}{3}\eta_{a[c}\eta_{d]b}h^{fg}P_{fg} \\ &- 4\eta_{[a\{c}h_{d\}b]}P^f{}_f - 4\mathcal{D}_{[a}\mathcal{D}_{\{c}h_{d\}b]} - 2\eta_{[c\{a}\mathcal{D}^f\mathcal{D}_{b\}}h_{d]f} \\ &- 2\eta_{[a\{c}\mathcal{D}^f\mathcal{D}_{d\}}h_{b]f} + \frac{2}{3}\eta_{a[c}\eta_{d]b}\mathcal{D}^f\mathcal{D}^g h_{fg} + 2\eta_{[a\{c}\Box h_{d\}b]} \end{aligned}$$

$${}^{(2)}C_{abcd} = \;\ldots(\text{ insert mess })\ldots$$

- Schouten tensor: $P_{ab} = \frac{1}{2}R_{ab} - \frac{1}{12}\eta_{ab}R$
- Invariant under linearised diffeomorphisms (gauge) and Weyl transformations

$$\begin{cases} \delta_\xi h_{ab} &= \mathcal{D}_{(a}\xi_{b)} - \frac{1}{4}\eta_{ab}\mathcal{D}^c\xi_c \\ \delta_\sigma h_{ab} &= 0 \\ \delta_\sigma e_a{}^m &= \sigma e_a{}^m \end{cases} \quad\Longrightarrow\quad \delta_\xi S^{(\text{Lin})}_{\text{CG}} = \delta_\sigma S^{(\text{Lin})}_{\text{CG}} = 0$$

Extension to higher-spin ?

- How to generalise top down approach to higher-spin?
- Non-linear CHS induced action: A.A. Tseytlin (2002)

  A. Segal (2002)

  X. Bekaert, E. Joung & J. Mourad (2011)

- Linearisation procedure with non-trivial background metric is non-trivial

  M. Grigoriev & A.A. Tseytlin (2016)

  M. Beccaria & A.A. Tseytlin (2017)

- **Bottom up approach**: Construct the linearised actions based on symmetry principles (gauge & Weyl invariance) + flat-space limit
- **Idea**: Start with minimal lift of flat-space model and add non-minimal (NM) corrections which ensure Weyl and gauge invariance

$$S_{\text{CHS}}[h,\eta] \quad \longmapsto \quad S_{\text{CHS}}[h,g] \; + \; S_{\text{NM}}$$

- **Complications**:
  1. Conformal higher-spin = higher-derivatives
  2. Curvature dependent terms
  3. Many possible terms

Conformal gravity: Gauging the conformal algebra

- **Solution**: Make local conformal symmetry manifest by enriching the local structure group of the space-time manifold to the conformal algebra
  M. Kaku, P.K. Townsend & P. van Nieuwenhuizen (1977)
- Modern formulation: D. Butter (2009)
  D. Butter, SMK, J. Novak & G. Tartaglino-Mazzucchelli (2013)
- Conformal algebra $\mathfrak{so}(4,2)$:

$$
\begin{aligned}
[M_{ab}, M_{cd}] &= 2\eta_{c[a}M_{b]d} - 2\eta_{d[a}M_{b]c} \ , & [\mathbb{D}, K_a] &= -K_a \ , \\
[M_{ab}, K_c] &= 2\eta_{c[a}K_{b]} \ , & [\mathbb{D}, P_a] &= P_a \ , \\
[M_{ab}, P_c] &= 2\eta_{c[a}P_{b]} \ , & [K_a, P_b] &= 2\eta_{ab}\mathbb{D} + 2M_{ab} \ .
\end{aligned}
$$

- 'Gauging' $\mathfrak{so}(4,2)$: For each generator associate a connection one-form,

$$
P_a \Longleftrightarrow e_a{}^m(x) \ , \quad M_{ab} \Longleftrightarrow \Omega_{abc}(x) \ , \quad K_a \Longleftrightarrow \mathfrak{f}_{ab}(x) \ , \quad \mathbb{D} \Longleftrightarrow \mathfrak{b}_a(x)
$$

- Introduce conformal covariant derivative:

$$
\underbrace{\nabla_a = e_a{}^m\partial_m - \frac{1}{2}\Omega_a{}^{bc}M_{bc} - \mathfrak{f}_a{}^bK_b - \mathfrak{b}_a\mathbb{D}}_{\text{Conformal covariant derivative}} \ , \qquad \underbrace{\mathcal{D}_a = e_a{}^m\partial_m - \frac{1}{2}\omega_a{}^{bc}M_{bc}}_{\text{Lorentz covariant derivative}}
$$

Conformal gravity: Constraining the algebra

- Commutator of covariant derivatives:

$$\big[\nabla_a, \nabla_b\big] = -\mathcal{T}_{ab}{}^c\nabla_c - \frac{1}{2}\mathcal{R}(M)_{ab}{}^{cd}M_{cd} - \mathcal{R}(K)_{ab}{}^cK_c - \mathcal{R}_{ab}(\mathbb{D})\mathbb{D}$$

- There are four independent fields instead of one: $\big\{e_m{}^a,\ \Omega_a{}^{bc},\ \mathfrak{f}_a{}^b, \mathfrak{b}_a\big\}$
- Need to impose covariant constraints amongst 'extra' connections:

(A) $\mathcal{T}_{ab}{}^c = 0 \quad \Longrightarrow \quad \Omega_{abc} = \Omega_{abc}(e, \mathfrak{b})$

(B) $\eta^{bd}\mathcal{R}(M)_{abcd} = 0 \quad \Longrightarrow \quad \mathfrak{f}_{ab} = \mathfrak{f}_{ab}(e, \mathfrak{b})$

- Under these constraints the algebra is determined by the Weyl tensor

$$\big[\nabla_a, \nabla_b\big] = -\frac{1}{2}C_{ab}{}^{cd}M_{cd} + \frac{1}{2}\nabla^c C_{abc}{}^d K_d$$

- Conformal covariant derivatives commute in conformally-flat space-times!

$$C_{abcd} = 0 \quad \Longrightarrow \quad \big[\nabla_a, \nabla_b\big] = 0$$

- Degauging: Gauge away $\mathfrak{b}_a$ connection using K-symmetry:

$$\mathfrak{b}_a = 0 \quad \Longrightarrow \quad \left\{ \begin{array}{c} \Omega_{abc}(e, \mathfrak{b}) = \Omega_{abc}(e) \\ \mathfrak{f}_{ab}(e, \mathfrak{b}) = \mathfrak{f}_{ab}(e) \end{array} \right. \quad \Longrightarrow \quad \nabla_a = \mathcal{D}_a + \frac{1}{2}P_a{}^bK_b$$

Construction of conformal invariants

- The field $\Phi(x)$ is said to be primary with Weyl weight Δ if it satisfies

$$K_a \Phi = 0 \ , \qquad \mathbb{D}\Phi = \Delta\Phi \qquad (\Delta \in \mathbb{R})$$

- Upon degauging can show these are equivalent to Weyl transformations

$$\begin{cases} K_a \Phi = 0 \\ \mathbb{D}\Phi = \Delta\Phi \end{cases} \quad \overset{\mathfrak{b}_a = 0}{\Longrightarrow} \quad \delta_\sigma \Phi = \Delta\sigma\Phi$$

- **<u>Aim</u>**: To construct an action $S = \int \mathrm{d}^4x\, e\, \mathcal{L}$ from a primary Lagrangian $\mathcal{L}$ with weight 4 $\Longrightarrow$ invariance under scale transformations

$$\begin{cases} K_a \mathcal{L} = 0 \\ \mathbb{D}\mathcal{L} = 4\mathcal{L} \end{cases} \quad \Longrightarrow \quad \delta_{K,\mathbb{D}} S = 0 \quad \overset{\mathfrak{b}_a = 0}{\Longrightarrow} \quad \delta_\sigma S = 0$$

- CHS gauge fields are primary with fixed Weyl weight Δ_s

$$\delta_\xi h_{a(s)} = \nabla_{(a_1}\xi_{a_2 \dots a_s)} - (\text{traces}) \ , \qquad K_b h_{a(s)} = 0 \ , \qquad \mathbb{D} h_{a(s)} = \Delta_s h_{a(s)}$$

- Gauge symmetry fixes Δ_s uniquely: $\big[K_a, \nabla_b\big] = 2\eta_{ab}\mathbb{D} + 2M_{ab}$

$$0 = K_b\big(h_{a(s)} + \delta_\xi h_{a(s)}\big) \quad \Longrightarrow \quad \Delta_s = 2 - s$$

Two-component spinor notation

- Convert to two-component spinor notation: $\mathsf{SO}_0(3,1) \cong \mathsf{SL}(2,\mathbb{C})/\mathbb{Z}_2$

$$h_{a(s)} \quad \longmapsto \quad h_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} = (\sigma^{a_1})_{\alpha_1\dot\alpha_1}\cdots(\sigma^{a_s})_{\alpha_s\dot\alpha_s} h_{a_1\dots a_s} = h_{\alpha(s)\dot\alpha(s)}$$

- CHS fields in two-component spinor notation are complex & symmetric:

	HS field	Values of (m,n)
Generic 'spin' $s=\frac{1}{2}(m+n)$	$\big(h_{\alpha(m)\dot\alpha(n)}, \bar h_{\alpha(n)\dot\alpha(m)}\big)$	$(m,n)\in\mathbb{Z}_+$
Bosonic spin-s	$h_{\alpha(s)\dot\alpha(s)} = \bar h_{\alpha(s)\dot\alpha(s)}$	$(m,n)=(s,s)$
Fermionic spin-$(s+\frac{1}{2})$	$\big(h_{\alpha(s+1)\dot\alpha(s)}, \bar h_{\alpha(s)\dot\alpha(s+1)}\big)$	$(m,n)=(s+1,s)$

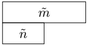
- Properties of CHS fields in two-component spinor notation

$$\delta_\xi h_{\alpha(m)\dot\alpha(n)} = \nabla_{(\alpha_1(\dot\alpha_1}\xi_{\alpha_2\dots\alpha_m)\dot\alpha_2\dots\alpha_n)} ~,$$

$$K_{\beta\dot\beta} h_{\alpha(m)\dot\alpha(n)} = 0 ~, \qquad \mathbb{D} h_{\alpha(m)\dot\alpha(n)} = \Big(2-\frac{1}{2}(m+n)\Big) h_{\alpha(m)\dot\alpha(n)}$$

- Side note: one-to-one correspondence with traceless two row Young diagram

$$\big(h_{\alpha(m)\dot\alpha(n)}, \bar h_{\alpha(n)\dot\alpha(m)}\big) \quad\Longleftrightarrow\quad h_{a(\tilde m),b(\tilde n)} \quad\Longleftrightarrow$$



Free CHS fields in 4D Minkowski

- Free CHS action in 4D Minkowski space:

$$S^{(m,n)}_{\text{CHS}}[h,\bar{h}] = \text{i}^{m+n} \int \text{d}^4x \, \hat{C}^{\alpha(m+n)}(h) \check{C}_{\alpha(m+n)}(\bar{h}) + \text{c.c.} \qquad (\star)$$

- 'Higher-spin Weyl' tensors:

$$\hat{C}_{\alpha(m+n)}(h) = \partial_{(\alpha_1}{}^{\dot\beta_1} \dots \partial_{\alpha_n}{}^{\dot\beta_n} h_{\alpha_{n+1}\dots\alpha_{n+m})\dot\beta_1\dots\dot\beta_n}$$
$$\check{C}_{\alpha(m+n)}(\bar{h}) = \partial_{(\alpha_1}{}^{\dot\beta_1} \dots \partial_{\alpha_m}{}^{\dot\beta_m} \bar{h}_{\alpha_{m+1}\dots\alpha_{m+n})\dot\beta_1\dots\dot\beta_m}$$

- Gauge invariance:

$$\delta_\xi h_{\alpha(m)\dot\alpha(n)} = \partial_{(\alpha_1(\dot\alpha_1}\xi_{\alpha_2\dots\alpha_m)\dot\alpha_2\dots\dot\alpha_n)} \qquad \Longrightarrow \qquad \begin{cases} \delta_\xi \hat{C}_{\alpha(m+n)} = 0 \\ \delta_\xi \check{C}_{\alpha(m+n)} = 0 \end{cases}$$

- How to lift to curved space in a Weyl and gauge invariant way?
- **Hint**: Curved-space model must reduce to $(\star)$ in flat-space limit

CHS fields on conformally-flat backgrounds

- Restrict to conformally-flat backgrounds $\implies$ $[\nabla_a, \nabla_b] = 0$
- Minimally lifted HS Weyl tensors:

$$\hat{\mathfrak{C}}_{\alpha(m+n)}(h) := \nabla_{(\alpha_1}{}^{\dot\beta_1} \dots \nabla_{\alpha_n}{}^{\dot\beta_n} h_{\alpha_{n+1}\dots\alpha_{n+m})\dot\beta_1\dots\dot\beta_n}$$

$$\check{\mathfrak{C}}_{\alpha(m+n)}(\bar h) := \nabla_{(\alpha_1}{}^{\dot\beta_1} \dots \nabla_{\alpha_m}{}^{\dot\beta_m} \bar h_{\alpha_{m+1}\dots\alpha_{m+n})\dot\beta_1\dots\dot\beta_m}$$

- HS Weyl tensors are gauge invariant, primary & have correct Weyl weight

$$\delta_\xi \hat{\mathfrak{C}}_{\alpha(m+n)} = \delta_\xi \check{\mathfrak{C}}_{\alpha(m+n)} = 0 \ , \qquad K_{\beta\dot\beta} \hat{\mathfrak{C}}_{\alpha(m+n)} = K_{\beta\dot\beta} \check{\mathfrak{C}}_{\alpha(m+n)} = 0$$

$$\mathbb{D}\hat{\mathfrak{C}}_{\alpha(m+n)} = \Big(2 - \frac{1}{2}(m-n)\Big)\hat{\mathfrak{C}}_{\alpha(m+n)} \ , \qquad \mathbb{D}\check{\mathfrak{C}}_{\alpha(m+n)} = \Big(2 + \frac{1}{2}(m-n)\Big)\check{\mathfrak{C}}_{\alpha(m+n)}$$

- The associated action is gauge invariant and primary:

$$\boxed{\mathcal{S}^{(m,n)}_{\text{Skeleton}} = \mathrm{i}^{m+n} \int \mathrm{d}^4x\, e\, \hat{\mathfrak{C}}^{\alpha(m+n)}(h) \check{\mathfrak{C}}_{\alpha(m+n)}(\bar h) + \text{c.c.} \ , \qquad \delta_{\xi,K,\mathbb{D}} \mathcal{S}^{(m,n)}_{\text{CHS}} = 0}$$

- Valid for all conformally-flat geometries: SMK & M.P. (2019)

Generalised CHS fields on conformally-flat backgrounds

- 'Generalised' CHS field $h^{(l)}_{\alpha(m)\dot\alpha(n)}$ with 'depth l' gauge transformations

$$\delta_\xi h^{(l)}_{\alpha(m)\dot\alpha(n)} = \underbrace{\nabla_{(\alpha_1(\dot\alpha_1}\cdots\nabla_{\alpha_l\dot\alpha_l}}_{l-\text{derivatives}}\xi_{\alpha_{l+1}\dots\alpha_m)\dot\alpha_{l+1}\dots\dot\alpha_n)}~, \qquad 1\leq l\leq \min(m,n)$$

- Generalised HS Weyl tensors:

$$\hat{\mathfrak{C}}^{(l)}_{\alpha(m+n-l+1)\dot\alpha(l-1)} = \nabla_{(\alpha_1}{}^{\dot\beta_1}\cdots\nabla_{\alpha_{n-l+1}}{}^{\dot\beta_{n-l+1}}h^{(l)}_{\alpha_{n-l+2}\dots\alpha_{m+n-l+1})\dot\beta(n-l+1)\dot\alpha(l-1)}$$

$$\check{\mathfrak{C}}^{(l)}_{\alpha(m+n-l+1)\dot\alpha(l-1)} = \nabla_{(\alpha_1}{}^{\dot\beta_1}\cdots\nabla_{\alpha_{m-l+1}}{}^{\dot\beta_{m-l+1}}\bar{h}^{(l)}_{\alpha_{m-l+2}\dots\alpha_{m+n-l+1})\dot\beta(m-l+1)\dot\alpha(l-1)}$$

- They are primary in arbitrary background

$$K_{\beta\dot\beta}\hat{\mathfrak{C}}^{(l)}_{\alpha(m+n-l+1)\dot\alpha(l-1)} = K_{\beta\dot\beta}\check{\mathfrak{C}}^{(l)}_{\alpha(m+n-l+1)\dot\alpha(l-1)} = 0$$

- Gauge invariant in conformally-flat background:

$$\delta_\xi\hat{\mathfrak{C}}^{(l)}_{\alpha(m+n-l+1)\dot\alpha(l-1)} = 0~, \qquad \delta_\xi\check{\mathfrak{C}}^{(l)}_{\alpha(m+n-l+1)\dot\alpha(l-1)} = 0$$

- Generalised action in conformally-flat space: SMK & M.P. (2019)

$$S^{(m,n,l)}_{\text{Skeleton}} = \mathrm{i}^{m+n}\int \mathrm{d}^4x\, e\,\hat{\mathfrak{C}}^{\alpha(m+n-l+1)\dot\alpha(l-1)}_{(l)}\check{\mathfrak{C}}^{(l)}_{\alpha(m+n-l+1)\dot\alpha(l-1)} + \text{c.c.}$$

Conformal non-gauge fields

- Introduce tensor fields $\big(\chi_{\alpha(m)\dot\alpha(n)},\ \bar\chi_{\alpha(n)\dot\alpha(m)}\big)$ with $m \geq n \geq 0$ satisfying

$$K_{\beta\dot\beta}\chi_{\alpha(m)\dot\alpha(n)} = 0 \ , \qquad \mathbb{D}\chi_{\alpha(m)\dot\alpha(n)} = \Big(2 - \frac{1}{2}(m-n)\Big)\chi_{\alpha(m)\dot\alpha(n)}$$

- From $\chi_{\alpha(m)\dot\alpha(n)}$ can construct the descendent

$$\mathcal{F}_{\alpha(n)\dot\alpha(m)}(\chi) = \nabla_{(\dot\alpha_1}{}^{\beta_1}\cdots\nabla_{\dot\alpha_{m-n}}{}^{\beta_{m-n}}\chi_{\alpha(n)\beta(m-n)\dot\alpha_{m-n+1}\dots\dot\alpha_m)}$$

- The descendent is primary in a generic background

$$K_{\beta\dot\beta}\mathcal{F}_{\alpha(n)\dot\alpha(m)}(\chi) = 0 \ , \qquad \mathbb{D}\mathcal{F}_{\alpha(n)\dot\alpha(m)}(\chi) = \Big(2 + \frac{1}{2}(m-n)\Big)\mathcal{F}_{\alpha(n)\dot\alpha(m)}(\chi)$$

- Can in turn construct a primary action functional

$$\mathcal{S}_{\rm NG}^{(m,n)}[\chi,\bar\chi] = \frac{1}{2}{\rm i}^{m+n}\int {\rm d}^4x\, e\, \bar\chi^{\alpha(n)\dot\alpha(m)}\mathcal{F}_{\alpha(n)\dot\alpha(m)}(\chi) + {\rm c.c.} \ , \qquad \delta_{\mathbb{D},K}\mathcal{S}_{\rm NG}^{(m,n)} = 0$$

- Why non-gauge? Not compatible with usual HS gauge symmetry. Recall:

$$\delta_\xi h_{\alpha(m)\dot\alpha(n)} = \nabla_{(\alpha_1(\dot\alpha_1}\xi_{\alpha_2\dots\alpha_m)\dot\alpha_2\dots\alpha_n)} \quad \Leftrightarrow \quad \mathbb{D}h_{\alpha(m)\dot\alpha(n)} = \Big(2 - \frac{1}{2}(m+n)\Big)h_{\alpha(m)\dot\alpha(n)}$$

SMK, M.P. & E.S.N. Raptakis (2020)

CHS models in Bach-flat backgrounds

- Can we extend these CHS models to more general curved backgrounds?
- Non-vanishing Weyl tensor: $C_{abcd} \iff \big(C_{\alpha(4)},\ \bar{C}_{\dot\alpha(4)}\big)$

$$\begin{aligned}\big[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}\big] &= -\big(\varepsilon_{\dot\alpha\dot\beta}C_{\alpha\beta\gamma\delta}M^{\gamma\delta}+\varepsilon_{\alpha\beta}\bar{C}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\bar{M}^{\dot\gamma\dot\delta}\big)\\ &\quad -\frac{1}{4}\big(\varepsilon_{\dot\alpha\dot\beta}\nabla^{\delta\dot\gamma}C_{\alpha\beta\delta}{}^{\gamma}+\varepsilon_{\alpha\beta}\nabla^{\gamma\dot\delta}\bar{C}_{\dot\alpha\dot\beta\dot\delta}{}^{\dot\gamma}\big)K_{\gamma\dot\gamma}\end{aligned}$$

- **Problem**: CHS gauge symmetry is broken

$$\big[\nabla_a,\nabla_b\big]\neq 0 \qquad \Longrightarrow \qquad \delta_\xi \mathcal{S}^{(m,n)}_{\text{Skeleton}}=\mathcal{O}(C)$$

- What are the necessary constraints on the curvature?

$$\text{Conformal gravity EoM:} \qquad B_{ab}=0 \qquad \text{(Bach-flat)}$$

$$B_{ab}=\mathcal{D}^c\mathcal{D}^dC_{cabd}+\frac{1}{2}R^{cd}C_{cabd} \qquad \iff \qquad B_{\alpha(2)\dot\alpha(2)}=\nabla_{(\dot\alpha_1}{}^{\beta}\nabla_{\dot\alpha_2)}{}^{\beta}C_{\alpha(2)\beta(2)}$$

- **Conformally-flat limit**: Bach-flat model must reduce to skeleton

(Naively) $$\mathcal{S}^{(m,n)}_{\text{CHS}} = \underbrace{\mathcal{S}^{(m,n)}_{\text{SK}}[h,\bar h]}_{\text{Conformally-flat action}} + \underbrace{\mathcal{S}^{(m,n)}_{\text{NM}}[h,\bar h, C_{abcd}]}_{\text{Non-minimal action}}$$

Example: Conformal graviton

- Conformal graviton properties

$$h_{\alpha(2)\dot\alpha(2)} = \bar h_{\alpha(2)\dot\alpha(2)}~, \qquad \delta_\xi h_{\alpha(2)\dot\alpha(2)} = \nabla_{(\alpha_1(\dot\alpha_1}\xi_{\alpha_2)\dot\alpha_2)}$$
$$K_{\beta\dot\beta} h_{\alpha(2)\dot\alpha(2)} = 0~, \qquad \mathbb{D} h_{\alpha(2)\dot\alpha(2)} = 0$$

- The skeleton sector is $(m = n = 2)$

$$S_{\rm SK} = \int {\rm d}^4x\, e\, \mathfrak{C}^{\alpha(4)}\mathfrak{C}_{\alpha(4)} + {\rm c.c.}~, \qquad \mathfrak{C}_{\alpha(4)} = \nabla_{(\alpha_1}{}^{\dot\beta_1}\nabla_{\alpha_2}{}^{\dot\beta_2} h_{\alpha_3\alpha_4)\dot\beta(2)}$$

- The non-minimal sector is

$$S_{\rm NM} = \int {\rm d}^4x\, e\, h^{\alpha(2)\dot\alpha(2)}\Big\{ 2C_\alpha{}^{\beta(3)}\nabla_{\beta\dot\alpha}\nabla_\beta{}^{\dot\gamma} h_{\beta\alpha\dot\alpha\dot\gamma} + \nabla^{\gamma\dot\gamma}C_{\alpha(2)}{}^{\beta(2)}\nabla_{\gamma\dot\gamma}h_{\beta(2)\dot\alpha(2)}$$
$$+4\nabla_{\beta\dot\alpha}C_\alpha{}^{\beta(3)}\nabla_\alpha{}^{\dot\gamma}h_{\beta(2)\dot\alpha\dot\gamma} + 2\nabla_\beta{}^{\dot\gamma}C_{\alpha(2)}{}^{\beta(2)}\nabla_{\dot\gamma}{}^{\gamma}h_{\beta\gamma\dot\alpha(2)} - 2h_{\beta(2)\dot\alpha(2)}\Box_c C_{\alpha(2)}{}^{\beta(2)}$$
$$-2C_{\alpha(2)}{}^{\beta(2)}\Box_c h_{\beta(2)\dot\alpha(2)} + 2h_{\gamma\beta\dot\gamma\dot\alpha}\nabla_{\dot\alpha}{}^{\gamma}\nabla_\beta{}^{\dot\gamma}C_{\alpha(2)}{}^{\beta(2)} + 2C_{\alpha(2)}{}^{\gamma(2)}C_{\gamma(2)}{}^{\beta(2)}h_{\beta(2)\dot\alpha(2)}$$
$$+2C_{\alpha\gamma(2)}{}^{\beta}C_\alpha{}^{\beta\gamma(2)}h_{\beta(2)\dot\alpha(2)} + 2C_{\alpha(2)}{}^{\beta(2)}\bar C_{\dot\alpha(2)}{}^{\dot\beta(2)}h_{\beta(2)\dot\beta(2)}\Big\} + {\rm c.c.}$$

- Non-minimal primary tensor field: $K_{\beta\dot\beta}\mathfrak{J}_{\alpha(2)\dot\alpha(2)}(h) = 0$
- Their sum is gauge invariant on a Bach-flat background

$$S_{\rm CHS} = S_{\rm SK} + S_{\rm NM}~, \qquad \delta_\xi S_{\rm CHS}\Big|_{B_{ab}=0} = 0$$

Example: Spin-2 maximal depth

- Maximal depth spin-2 field is real with depth 2 gauge transformations

$$h_{\alpha(2)\dot\alpha(2)} = \bar h_{\alpha(2)\dot\alpha(2)} \ , \qquad \delta_\xi h_{\alpha(2)\dot\alpha(2)} = \nabla_{(\alpha_1(\dot\alpha_1}\nabla_{\alpha_2)\dot\alpha_2)}\xi$$

$$K_{\beta\dot\beta} h_{\alpha(2)\dot\alpha(2)} = 0 \ , \qquad \mathbb{D} h_{\alpha(2)\dot\alpha(2)} = h_{\alpha(2)\dot\alpha(2)}$$

- The skeleton sector is $(m = n = l = 2)$

$$S_{\text{SK}} = \int \mathrm{d}^4x\, e\, \mathfrak{C}^{\alpha(3)\dot\alpha}\mathfrak{C}_{\alpha(3)\dot\alpha} + \text{c.c.} \ , \qquad \mathfrak{C}_{\alpha(3)\dot\alpha} = \nabla_{(\alpha_1}{}^{\dot\beta} h_{\alpha_2\alpha_3)\dot\alpha\dot\beta}$$

- The non-minimal sector is

$$S_{\text{NM}} = -\int \mathrm{d}^4x\, e\, C^{\alpha(2)\beta(2)} h_{\alpha(2)}{}^{\dot\alpha(2)} h_{\beta(2)\dot\alpha(2)} + \text{c.c.}$$

- Their sum is gauge invariant on a Bach-flat background

$$S_{\text{CHS}} = S_{\text{SK}} + S_{\text{NM}} \ , \qquad \delta_\xi S_{\text{CHS}}\big|_{B_{ab}=0} = 0$$ SMK & M.P. (2019)

- Gauge invariant model on Einstein backgrounds:

$$\delta_\xi h_{ab} = (\mathcal{D}_a\mathcal{D}_b - \frac{1}{4}\eta_{ab}\Box)\xi$$ M. Beccaria & A. Tseytlin (2015)

- Correct gauge transformations (after degauging)

$$\mathfrak{b}_a = 0 \quad \Longrightarrow \quad \delta_\xi h_{ab} = \Big(\mathcal{D}_a\mathcal{D}_b - \frac{1}{2}R_{ab}\Big)\xi - \frac{1}{4}\eta_{ab}\Big(\Box - \frac{1}{2}R\Big)\xi$$

The plot thickens: Enter, lower-spin couplings

Spin-3 story so far (incomplete):

- Quadratic spin-3 action with gauge invariance to second order in curvature

$$S^{(\text{Linear})}_{\text{Spin-3}}[h^2, C] = S_{\text{SK}}[h^2, C^0] + S_{\text{NM}}[h^2, C^1] \qquad \delta_{\xi,\sigma} S^{(\text{Linear})}_{\text{Spin-3}} = \mathcal{O}(C^2)$$

T. Nutma & M. Taronna (2014)

- Geometric construction of primary spin-3 descendent (linear in curvature)

R. Manvelyan & G. Poghosyan (2018)

- Conjectured necessity of (spin-3)−(spin-1) coupling for full gauge invariance

$$S^{(\text{Exact})}_{\text{Spin-3}}[h, A] = \underbrace{S_{33}[h]}_{\text{Pure spin-3}} + \underbrace{S_{13}[h, A]}_{\text{Spin 1-3 coupling}} + \underbrace{S_{11}[A]}_{\text{Pure spin-1}}$$

M. Grigoriev & A. Tseytlin (2016)

- Explicit calculation of spin 1−3 mixing terms (exact)

M. Beccaria & A. Tseytlin (2017)

- Missing:
  - 2^{nd} & 3^{rd} order curvature corrections to pure spin-3 sector
  - Relative coefficient between spin-1 and spin-3 sectors
  - Other lower-spin fields?

Example: Spin-3 maximal depth

- Maximal depth spin-3 field is real with depth 3 gauge transformations

$$h_{\alpha(3)\dot\alpha(3)} = \bar h_{\alpha(3)\dot\alpha(3)} \ , \qquad \delta_\xi h_{\alpha(3)\dot\alpha(3)} = \nabla_{(\alpha_1(\dot\alpha_1}\nabla_{\alpha_2\dot\alpha_2}\nabla_{\alpha_3)\dot\alpha_3)}\xi$$
$$K_{\beta\dot\beta}h_{\alpha(3)\dot\alpha(3)} = 0 \ , \qquad \mathbb{D}h_{\alpha(3)\dot\alpha(3)} = h_{\alpha(3)\dot\alpha(3)}$$

- The skeleton sector is $(m = n = l = 3)$

$$S_{\rm SK} = -\int {\rm d}^4x\, e\, \mathfrak{C}^{\alpha(4)\dot\alpha(2)}\mathfrak{C}_{\alpha(4)\dot\alpha(2)} + {\rm c.c.} \ , \qquad \mathfrak{C}_{\alpha(4)\dot\alpha(2)} = \nabla_{(\alpha_1}{}^{\dot\beta}h_{\alpha_2\alpha_3\alpha_4)\dot\alpha(2)\dot\beta}$$

- The non-minimal sector is

$$S_{\rm NM} = 2\int {\rm d}^4x\, e\, h^{\gamma\alpha(2)\dot\alpha(3)}C_{\alpha(2)}{}^{\beta(2)}h_{\beta(2)\gamma\dot\alpha(3)} + c.c.$$

- Sum is gauge invariant on a Bach-flat background only to first order in C_{abcd}

$$\delta_\xi\big(S_{\rm SK} + S_{\rm NM}\big)\,\Big|_{B_{ab}=0} = \mathcal{O}(C^2)$$

Example: Spin-3 maximal depth

- **Solution**: Introduce 'lower-spin' non-gauge fields to kill $\mathcal{O}(C^2)$ terms

$$\chi_{\alpha(3)\dot\alpha}: \qquad \delta_\xi \chi_{\alpha(3)\dot\alpha} = C_{\alpha(3)}{}^{\beta}\nabla_{\beta\dot\alpha}\xi - 2\nabla_{\beta\dot\alpha}C_{\alpha(3)}{}^{\beta}\xi$$

$$\varphi_{\alpha(4)}: \qquad \delta_\xi \varphi_{\alpha(4)} = C_{\alpha(4)}\xi$$

- Couple lower-spin fields to spin-3 field:

$$S_{\text{CHS}} = \underbrace{S_{\text{SK}} + S_{\text{NM}}}_{\text{pure spin-3 sector}} + \underbrace{S_{h\chi} + S_{h\bar\varphi}}_{\text{lower-spin coupling}} + \underbrace{S_{\chi\bar\chi} + S_{\varphi\bar\varphi}}_{\text{lower-spin kinetic}}$$

$$S_{h\chi} = \frac{16}{3}\int \mathrm{d}^4x\, e\, h^{\alpha(3)\dot\alpha(3)}\bar C_{\dot\alpha(3)}{}^{\dot\beta}\chi_{\alpha(3)\dot\beta} + \text{c.c.}$$

$$S_{h\bar\varphi} = -\frac{8}{3}\int \mathrm{d}^4x\, e\, h^{\alpha(3)\dot\alpha(3)}\Big\{C_{\alpha(3)}{}^{\beta}\nabla_{\beta\dot\beta}\bar\varphi_{\dot\alpha(3)}{}^{\dot\beta} - 3\nabla_{\beta\dot\beta}C_{\alpha(3)}{}^{\beta}\bar\varphi_{\dot\alpha(3)}{}^{\dot\beta}\Big\} + \text{c.c.}$$

$$S_{\chi\bar\chi} = \frac{8}{3}\int \mathrm{d}^4x\, e\, \chi^{\alpha(3)\dot\alpha}\nabla_\alpha{}^{\dot\alpha}\nabla_\alpha{}^{\dot\alpha}\bar\chi_{\alpha\dot\alpha(3)} + \text{c.c.}$$

$$S_{\varphi\bar\varphi} = -\frac{4}{3}\int \mathrm{d}^4x\, e\, \bar\varphi^{\dot\alpha(4)}\nabla_{\dot\alpha}{}^{\alpha}\nabla_{\dot\alpha}{}^{\alpha}\nabla_{\dot\alpha}{}^{\alpha}\nabla_{\dot\alpha}{}^{\alpha}\varphi_{\alpha(4)} + \text{c.c.}$$

- Combined action is primary and gauge invariant on Bach-flat background

$$\delta_{K,\mathbb{D}}S_{\text{CHS}} = 0 \qquad \delta_\xi S_{\text{CHS}}\Big|_{B_{ab}=0} = 0$$

SMK & M.P. (2019)

Example: Spin-5/2 maximal depth

- Maximal depth spin-5/2 $(\psi_{\alpha(3)\dot\alpha(2)},\ \bar\psi_{\alpha(2)\dot\alpha(3)})$ field has the properties

$$\delta_\lambda \psi_{\alpha(3)\dot\alpha(2)} = \nabla_{(\alpha_1(\dot\alpha_1}\nabla_{\alpha_2\dot\alpha_2)}\lambda_{\alpha_3)}$$

$$K_{\beta\dot\beta}\psi_{\alpha(3)\dot\alpha(2)} = 0\ , \qquad \mathbb{D}\psi_{\alpha(3)\dot\alpha(2)} = \frac{1}{2}\psi_{\alpha(3)\dot\alpha(2)}$$

- The skeleton sector is composed of the two generalised Weyl tensors

$$S_{\text{SK}} = \text{i}\int \text{d}^4x\, e\, \hat{\mathfrak{C}}^{\alpha(4)\dot\alpha}(\psi)\check{\mathfrak{C}}_{\alpha(4)\dot\alpha}(\bar\psi) + \text{c.c.}\ ,$$

$$\hat{\mathfrak{C}}_{\alpha(4)\dot\alpha}(\psi) = \nabla_{(\alpha_1}{}^{\dot\beta}\psi_{\alpha_2\alpha_3\alpha_4)\dot\alpha\dot\beta}\ , \qquad \check{\mathfrak{C}}_{\alpha(4)\dot\alpha}(\bar\psi) = \nabla_{(\alpha_1}{}^{\dot\beta}\nabla_{\alpha_2}{}^{\dot\beta}\bar\psi_{\alpha_3\alpha_4)\dot\alpha\dot\beta(2)}$$

- The non-minimal sector is

$$S_{\text{NM}} = \text{i}\int \text{d}^4x\, e\, \psi^{\alpha(3)\dot\alpha(2)}\Big\{\frac{5}{4}C_{\alpha(3)}{}^{\beta}\nabla^{\beta\dot\beta}\bar\psi_{\beta(2)\dot\beta\dot\alpha(2)} - \nabla^{\beta\dot\beta}C_{\alpha(3)}{}^{\beta}\bar\psi_{\beta(2)\dot\beta\dot\alpha(2)}$$
$$- 3C_{\alpha(2)}{}^{\beta(2)}\nabla_{\alpha}{}^{\dot\beta}\bar\psi_{\beta(2)\dot\beta\dot\alpha(2)}\Big\} + \text{c.c.}$$

- Sum is gauge invariant on a Bach-flat background only to first order in C_{abcd}

$$\delta_\lambda\big(S_{\text{SK}} + S_{\text{NM}}\big)\,|_{B_{ab}=0} = \mathcal{O}(C^2)$$

Example: Spin-5/2 maximal depth

- Introduce lower-spin non-gauge fields to kill $\mathcal{O}(C^2)$ terms

$$\chi_{\alpha(2)\dot\alpha}: \qquad \delta_\lambda \chi_{\alpha(2)\dot\alpha} = C_{\alpha(2)}{}^{\beta(2)} \nabla_{\beta\dot\alpha}\lambda_\beta - \nabla_{\beta\dot\alpha} C_{\alpha(2)}{}^{\beta(2)}\lambda_\beta$$

$$\varphi_{\alpha(3)}: \qquad \delta_\xi \varphi_{\alpha(3)} = C_{\alpha(3)}{}^{\beta}\lambda_\beta$$

- Couple lower-spin fields to spin-(5/2) field:

$$S_{\text{CHS}} = \underbrace{S_{\text{SK}} + S_{\text{NM}}}_{\text{pure spin-(5/2) sector}} + \underbrace{S_{\bar\psi\chi} + S_{\psi\bar\varphi}}_{\text{lower-spin coupling}} + \underbrace{S_{\chi\bar\chi} + S_{\varphi\bar\varphi}}_{\text{lower-spin kinetic}}$$

$$S_{\bar\psi\chi} = -\frac{37}{24}\text{i}\int \text{d}^4x\, e\, \bar\psi^{\alpha(2)\dot\alpha(3)} \bar C_{\dot\alpha(3)}{}^{\dot\beta}\chi_{\alpha(2)\dot\beta} + \text{c.c.}$$

$$S_{\psi\bar\varphi} = \frac{17}{24}\text{i}\int \text{d}^4x\, e\, \psi^{\alpha(3)\dot\alpha(2)}\Big\{C_{\alpha(3)}{}^{\beta}\nabla_\beta{}^{\dot\beta}\bar\varphi_{\dot\alpha(2)\dot\beta} - 2\nabla_\beta{}^{\dot\beta} C_{\alpha(3)}{}^{\beta}\bar\varphi_{\dot\alpha(3)\dot\beta}\Big\} + \text{c.c.}$$

$$S_{\chi\bar\chi} = \frac{37}{48}\text{i}\int \text{d}^4x\, e\, \chi^{\alpha(2)\dot\alpha}\nabla_\alpha{}^{\dot\alpha}\bar\chi_{\alpha\dot\alpha(2)} + \text{c.c.}$$

$$S_{\varphi\bar\varphi} = \frac{17}{48}\text{i}\int \text{d}^4x\, e\, \bar\varphi^{\dot\alpha(3)}\nabla_{\dot\alpha}{}^{\alpha}\nabla_{\dot\alpha}{}^{\alpha}\nabla_{\dot\alpha}{}^{\alpha}\varphi_{\alpha(3)} + \text{c.c.}$$

- Combined action is primary and gauge invariant on Bach-flat background

$$\delta_{K,\mathbb{D}} S_{\text{CHS}} = 0 \qquad \delta_\xi S_{\text{CHS}}\big|_{B_{ab}=0} = 0$$

SMK & M.P. (2019)

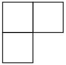
Conformal pseudo-graviton (hook field)

- What about minimal depth (aka usual) gauge fields?
- Conformal pseudo-graviton has similar properties as conformal graviton

$$\delta_\xi h_{\alpha(3)\dot\alpha} = \nabla_{(\alpha_1\dot\alpha}\xi_{\alpha_2\alpha_3)}$$
$$K_{\beta\dot\beta}h_{\alpha(3)\dot\alpha} = 0 \ , \qquad \mathbb{D}h_{\alpha(3)\dot\alpha} = 0$$

- Also known as conformal hook field

$$\left(h_{\alpha(3)\dot\alpha},\ \bar h_{\alpha(3)\dot\alpha}\right) \qquad \Longleftrightarrow \qquad \begin{cases} 0 = h_{abc} + h_{bca} + h_{cab} \\ 0 = h_{abc} + h_{bac} \\ 0 = h_{ab}{}^{b} \end{cases} \qquad \Longleftrightarrow$$



- Two-derivative models in $\mathbb{M}^4$:
  T. Curtright & P. Freund (1980) and T. Curtright (1985)
- Four-derivative non-conformal models in $\mathbb{M}^d$ and AdS_d:
  E. Joung & K. Mkrtchyan (2016)
- Extra gauge symmetry (incompatible with conformal symmetry):

$$\delta_\theta h_{\alpha(3)\dot\alpha} = \partial_{(\alpha_1}{}^{\dot\beta}\theta_{\alpha_2\alpha_3)\dot\alpha\dot\beta}$$

Conformal pseudo-graviton (hook field)

- The skeleton and non-minimal sectors are

$$S_{\rm SK} = \int {\rm d}^4x\, e\, \hat{\mathfrak{C}}^{\alpha(4)} \check{\mathfrak{C}}_{\alpha(4)} + {\rm c.c.}\ , \qquad \begin{cases} \hat{\mathfrak{C}}_{\alpha(4)} = \nabla_{(\alpha_1}{}^{\dot\beta} h_{\alpha_2\alpha_3\alpha_4)\dot\beta} \\ \check{\mathfrak{C}}_{\alpha(4)} = \nabla_{(\alpha_1}{}^{\dot\beta_1} \nabla_{\alpha_2}{}^{\dot\beta_2} \nabla_{\alpha_3}{}^{\dot\beta_3} \bar{h}_{\alpha_4)\dot\beta(3)} \end{cases}$$

$$S_{\rm NM} = -\frac{1}{2}\int {\rm d}^4x\, e\, h^{\alpha(3)\dot\alpha}\Big\{ 5C_{\alpha(3)}{}^{\gamma}\nabla_{\gamma}{}^{\dot\beta}\nabla^{\beta\dot\beta}\bar{h}_{\beta\dot\beta(2)\dot\alpha} - 6\nabla_{\gamma}{}^{\dot\beta}C_{\alpha(2)}{}^{\beta\gamma}\nabla_{\alpha}{}^{\dot\beta}\bar{h}_{\beta\dot\beta(2)\dot\alpha}$$
$$+ \nabla_{\gamma}{}^{\dot\beta}C_{\alpha(3)}{}^{\gamma}\nabla^{\beta\dot\beta}\bar{h}_{\beta\dot\beta(2)\dot\alpha} - 4\nabla^{\beta\dot\beta}\nabla_{\gamma}{}^{\dot\beta}C_{\alpha(3)}{}^{\gamma}\bar{h}_{\beta\dot\beta(2)\dot\alpha} - 2C_{\alpha(3)}{}^{\beta}\bar{C}_{\dot\alpha(3)}{}^{\dot\beta}\bar{h}_{\beta\dot\beta(3)}$$
$$+ 2\nabla^{\delta\dot\beta}C_{\alpha(3)}{}^{\beta}\nabla_{\delta}{}^{\dot\beta}\bar{h}_{\beta\dot\beta(2)\dot\alpha} + 6C_{\alpha(2)}{}^{\gamma(2)}\nabla_{\gamma}{}^{\dot\beta}\nabla_{\gamma}{}^{\dot\beta}\bar{h}_{\alpha\dot\alpha\dot\beta(2)}\Big\} + {\rm c.c.}$$

- Gauge variation is of second order: $\delta_\xi\big(S_{\rm SK} + S_{\rm NM}\big) = \mathcal{O}(C^2)$
- Couple to lower-spin field $(\chi_{\alpha(2)},\ \bar\chi_{\dot\alpha(2)})$ with $\delta_\xi\chi_{\alpha(2)} = C_{\alpha(2)}{}^{\beta(2)}\xi_{\beta(2)}$

$$S_{h\bar\chi} = -2\int {\rm d}^4x\, e\, h^{\alpha(3)\dot\alpha}\Big\{C_{\alpha(3)}{}^{\gamma}\nabla_{\gamma}{}^{\dot\beta}\bar\chi_{\dot\beta\dot\alpha} - \nabla_{\gamma}{}^{\dot\beta}C_{\alpha(3)}{}^{\gamma}\bar\chi_{\dot\beta\dot\alpha}\Big\} + {\rm c.c.}\ ,$$
$$S_{\chi\bar\chi} = \int {\rm d}^4x\, e\, \bar\chi^{\dot\alpha(2)}\nabla_{\dot\alpha}{}^{\alpha}\nabla_{\dot\alpha}{}^{\alpha}\chi_{\alpha(2)} + {\rm c.c.}$$

- The total action is primary and gauge invariant on Bach-flat background

$$S_{\rm CHS} = S_{\rm SK} + S_{\rm NM} + S_{h\bar\chi} + S_{\chi\bar\chi}\ , \qquad \delta_{K,\mathbb{D}}S_{\rm CHS} = \delta_\xi S_{\rm CHS} = 0$$

SMK, M.P. & E.S.N. Raptakis (2020)

Superconformal pseudo-graviton multiplet

- SMK, R. Manvelyan & S. Theisen (2017): $\mathcal{N}=1$ SCHS models in $\mathbb{M}^{4|4}$
- Pseudo-graviton supermultiplet $\Upsilon_{\alpha(2)}$ primary weight -2 superfield

$$\delta_{\zeta,\lambda}\Upsilon_{\alpha(2)} = \nabla_{(\alpha_1}\zeta_{\alpha_2)} + \lambda_{\alpha(2)}~, \qquad \bar{\nabla}_{\dot\beta}\lambda_{\alpha(2)} = 0$$

- At component level $\Upsilon_{\alpha(2)}(\theta,\bar\theta) = \theta^\beta\bar\theta^{\dot\beta}h_{\alpha(2)\beta\dot\beta} + \theta^2\bar\theta^2\chi_{\alpha(2)} + \cdots$

$$\delta_\xi h_{\alpha(3)\dot\alpha} = \nabla_{(\alpha_1\dot\alpha}\xi_{\alpha_2\alpha_3)}~, \qquad \delta_\xi\chi_{\alpha(2)} = C_{\alpha(2)}{}^{\beta(2)}\xi_{\beta(2)}$$

- Again: Skeleton + non-minimal sector is gauge invariant to second order

$$S_{\Upsilon\bar\Upsilon} = S_{\rm SK}[\Upsilon,\bar\Upsilon] + S_{\rm NM}[\Upsilon,\bar\Upsilon,W_{\alpha\beta\gamma}]~, \qquad \delta_{\zeta,\lambda}S_{\Upsilon\bar\Upsilon}\big|_{B_{\alpha\dot\alpha}=0} = \mathcal{O}(W^2)$$

- Kill $\mathcal{O}(W^2)$ terms via coupling to LS chiral superfield Ω_α: $\delta_\lambda\Omega_\alpha = W_\alpha{}^{\beta(2)}\lambda_{\beta(2)}$

$$S_{\rm SCHS} = S_{\Upsilon\bar\Upsilon} + S_{\Upsilon\bar\Omega} + S_{\Omega\bar\Omega} \quad\Longrightarrow\quad \delta_{\zeta,\lambda}S_{\rm SCHS}\big|_{B_{\alpha\dot\alpha}=0} = 0$$

- **Outcome**: Can extract non-supersymmetric gauge invariant model for $h_{\alpha(3)\dot\alpha}$

$$S_{\rm SCHS} = S_{h\bar h} + S_{h\bar\chi} + S_{\chi\bar\chi} + \cdots$$ SMK, M.P. & E.S.N. Raptakis (2020)

Recipe for conformal spin-3 model

Can deduce minimal ingredients for gauge invariant conformal spin-3 model using supersymmetric arguments: SMK, M.P. & E.S.N. Raptakis (2020)

- Assume existence of superconformal gauge model for superfield $H_{\alpha(2)\dot\alpha(2)}$
- Spin-3 supermultiplet $H_{\alpha(2)\dot\alpha(2)}$ real primary superfield with weight -2

$$\delta_\zeta H_{\alpha(2)\dot\alpha(2)} = \bar\nabla_{(\dot\alpha_1}\zeta_{\alpha(2)\dot\alpha_2)} - \nabla_{(\alpha_1}\bar\zeta_{\alpha_2)\dot\alpha(2)}$$

- Component level: Contains both spin-3 and spin-2 conformal fields

$$H_{\alpha(2)\dot\alpha(2)}(\theta,\bar\theta) = \theta^\beta\bar\theta^{\dot\beta} h_{\beta\alpha(2)\dot\beta\dot\alpha(2)} + \theta^2\bar\theta^2 h_{\alpha(2)\dot\alpha(2)} + (\text{fermionic fields})$$

- Their gauge transformations are entangled ($\implies$ coupled in action)

$$\delta_\xi h_{\alpha(3)\dot\alpha(3)} = \nabla_{(\alpha_1(\dot\alpha_1}\xi_{\alpha_2\alpha_3)\dot\alpha_2\dot\alpha_3)}$$
$$\delta_\xi h_{\alpha(2)\dot\alpha(2)} = \nabla_{(\alpha_1(\dot\alpha_1}\xi_{\alpha_2)\dot\alpha_2)} + C_{\alpha(2)}{}^{\beta(2)}\xi_{\beta(2)\dot\alpha(2)} + \text{c.c.}$$

- Experience suggests that gauge invariance of $H_{\alpha(2)\dot\alpha(2)}$ model requires

Lower-spin supermultiplets: $\boxed{H_{\alpha\dot\alpha}}$ & H

Recipe for conformal spin-3 model

- Spin-3 $H_{\alpha(2)\dot{\alpha}(2)}$ and spin-2 $H_{\alpha\dot{\alpha}}$ supermultiplets are entangled

$$\delta_\zeta H_{\alpha\dot{\alpha}} = \bar{\nabla}_{\dot{\alpha}}\zeta_\alpha + W_\alpha{}^{\beta(2)}\zeta_{\beta(2)\dot{\alpha}} + \text{c.c.}$$

- Component level: $H_{\alpha\dot{\alpha}}$ contains both spin-2 and spin-1 conformal fields

$$H_{\alpha\dot{\alpha}}(\theta,\bar{\theta}) = \theta^\beta\bar{\theta}^{\dot{\beta}} h_{\beta\alpha\dot{\beta}\dot{\alpha}} + \theta^2\bar{\theta}^2 h_{\alpha\dot{\alpha}} + (\text{fermionic fields})$$

- Spin-2 and spin-1 fields become entangled with spin-3 field

$$\delta_\xi h_{\alpha(3)\dot{\alpha}(3)} = \nabla_{(\alpha_1(\dot{\alpha}_1}\xi_{\alpha_2)\dot{\alpha}_2)} + C_{\alpha(2)}{}^{\beta(2)}\xi_{\beta(2)\dot{\alpha}(2)} + \text{c.c.}$$

$$\delta_\xi h_{\alpha\dot{\alpha}} = \nabla_{\alpha\dot{\alpha}}\xi + C_\alpha{}^{\beta(3)}\nabla_\beta{}^{\dot{\beta}}\xi_{\beta(2)\dot{\alpha}\dot{\beta}} - 3\nabla_\beta{}^{\dot{\beta}}C_\alpha{}^{\beta(3)}\xi_{\beta(2)\dot{\alpha}\dot{\beta}} + \text{c.c.}$$

- **Outcome**: Gauge invariant model for conformal spin-3 must take the form

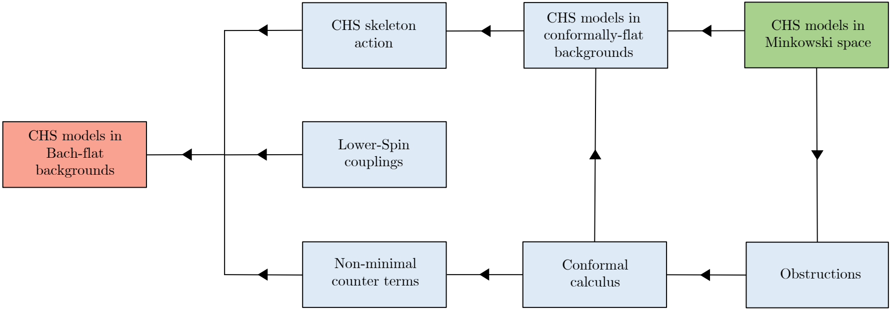
$S_{\text{Spin-3}} = S_{33} + S_{22} + S_{11} + S_{23} + S_{13} + \cdots$ SMK, M.P. & E. Raptakis (2020)

- Conjectured by Grigoriev & Tseytlin, Spin-2 coupling is new
- Can repeat supersymmetric argument for any fixed spin $s_0 \geq 3$ with outcome:

$$S_{\text{Spin-}s_0} = \sum_{s=1}^{s_0} S_{s,s} + \sum_{s=1}^{s_0}\sum_{s'<s} S_{s,s'} + \cdots \qquad \big(\text{Truncated Tower} \quad \forall s \leq s_0\big)$$

Conclusion

The bottom up approach:



- Possible future directions:
  - **Spin-3 CHS model to all orders in background curvature**
  - Supersymmetric extensions
  - Compute anomaly contributions
  - Top down approach for fermion CHS fields