GENERAL RELATIVITY HOMEWORK – WEEK 5

Exercise 1. *Consider the three possible Lorentz-invariant Lagrangian terms:*

$$\mathcal{L}_1 = \partial_\mu A_\nu \partial^\mu A^\nu \ ; \quad \mathcal{L}_2 = \partial_\mu A_\nu \partial^\nu A^\mu \ ; \quad \mathcal{L}_3 = (\partial_\mu A^\mu)^2 \ . \tag{1}$$

For each of these, compute the derivatives $\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}$, *and the contribution* $\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}$ *to the Euler-Lagrange equations. Are any of your answers the same?*

Exercise 2. *In this exercise, we'll derive half of the Maxwell equations from the relativistic Lagrangian of the electromagnetic field (in the presence of charges and currents):*

$$\mathcal{L} = A_\mu j^\mu - \frac{\epsilon_0}{2}(\mathcal{L}_1 - \mathcal{L}_2) = A_\mu j^\mu - \frac{\epsilon_0}{4} F_{\mu\nu} F^{\mu\nu} \ . \tag{2}$$

1. *Find the Euler-Lagrange field equations* $\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{\partial \mathcal{L}}{\partial A_\mu}$.
2. *Write them in terms of* $F_{\mu\nu}$ *(i.e. without any "naked"* A_μ*'s).*
3. *What is the equations' t component? Write it in terms of the electric field* $\mathbf{E}$ *and the charge density* ρ.
4. *What is the equations' x component? Write it in terms of the electric field* $\mathbf{E}$, *the magnetic field* $\mathbf{B}$ *and the current density* $\mathbf{j}$.

Exercise 3. *The other half of the Maxwell equations comes from the identity* $\partial_{[\mu} F_{\nu\rho]} = 2\partial_{[\mu}\partial_\nu A_{\rho]} = 0$ *(remember that partial derivatives commute* $\partial_{[\mu}\partial_{\nu]} = 0$*!)*

1. *Write the xyz component of* $\partial_{[\mu} F_{\nu\rho]} = 0$ *in terms of the magnetic field* $\mathbf{B}$.
2. *Write the txy component of* $\partial_{[\mu} F_{\nu\rho]} = 0$ *in terms of the electric field* $\mathbf{E}$ *and the magnetic field* $\mathbf{B}$.