∞-dim Lie algebras & higher spins

- A "Cartan" approach to higher-spin gauge theories:
  - *1987*: proposal for a **higher-spin algebra** in AdS_4 — Fradkin, Vasiliev
  - *1990*: procedure to implement its **gauging** → Vasiliev's equations — Vasiliev
  - 2003: higher-spin algebras and interacting e.o.m. in AdS_D — Eastwood; Vasiliev

- Other recent (and less recent) developments
  - 3D HS algebras → Chern-Simons gauge theories (& matter couplings)

    Blencowe (1989); Porkushkin, Vasiliev (1999) & many others…
  - HS algebras for mixed symmetry and partially-massless fields

    Boulanger, Skvortsov (2011); Joung, Mkrtchyan (2016)

Higher spins & (A)dS

- Why (massless) HS fields like (A)dS?
  - Long-range HS interactions:
    - in flat-space → trivial S-matrix   Weinberg (1964)
    - in AdS → free CFT boundary correlators → "soluble" AdS/CFT

Sezgin, Sundell (2002); Klebanov, Polyakov (2002); Maldacena, Zhiboedov (2011) et al.

- May Minkowski still play a role?
  - Is String Theory a broken phase of a HS gauge theory?
  - Models with trivial S-matrix, but non-trivial interactions (*& symmetries*)?

Skvortsov, Tran, Tsulaia (2018); A.C., Francia, Heissenberg (2017)

- Outlook: *"non-AdS" holography with higher spins*

see e.g. Ponomarev (2021)

Higher-spin algebras

- Key ingredient in building HS theories and studying HS holography
- **What is a HS algebra?** *Lie algebra on traceless Killing tensors*
  - Poincaré & (A)dS algebras: isometries of the vacuum

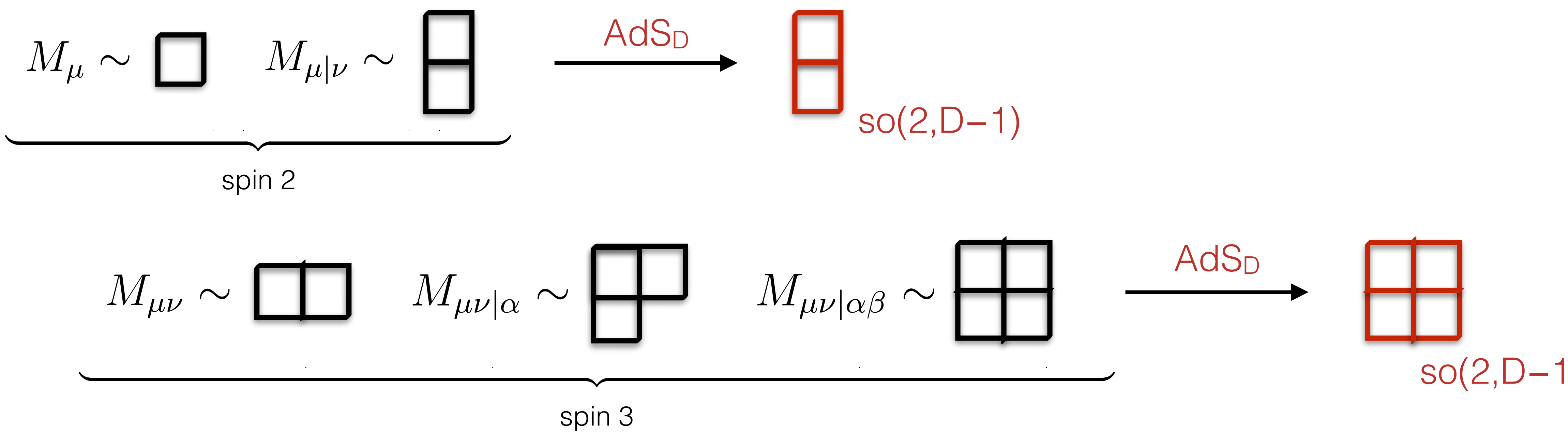
HS "isometries" of the vacuum

- Fronsdal's gauge transf.: $\delta\varphi_{\mu_1\cdots\mu_s} = \bar{\nabla}_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)} + \mathcal{O}(\varphi)$
- Vacuum-preserving symm.: $\bar{\nabla}_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)} = 0$
- Solution (in Minkowski): $\epsilon_{\mu_1\cdots\mu_{s-1}} = \sum_{k=0}^{s-1} M_{\mu_1\cdots\mu_{s-1}|\nu_1\cdots\nu_k} x^{\nu_1}\cdots x^{\nu_k}$

$$\epsilon_{\mu_1\cdots\mu_{s-3}\lambda}{}^{\lambda} = 0$$

Higher-spin algebras

- Vector space of traceless Killing tensors:



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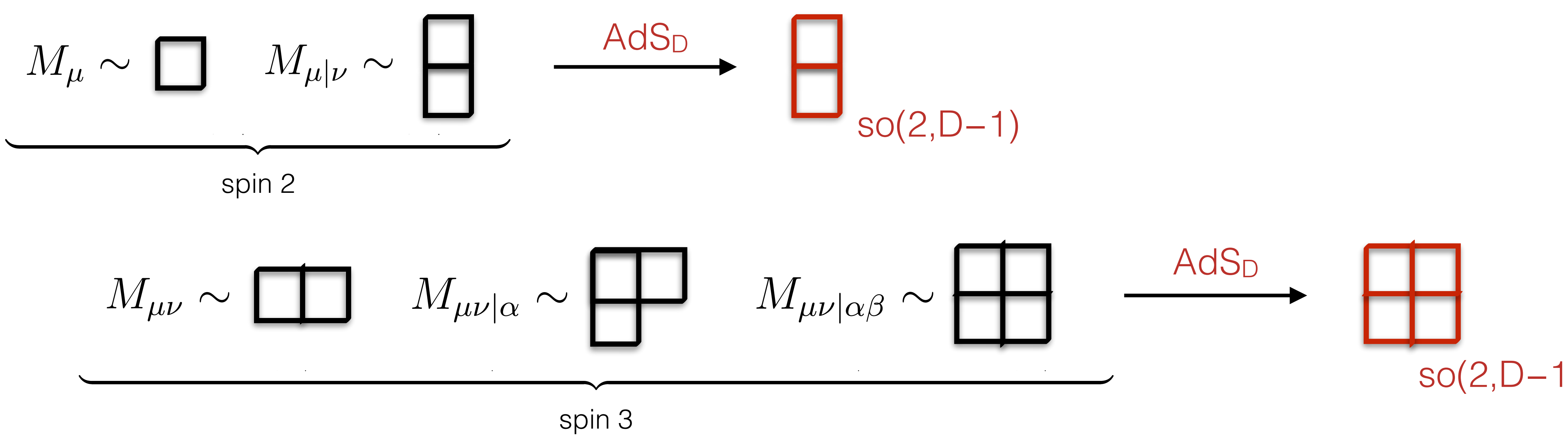
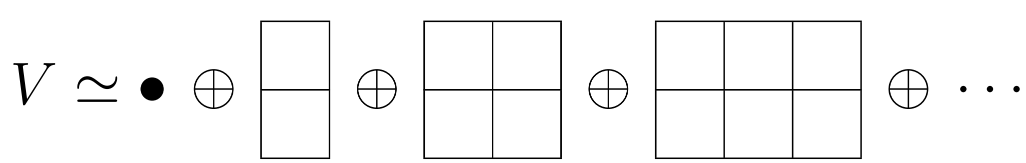


Eastwood-Vasiliev algebras in any D: non-Abelian Lie algebras on V including a so(2,D−1) subalgebra

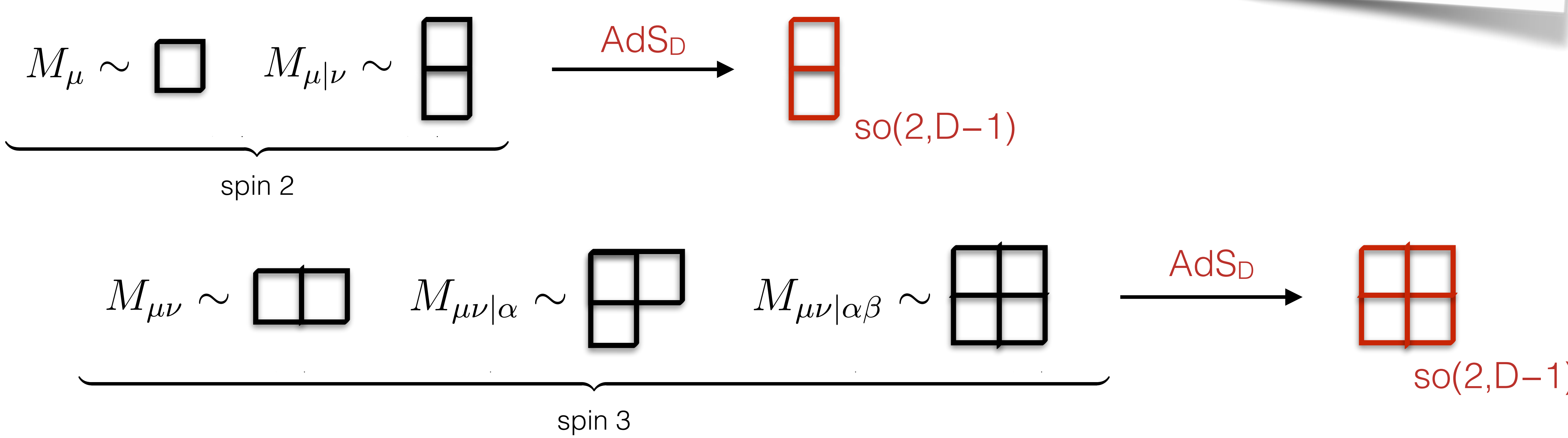


Fradkin, Vasiliev (1987);
Eastwood (2002);
Segal (2002);
Vasiliev (2003)

Higher-spin algebras

so(2,D−1): isometries of AdS$_D$ & *conformal symmetries (in D−1)*

- Vector space of traceless Killing te



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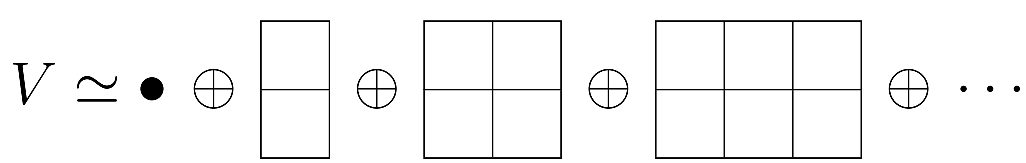

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Notable so(2,D−1) Inönü-Wigner contractions

AdS isometries $\qquad$ Minkowski isometries

$$so(2,D-1) \xrightarrow{\Lambda \to 0} iso(1,D-1)$$

Notable so(2,D−1) Inönü-Wigner contractions

AdS isometries

$so(2,D-1)$

conformal algebra

$\Lambda \to 0$

$\longrightarrow$

$c \to 0$

Minkowski isometries

$iso(1,D-1)$

conformal Carroll algebra

Notable so(2,D−1) Inönü-Wigner contractions

AdS isometries

so(2,D−1)

conformal algebra

$\Lambda \to 0$

$c \to 0$

Minkowski isometries

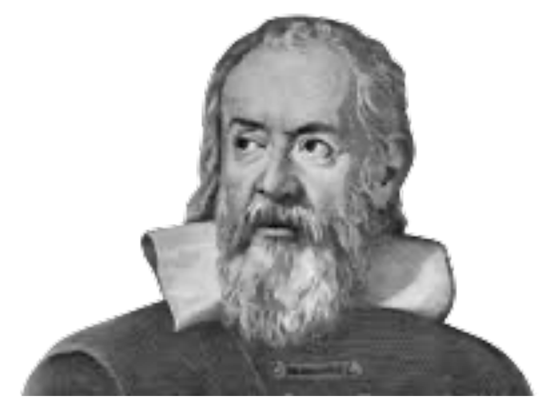
iso(1,D−1)

conformal Carroll algebra

$c \to \infty$

gca_{D-1}

conformal Galilei algebra



What about higher-spin algebras?

Goals & strategy/hypotheses

- **Goal:** classify Lie algebras defined on the vector space V (traceless Killing tensors) that

    1. contain a Poincaré subalgebra, **iso(1,D−1)**
    2. contain a conformal Galilei subalgebra, $\mathbf{gca}_{D-1}$

  ...and discuss their properties

- **Strategy:** look for *coset algebras*, obtained by quotienting out an ideal from the universal enveloping algebras of iso(1,D−1) or gca_{D-1} (bonus: "good" Lorentz transf. for free) Eastwood (2002)

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Eastwood (2002)

↑ partial classification, still with interesting examples!

HS algebras in AdS_D

Conformal HS algebras in D−1 dimensions

Coset construction of HS algebras

- so(2,D−1) algebra: $[J_{AB}\,,\,J_{CD}] = \tilde{\eta}_{AC}\,J_{BD} - \tilde{\eta}_{BC}\,J_{AD} - \tilde{\eta}_{AD}\,J_{BC} + \tilde{\eta}_{BD}\,J_{AC}$

- Quadratic products of the generators

$$J_{A(B} \odot J_{C)D} - \text{traces} \sim \yng(2,2) \qquad\qquad C_2 \equiv \frac{1}{2}\,J_{AB} \odot J^{BA} \sim \bullet$$

$$\mathcal{I}_{AB} \equiv J_{C(A} \odot J_{B)}{}^{C} - \frac{2}{D+1}\,\tilde{\eta}_{AB}\,C_2 \sim \yng(2) \qquad\qquad \mathcal{I}_{ABCD} \equiv J_{[AB} \odot J_{CD]} \sim \yng(1,1,1,1)$$

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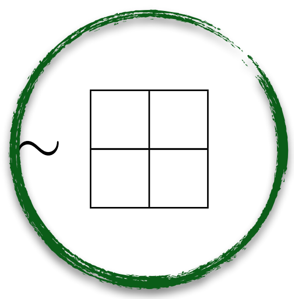
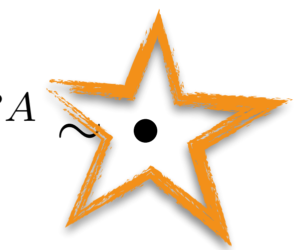
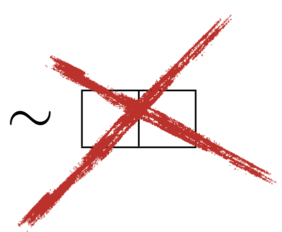
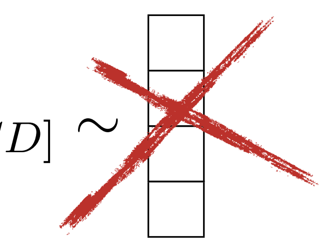
$J_{A(B} \odot J_{C)D} - \text{traces} \sim$ 

$C_2 \equiv \frac{1}{2}\,J_{AB} \odot J^{BA} \sim \bullet$ 

$\mathcal{I}_{AB} \equiv J_{C(A} \odot J_{B)}{}^{C} - \frac{2}{D+1}\,\tilde{\eta}_{AB}\,C_2 \sim$ 

$\mathcal{I}_{ABCD} \equiv J_{[AB} \odot J_{CD]} \sim$ 

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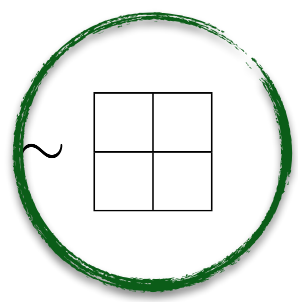

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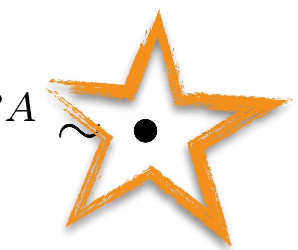

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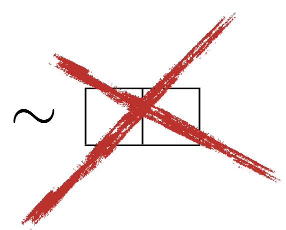

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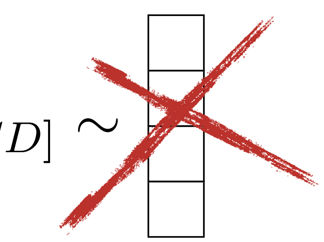

- Eastwood-Vasiliev algebras:

$$\mathfrak{hs}_D = \frac{\mathcal{U}(\mathfrak{so}(2,D-1))}{\langle \mathcal{I}_{AB} \oplus \mathcal{I}_{ABCD} \rangle} \;\Rightarrow\; C_2 \sim -\frac{(D+1)(D-3)}{4}\,id$$

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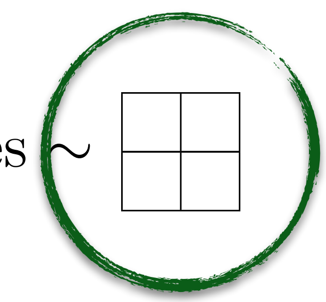
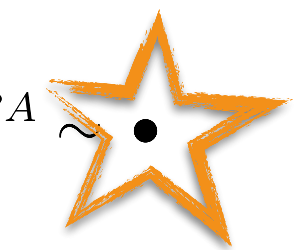
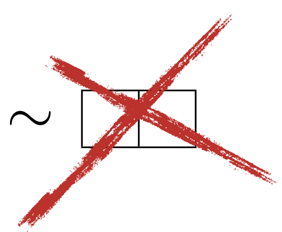
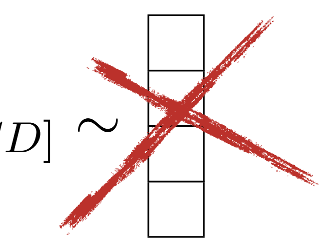
$$J_{A(B} \odot J_{C)D} - \text{traces} \sim$$ 

$$C_2 \equiv \frac{1}{2}\, J_{AB} \odot J^{BA} \sim \bullet$$ 

$$\mathcal{I}_{AB} \equiv J_{C(A} \odot J_{B)}{}^{C} - \frac{2}{D+1}\,\tilde{\eta}_{AB}\,C_2 \sim$$ 

$$\mathcal{I}_{ABCD} \equiv J_{[AB} \odot J_{CD]} \sim$$ 

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Iazeolla, Sundel (2008)

scalar singleton module

Special cases: D=3

- so(2,2) algebra: $[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}\,, \quad [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n}\,, \quad [\mathcal{L}_m, \bar{\mathcal{L}}_n] = 0$
- Ideal to be factored out: $\mathcal{I}_{AB} \sim 0 \quad \Rightarrow \quad \mathcal{L}_m \bar{\mathcal{L}}_n \sim 0$
- No need to factor out $W \equiv \frac{1}{8}\,\varepsilon^{ABCD}\mathcal{I}_{ABCD}$ but $W^2 \sim \frac{1}{4}\,(C_2)^2$
- Still, better to get rid of C_2: $C_2 = 2\left(\mathcal{L}^2 + \bar{\mathcal{L}}^2\right) \sim \frac{\lambda^2-1}{2}\,id$

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- Not factorising $\mathcal{I}_{ABCD}$ gives a one-parameter family of HS algebras

$$\mathfrak{hs}_3[\lambda] = id \oplus W \oplus \mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda] \quad \text{with} \quad \mathbb{1} \oplus \mathfrak{hs}[\lambda] = \frac{\mathcal{U}(\mathfrak{sl}\,(2,\mathbb{R}))}{\left\langle C_2 - \frac{\lambda^2-1}{4}\,\mathbb{1} \right\rangle}$$

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- Are we evaluating *U*(so(2,2)) on which module?

- Simple answer for $\lambda \in \mathbb{N}$:

$$\mathcal{L}_m = \begin{pmatrix} l_m & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\mathcal{L}}_m = \begin{pmatrix} 0 & 0 \\ 0 & \bar{l}_m \end{pmatrix}$$ with l_m *N×N irrep of* $\mathfrak{sl}\,(2,\mathbb{R})$

$$\Rightarrow \quad C_2 = \frac{N^2-1}{2}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad W = \frac{N^2-1}{4}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

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Casimir operator not proportional to the identity

Special cases: D=5

- Again, no need to factor out $\mathcal{I}_{ABCD}$

- Ideal:

$$\mathcal{I}_{AB} \equiv J_{C(A}J_{B)}{}^{C} - \frac{1}{3}\,\tilde{\eta}_{AB}\,C_2\,,$$

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mixing terms with different # of J_{AB}

- One parameter family of HS algebras

Boulanger, Skvortsov (2011)

$$\mathfrak{hs}_5[\lambda] = \frac{\mathcal{U}(\mathfrak{so}(2,4))}{\langle \mathcal{I}_{AB} \oplus \mathcal{I}^{\lambda}_{ABDC} \rangle} \quad \Rightarrow \quad C_2 \sim 3\left(\lambda^2 - 1\right) id$$

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Warming up in 3D

Carrollian and Galilean 3D HS algebras

- iso(1,2) and gca_2 are isomorphic

Bagchi, Gopakumar, Mandal, Miwa (2009)

- $\mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda]$ algebra:

Bergshoeff, Blencowe, Stelle; Bordemann, Hoppe, Schaller; Fradkin, Linetsky; Pope, Romans, Shen (1990)

$$P_m^{(s)} \equiv \epsilon \left(\mathcal{L}_m^{(s)} - \bar{\mathcal{L}}_m^{(s)} \right) , \qquad L_m^{(s)} \equiv \mathcal{L}_m^{(s)} + \bar{\mathcal{L}}_m^{(s)}$$

$$\begin{cases} \mathcal{L}^{(s)}_{\pm(s-1)} \equiv \left(\mathcal{L}_{\pm 1} \right)^{s-1} \\ \mathcal{L}^{(s)}_{m\mp 1} \equiv \frac{\mp 1}{s \pm m - 1} \left[\mathcal{L}_{\mp 1} , \mathcal{L}_m^{(s)} \right] \end{cases}$$

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$$\left[P_m^{(s)}, P_n^{(t)} \right] = \epsilon^2 \sum_{\substack{u=|s-t|+2 \\ s+t+u \text{ even}}}^{s+t-2} g_{s+t-u}^{st}(m,n;\lambda) L_{m+n}^{(u)} \, ,$$

$$\left[L_m^{(s)}, P_n^{(t)} \right] = \sum_{\substack{u=|s-t|+2 \\ s+t+u \text{ even}}}^{s+t-2} g_{s+t-u}^{st}(m,n;\lambda) P_{m+n}^{(u)} \, ,$$

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- Carrollian (aka flat!) & Galilean limits defined by $\epsilon \to 0$

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Blencowe (1989); Afshar, Bagchi, Fareghbal, Grumiller, Rosseel (2013); Gonzalez, Matulich, Pino, Troncoso (2013); Ammon, Grumiller, Prohazka, Riegler, Wutte (2017)

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ihs[λ] algebra

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[P,P] ≃ ~~L~~

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Coset construction from *U*(iso(1,2))

- $\mathfrak{hs}[\lambda]$ generators: $\mathcal{L}^{(s)}_{\pm(s-1)} \equiv (\mathcal{L}_{\pm})^{s-1}$ & $\mathcal{L}^{(s)}_{m\mp1} \equiv \frac{\mp1}{s \pm m - 1}\,[\,\mathcal{L}_{\mp}, \mathcal{L}^{(s)}_{m}\,]$

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- ***Which option do you choose?***

A $\quad P^{(s)}_{\pm(s-1)} \equiv (P_\pm)^{s-1}$ & $L^{(s)}_{\pm(s-1)} \equiv (s-1)\,(P_\pm)^{s-2}\,L_\pm$

B $\quad L^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-1}$ & $P^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-2}\,P_\pm$

(other components fixed by [$L_\pm$, .])

Coset construction from *U*(iso(1,2))

- $\mathfrak{hs}[\lambda]$ generators: $\mathcal{L}^{(s)}_{\pm(s-1)} \equiv (\mathcal{L}_\pm)^{s-1}$ & $\mathcal{L}^{(s)}_{m\mp1} \equiv \dfrac{\mp1}{s \pm m - 1}\,[\,\mathcal{L}_\mp, \mathcal{L}^{(s)}_m\,]$

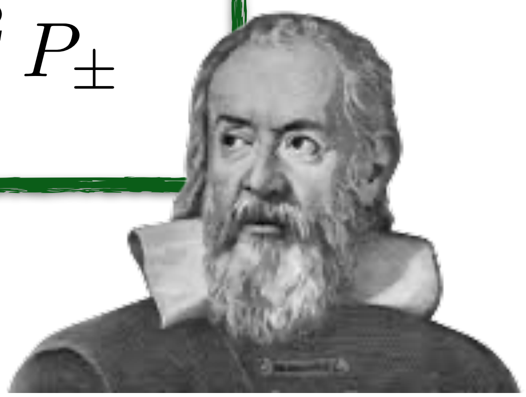
- We wish to get [P,P] ≃ 0, [L,P] ≃ P and [L,L] ≃ L

- ***Which option do you choose?***

commutators close, but we can only get ihs[∞]

A $$P^{(s)}_{\pm(s-1)} \equiv (P_\pm)^{s-1} \quad \& \quad L^{(s)}_{\pm(s-1)} \equiv (s-1)\,(P_\pm)^{s-2}\,L_\pm$$

B $$L^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-1} \quad \& \quad P^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-2}\,P_\pm$$



(other components fixed by [$L_\pm$, .])

Coset construction from *U*(iso(1,2))

see also Ammon, Pannier, Riegler (2009)

- HS generators: $L^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-1}$ & $P^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-2}\,P_\pm$ etc.

- Consistency conditions to recover the ihs[λ] commutators:

$$P_m P_n \sim 0 \qquad L_m P_n \sim P_m L_n \qquad L^2 - \frac{\lambda^2-1}{4}\,id \sim 0$$

Poincaré ideal

- $id \oplus W \oplus \mathfrak{ihs}_3[\lambda] = \mathcal{U}(\mathfrak{iso}(1,2))/\left\langle \mathcal{I} \right\rangle$

Coset construction from *U*(iso(1,2))

see also Ammon, Pannier, Riegler (2009)

- HS generators: $L^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-1}$ & $P^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-2}\, P_\pm$ etc.

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Poincaré ideal

- On which representation are we evaluating *U*(iso(1,2))?

Coset construction from *U*(iso(1,2))

see also Ammon, Pannier, Riegler (2009)

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Poincaré ideal

- On which representation are we evaluating *U*(iso(1,2))?

$$L_m = \begin{pmatrix} l_m & 0 \\ 0 & l_m \end{pmatrix}, \quad P_m = \begin{pmatrix} 0 & l_m \\ 0 & 0 \end{pmatrix}$$

with l_m *N* × *N* *irrep of* $\mathfrak{so}(1,2) \simeq \mathfrak{sl}(2,\mathbb{R})$

$$\Rightarrow \qquad L^2 = \begin{pmatrix} l^2 & 0 \\ 0 & l^2 \end{pmatrix} = \frac{N^2-1}{4}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad W = \begin{pmatrix} 0 & l^2 \\ 0 & 0 \end{pmatrix} = \frac{N^2-1}{4}\begin{pmatrix} 0 & \mathbb{1} \\ 0 & 0 \end{pmatrix}$$

From U(so(2,D)) to U(iso(1,2))

- so(2,2) ideal: $$\mathcal{I}_{AB} \sim 0 \;\Rightarrow\; \mathcal{L}_m \bar{\mathcal{L}}_n \sim 0 \;\Rightarrow\; \begin{cases} P_m P_n - L_m L_n \sim 0 \\ L_m P_n - P_m L_n \sim 0 \end{cases}$$

$$C_2 = L^2 + P^2 \sim 2\, L^2 \sim \frac{\lambda^2 - 1}{2}\, id$$

- Introducing the contraction parameter via $P_m \to \epsilon^{-1} P_m$

$$\begin{aligned} \epsilon^{-2} P_m P_n - L_m L_n &\sim 0 \\ \epsilon^{-1} \left(L_m P_n - P_m L_n \right) &\sim 0 \\ L^2 - \frac{\lambda^2 - 1}{4}\, id &\sim 0 \end{aligned} \quad \underset{\epsilon \to 0}{\Longrightarrow} \quad \begin{aligned} P_m P_n &\sim 0 \\ L_m P_n - P_m L_n &\sim 0 \\ L^2 - \frac{\lambda^2 - 1}{4}\, id &\sim 0 \end{aligned}$$

Poincaré ideal

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Poincaré ideal

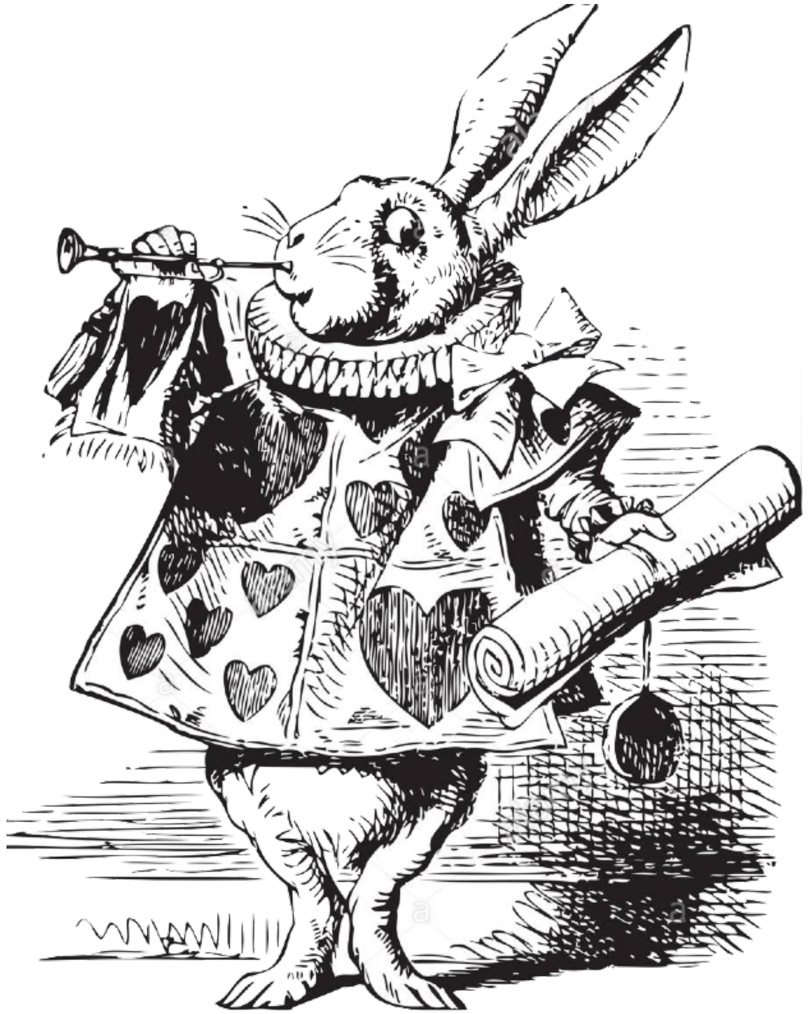
Key lessons from 3D

- The limit of the AdS_3 ideal is still an ideal

- Factoring it out from the Poincaré universal enveloping algebra gives a higher-spin algebra with the same spectrum as the AdS_3 one

Carrollian conformal HS algebras

(in any dimensions)



From $U(\mathrm{so}(2,D-1))$ to $U(\mathrm{iso}(1,D-1))$

- Now reverse the logic: look at how the contraction affects the so(2,D−1) ideal to *define* the iso(1,D−1) coset

From *U*(so(2,D−1)) to *U*(iso(1,D−1))

- Now reverse the logic: look at how the contraction affects the so(2,D−1) ideal to *define* the iso(1,D−1) coset

$$[J_{AB}\,,\,J_{CD}] = \tilde{\eta}_{AC}\,J_{BD} - \tilde{\eta}_{AD}\,J_{BC} - \tilde{\eta}_{BC}\,J_{AD} + \tilde{\eta}_{BD}\,J_{AC}$$

$$\mathcal{P}_a \equiv \epsilon\,J_{aD}\,, \qquad \mathcal{J}_{ab} \equiv J_{ab}$$

$$[\mathcal{J}_{ab}\,,\,\mathcal{J}_{cd}] = \eta_{ac}\,\mathcal{J}_{bd} - \eta_{ad}\,\mathcal{J}_{bc} - \eta_{bc}\,\mathcal{J}_{ad} + \eta_{bd}\,\mathcal{J}_{ac}\,,$$

$$[\mathcal{J}_{ab}\,,\,\mathcal{P}_c] = \eta_{ac}\,\mathcal{P}_b - \eta_{bc}\,\mathcal{P}_a\,,$$

$$[\mathcal{P}_a\,,\,\mathcal{P}_b] = -\,\epsilon^2\,\mathcal{J}_{ab}\,,$$

- Next step: branching so(2,D−1) → so(1,D−1) of the ideal

From U(so(2,D−1)) to U(iso(1,D−1))

- Branching so(2,D−1) → so(1,D−1) of the ideal

$$\mathcal{I}_{AB} \sim 0 \quad \Rightarrow \quad \begin{aligned} \mathcal{J}^2 - \frac{D-1}{2}\,\epsilon^{-2}\,\mathcal{P}^2 &\sim 0 \\ \epsilon^{-1}\,\{\mathcal{P}^b,\,\mathcal{J}_{ba}\} &\sim 0 \\ \mathcal{S}_{ab} + \epsilon^{-2}\,\mathcal{Q}_{ab} &\sim 0 \end{aligned}$$

$$\mathcal{Q}_{ab} \equiv \{\mathcal{P}_a, \mathcal{P}_b\} - \frac{2}{D}\,\eta_{ab}\mathcal{P}^2$$

$$\mathcal{S}_{ab} \equiv \{\mathcal{J}^c{}_a, \mathcal{J}_{bc}\} - \frac{4}{D}\,\eta_{ab}\mathcal{J}^2$$

$$\mathcal{I}_{ABCD} \sim 0 \quad \Rightarrow \quad \begin{aligned} \epsilon^{-1}\,\{\mathcal{J}_{[ab}\,,\mathcal{P}_{c]}\} &\sim 0 \\ \{\mathcal{J}_{[ab}\,,\,\mathcal{J}_{cd]}\} &\sim 0 \end{aligned}$$

$$C_2 \equiv \mathcal{J}^2 + \epsilon^{-2}\,\mathcal{P}^2 \sim -\frac{(D+1)(D-3)}{4}\,id \quad \Rightarrow \quad \begin{aligned} \mathcal{J}^2 &\sim \frac{D-1}{D+1}C_2 \sim -\frac{(D-1)(D-3)}{4}\,id \\ \epsilon^{-2}\,\mathcal{P}^2 &\sim \frac{2}{D+1}C_2 \sim -\frac{D-3}{2}\,id\,, \end{aligned}$$

Coset construction from *U*(iso(1,D–1))

- iso(1,D–1) ideal

$$\mathcal{P}_a\mathcal{P}_b \sim 0$$

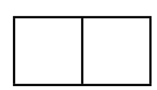
$$\mathcal{I}_a \equiv \{\mathcal{P}^b,\, \mathcal{J}_{ba}\} \sim 0$$

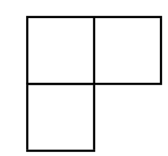
$$\mathcal{I}_{abc} \equiv \{\mathcal{J}_{[ab}\,,\mathcal{P}_{c]}\} \sim 0$$

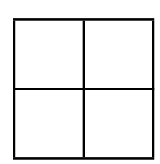
$$\mathcal{I}_{abcd} \equiv \{\mathcal{J}_{[ab}\,,\, \mathcal{J}_{cd]}\} \sim 0$$

$$\mathcal{J}^2 + \frac{(D-1)(D-3)}{4}\, id \sim 0$$

- Leftover quadratic combinations, i.e. spin-3 generators:

$$\mathcal{S}_{ab} \equiv \{\mathcal{J}^c{}_{(a}, \mathcal{J}_{b)c}\} - \text{tr.} \simeq$$ 

$$\mathcal{M}_{ab|c} \equiv \{\mathcal{J}_{a(c}, \mathcal{J}_{d)b}\} - \text{tr.} \simeq$$ 

$$\mathcal{K}_{ab|cd} \equiv \{\mathcal{P}_{(a}, \mathcal{J}_{b)c}\} - \text{tr.} \simeq$$ 

$$\mathfrak{ihs}_D \equiv \mathcal{U}(\mathfrak{iso}(1, D-1))/\langle \mathcal{I}_{\mathfrak{c}} \rangle$$

Some commutators…

- All generators transform as Lorentz tensors

$$[\mathcal{J}_{ab}, \mathcal{S}_{cd}] = \eta_{ac}\mathcal{S}_{bd} + \eta_{ad}\mathcal{S}_{bc} - \eta_{bc}\mathcal{S}_{ad} - \eta_{bd}\mathcal{S}_{ac}\,,$$

$$[\mathcal{J}_{ab}, \mathcal{M}_{cd|e}] = 2\,\eta_{a(c}\mathcal{M}_{d)b|e} + \eta_{ae}\mathcal{M}_{cd|b} - 2\,\eta_{b(c}\mathcal{M}_{d)a|e} - \eta_{be}\mathcal{M}_{cd|a}\,,$$

$$[\mathcal{J}_{ab}, \mathcal{K}_{cd|ef}] = 2\left(\eta_{a(c}\,\mathcal{K}_{d)b|ef} + \eta_{a(e}\,\mathcal{K}_{f)b|cd} - \eta_{b(c}\,\mathcal{K}_{d)a|ef} - \eta_{b(e}\,\mathcal{K}_{f)a|cd}\right)$$

- Commutators with translations:

$$[\mathcal{P}_a, \mathcal{S}_{bc}] = -2\,\mathcal{M}_{bc|a}\,,$$

$$[\mathcal{P}_a, \mathcal{M}_{bc|d}] = 0\,,$$

$$[\mathcal{P}_a, \mathcal{K}_{bc|de}] = -\,\eta_{ab}\,\mathcal{M}_{de|c} - \eta_{ac}\,\mathcal{M}_{de|b} - \eta_{ad}\,\mathcal{M}_{bc|e} - \eta_{ae}\,\mathcal{M}_{bc|d} - \frac{2}{D-2}\left(\eta_{d(b}\mathcal{M}_{c)e|a} + \eta_{e(b}\mathcal{M}_{c)d|a} - \eta_{bc}\mathcal{M}_{de|a} - \eta_{de}\mathcal{M}_{bc|a}\right)$$

Some commutators…

- All generators transform as Lorentz tensors

$$[\mathcal{J}_{ab}, \mathcal{S}_{cd}] = \eta_{ac}\mathcal{S}_{bd} + \eta_{ad}\mathcal{S}_{bc} - \eta_{bc}\mathcal{S}_{ad} - \eta_{bd}\mathcal{S}_{ac} \,,$$

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- Commutators with translations:

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The linearised curvatures do not reproduce those of Fradkin and Vasiliev

Structure of the algebra

- Higher-spin generators

$$\mathcal{Z}^{s,t} \equiv \begin{array}{|c|}\hline s-1 \\ \hline s-t-1 \\ \hline \end{array} \quad \text{with } t \in \{0,\dots,s-1\}$$

  - t even: no P's
  - t odd: one P

- Commutators with P

For D=4 see also
Fradkin, Vasiliev (1987)

$$\left[\mathcal{P}, \mathcal{Z}^{(s,t)}\right] \propto \mathcal{Z}^{(s,t-1)} + \eta\, \mathcal{Z}^{(s,t+1)} \quad \text{for } t \text{ even}$$

$$\left[\mathcal{P}, \mathcal{Z}^{(s,t)}\right] = 0 \quad \text{for } t \text{ odd}$$

- $\mathfrak{ihs}_D$ as Inönü-Wigner contraction of $\mathfrak{hs}_D$

$$\left[\mathcal{Z}^{(s_1,t_1)}, \mathcal{Z}^{(s_2,t_2)}\right] \propto \sum_{s_3,t_3} \mathcal{Z}^{(s_3,t_3)} \quad \text{with} \quad \begin{array}{l} s_1+s_2-s_3 \mod 2 = 0 \\ t_1+t_2-t_3 \mod 2 = 0 \end{array}$$

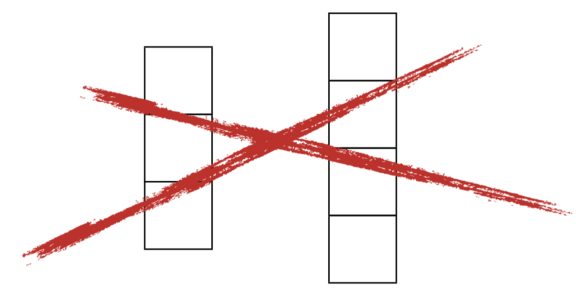
$$\Rightarrow \quad \mathcal{Z}^{(s,t)} \to \epsilon^{-1}\mathcal{Z}^{(s,t)} \quad \text{for } t \text{ odd}$$

Classification of consistent ideals

- Can one build other conformal Carrollian HS algebras from *U*(iso(1,D−1))?

- Portion of the ideal we need to quotient out:

$$\mathcal{I}_{ABCD} \sim 0 \quad \Rightarrow \quad \begin{array}{r} \epsilon^{-1}\,\{\mathcal{J}_{[ab}\,,\mathcal{P}_{c]}\} \sim 0 \\ \{\mathcal{J}_{[ab}\,,\,\mathcal{J}_{cd]}\} \sim 0 \end{array}$$



- Candidate spin-3 generators:

recall the 3D poll!

$$\{\mathcal{P}_\mu, \mathcal{P}_\nu\} - \text{tr.} \;\simeq\; \square\!\square \qquad \{\mathcal{J}^{\rho}{}_{(\mu}, \mathcal{J}_{\nu)\rho}\} - \text{tr.} \;\simeq\; \square\!\square$$

Classification of consistent ideals

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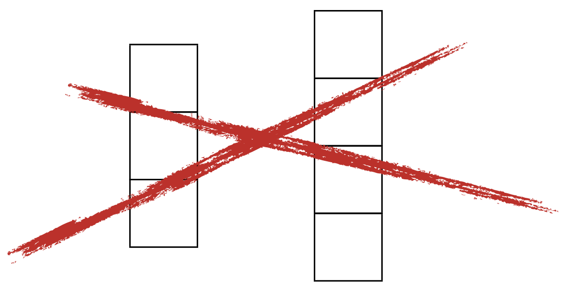

- Candidate spin-3 generators:

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$$\{\mathcal{P}_\mu, \mathcal{P}_\nu\} - \text{tr.} \;\; \simeq \;\; \square\!\square \qquad \{\mathcal{J}^\rho{}_{(\mu}, \mathcal{J}_{\nu)\rho}\} - \text{tr.} \;\; \simeq \;\; \square\!\square$$

- Can one use $P_\mu P_\nu$ as spin-3 generator?

$$[\mathcal{P}_\alpha\,,\, \mathcal{J}^\rho{}_{(\mu}\mathcal{J}_{\nu)\rho} - \frac{2}{D}\eta_{\mu\nu}\,\mathcal{J}^2] = \{\mathcal{J}_{\alpha(\mu}, \mathcal{P}_{\nu)}\} + \cdots$$

Classification of consistent ideals

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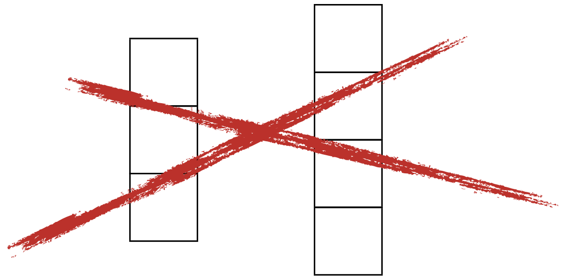

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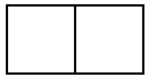
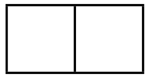
$$\{\mathcal{P}_\mu, \mathcal{P}_\nu\} - \text{tr.} \;\simeq$$  $$\{\mathcal{J}^\rho{}_{(\mu}, \mathcal{J}_{\nu)\rho}\} - \text{tr.} \;\simeq$$ 

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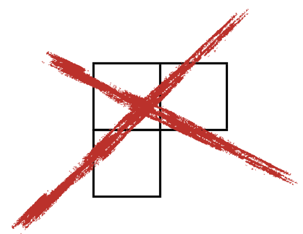
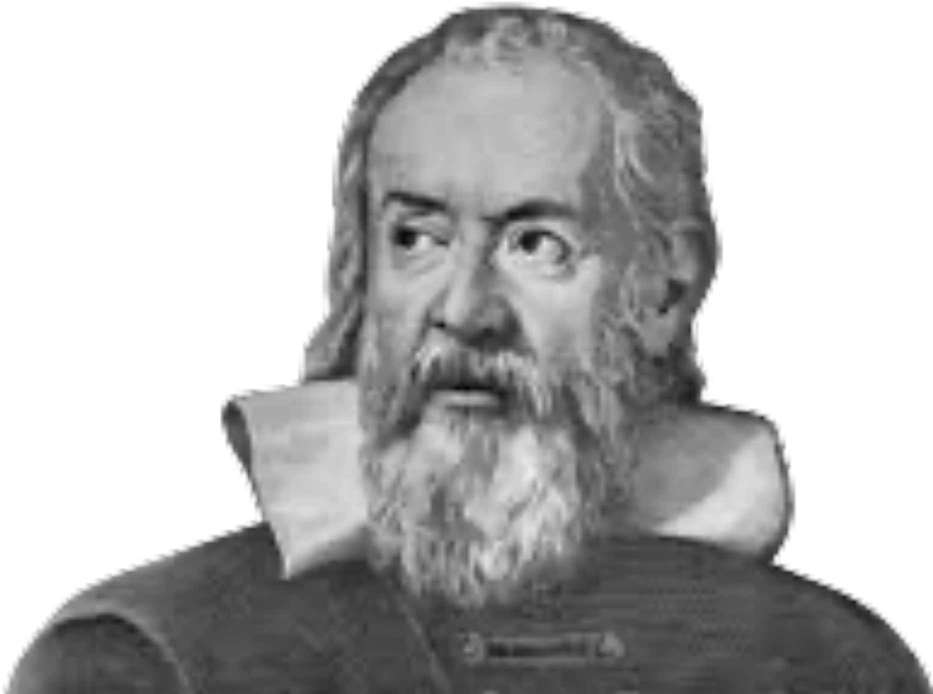


Galilean conformal HS algebras

(in any dimensions)



From U(so(2,D−1)) to U(iso(1,D−1))

- Same approach as for Carroll, but with a new splitting of so(2,D−1)

$$[J_{AB}\,,\,J_{CD}] = \tilde{\eta}_{AC}\,J_{BD} - \tilde{\eta}_{AD}\,J_{BC} - \tilde{\eta}_{BC}\,J_{AD} + \tilde{\eta}_{BD}\,J_{AC}$$

$$J_{ij} \longrightarrow \mathfrak{so}(D-2)$$

$$\bar{L}_- = H\,, \quad \bar{L}_0 = D\,, \quad \bar{L}_+ = K\,, \longrightarrow \mathfrak{sl}(2,\mathbb{R})$$

$$T_{i,-} = P_i\,, \quad T_{i,0} = B_i\,, \quad T_{i,+} = K_i$$

$$[J_{ij}, \bar{L}_m] = 0 \qquad [J_{ij}, T_{k,m}] = \delta_{ik}T_{j,m} - \delta_{jk}T_{i,m} \qquad [\bar{L}_m, T_{i,n}] = (m-n)T_{i,m+n}$$

$$[T_{i,m}, T_{j,n}] = \delta_{ij}(m-n)\bar{L}_{m+n} + \gamma_{mn}J_{ij}$$

From *U*(so(2,D−1)) to *U*(iso(1,D−1))

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$$[T_{i,m}, T_{j,n}] = \delta_{ij}(m-n)\bar L_{m+n} + \gamma_{mn}J_{ij}$$

Contraction: $T_{i,m} \to \epsilon^{-1}T_{i,m}$ with $\epsilon \to 0$

Bagchi, Gopakumar (2009)

The so(2,D–1) ideal

$\mathcal{I}_{AB} \sim 0$ $\qquad$ $\mathcal{I}_{ABCD} \sim 0$ $\qquad$ $C_2 \sim -\frac{(D+1)(D-3)}{4}\, id$ $\qquad$ or…

$$\gamma^{mn}\left\{T_{i,m},T_{j,n}\right\} - J_{k(i}J_{j)}{}^{k} - \frac{2}{D-2}\delta_{ij}\left(T^2 - J^2\right) \sim 0\,,$$

$$\delta^{ij}\left\{T_{i,m},T_{j,n}\right\} - \left\{\bar{L}_m,\bar{L}_n\right\} - \frac{2}{3}\gamma_{mn}\left(T^2 - \bar{L}^2\right) \sim 0\,,$$

$$6J^2 - 2(D-2)\bar{L}^2 - (D-5)T^2 \sim 0\,,$$

$$\left\{J_i{}^j,T_{j,m}\right\} + \gamma^{kn}(m-n)\left\{\bar{L}_k,T_{i,m+n}\right\} \sim 0\,,$$

$$\left\{J_{[ij},T_{k],m}\right\} \sim 0\,,$$

$$\gamma^{mn}\left\{\bar{L}_m,T_{i,n}\right\} \sim 0\,,$$

$$2\left\{T_{[i,m},T_{j],n}\right\} + (m-n)\left\{J_{ij},\bar{L}_{m+n}\right\} \sim 0\,,$$

$$J_{[ij}J_{kl]} \sim 0\,,$$

$$C_2 \equiv J^2 + \bar{L}^2 + T^2 \sim -\frac{(D+1)(D-3)}{2}\, id$$

The gca$_{D-1}$ ideal and Galilean HS algebras

$$
\begin{aligned}
\gamma^{mn}\left\{T_{i,m},T_{j,n}\right\}-\frac{2}{D-2}\delta_{ij}T^2 &\sim 0\,,\\
\delta^{ij}\left\{T_{i,m},T_{j,n}\right\}-\frac{2}{3}\gamma_{mn}T^2 &\sim 0\,,\\
J^2-\bar{L}^2 &\sim -\frac{(D-3)(D-5)}{4}\,id\,,\\
\left\{J_i{}^j,T_{j,m}\right\}+\gamma^{kn}(m-n)\left\{\bar{L}_k,T_{i,m+n}\right\} &\sim 0\,,\\
\left\{J_{[ij},T_{k],m}\right\} &\sim 0\,,\\
\gamma^{mn}\left\{\bar{L}_m,T_{i,n}\right\} &\sim 0\,,\\
\left\{T_{[i,m},T_{j],n}\right\} &\sim 0\,,\\
J_{[ij}J_{kl]} &\sim 0\,,\\
T^2 &\sim 0\,.
\end{aligned}
$$

- Galilean conformal HS algebra: $\mathfrak{ghs}_D \equiv \mathcal{U}(\mathfrak{gca}_{D-1})/\langle \mathcal{I}_{\mathfrak{g}}\rangle$

Carrollian and Galilean HS algebras in D=5

- In D=5 we start from a one-parameter family of algebras
  - Carrollian contraction: only one extra non-isomorphic algebra obtained in the limit $\lambda \to 0$
  - Galilean contraction: a 3D like structure emerges…

$$L_m = \{J_{31} + iJ_{12}\,,\, iJ_{23}\,,\, J_{31} - iJ_{12}\}$$

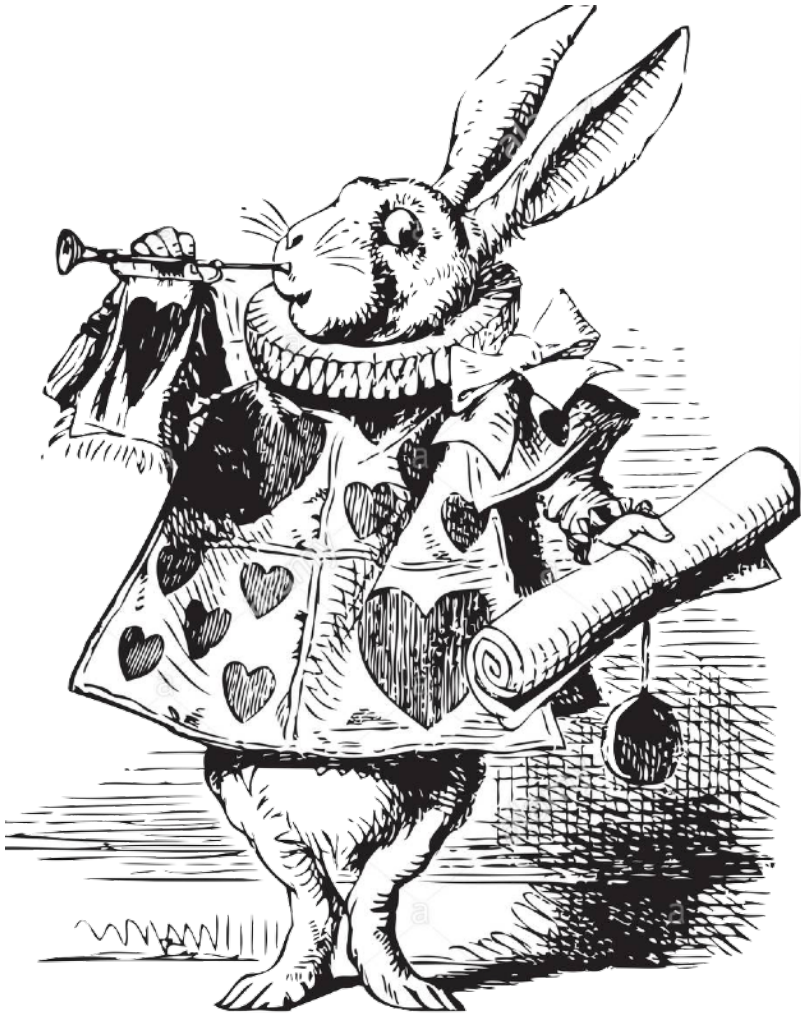
$$[L_m, L_n] = (m-n)\, L_{m+n}\,, \qquad [\bar{L}_m, \bar{L}_n] = (m-n)\, \bar{L}_{m+n}\,,$$

$$[L_m, T_{n,k}] = (m-n)\, T_{m+n,k}\,, \qquad [\bar{L}_m, T_{k,n}] = (m-n)\, T_{k,m+n}\,,$$

$$[T_{m,k}, T_{n,l}] = (m-n)\, \gamma_{kl} L_{m+n} + (k-l)\, \gamma_{mn} \bar{L}_{k+l}\,, \qquad [\bar{L}_m, L_m] = 0\,,$$

…but only one extra non-isomorphic algebra results from the *coset construction*

Other flat/Carrollian conformal HS algebras



“Geometric” algebras for Killing tensors?

- Why cannot we use the following bracket? Schouten (1940)
  - $$[v, w]^{\mu_1 \cdots \mu_{p+q-1}} \equiv \frac{(p+q-1)!}{p!q!} \left(p\, v^{\alpha(\mu_1 \cdots} \partial_\alpha w^{\cdots \mu_{p+q-1}} - q\, w^{\alpha(\mu_1 \cdots} \partial_\alpha v^{\cdots \mu_{p+q-1}} \right)$$
  - for p=1 and q=1 it coincides with the Lie bracket
  - the bracket of two Killing tensors is a Killing tensor
  - *the bracket of two traceless tensors isn't traceless*

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  - for p=1 and q=1 it coincides with the Lie bracket
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- Exception in D=3: $$(P^{(s)}_{\pm(s-1)})^{\mu_1\cdots\mu_{s-1}} \equiv \frac{(s-1)!}{(2\sqrt{2})^{s-2}}\,(P_{\pm1})^{\mu_1}\cdots(P_{\pm1})^{\mu_{s-1}},$$

  AC, Henneaux (2014)

  $$(L^{(s)}_{\pm(s-1)})^{\mu_1\cdots\mu_{s-1}} \equiv (s-1)\frac{(s-1)!}{(2\sqrt{2})^{s-2}}\,(P_{\pm1})^{(\mu_1}\cdots(P_{\pm1})^{\mu_{s-2}}(L_{\pm1})^{\mu_{s-1})}$$

$$[L^{(3)}_m, P^{(3)}_n]^{\mu\nu\rho} = (m-n)\left(2\,(P^{(4)}_{m+n})^{\mu\nu\rho} - \frac{2m^2+2n^2-mn-8}{20}\,\eta^{(\mu\nu}(P_{m+n})^{\rho)}\right)$$

ihs[∞] !

“Geometric” algebras for Killing tensors?

- Can we do something similar in any dimensions?

Basis of rank-2 Killing tensors	
$\mathcal{K}_{ab\|cd}{}^{\mu\nu} \equiv \mathcal{J}_{ac}{}^{(\mu}\mathcal{J}_{db}{}^{\nu)} + \mathcal{J}_{ad}{}^{(\mu}\mathcal{J}_{cb}{}^{\nu)} + \cdots\,,$ $\mathcal{M}_{ab\|c}{}^{\mu\nu} \equiv \mathcal{P}_a{}^{(\mu}\mathcal{J}_{bc}{}^{\nu)} + \mathcal{P}_b{}^{(\mu}\mathcal{J}_{ac}{}^{\nu)} + \cdots\,,$ $\mathcal{Q}_{ab}{}^{\mu\nu} \equiv 2\left(\mathcal{P}_a{}^{(\mu}\mathcal{P}_b{}^{\nu)} - \frac{1}{D}\,\eta_{ab}\eta^{cd}\mathcal{P}_c{}^{(\mu}\mathcal{P}_d{}^{\nu)}\right)$	$\mathcal{S}_{ab}{}^{\mu\nu} \equiv 2\left(\eta^{cd}\mathcal{J}_{ac}{}^{(\mu}\mathcal{J}_{db}{}^{\nu)} - \frac{1}{D}\,\eta_{ab}\eta^{cd}\eta^{ef}\mathcal{J}_{ce}{}^{(\mu}\mathcal{J}_{fd}{}^{\nu)}\right)$ $\mathcal{I}_a{}^{\mu\nu} \equiv 2\,\eta^{bc}\mathcal{P}_b{}^{(\mu}\mathcal{J}_{ca}{}^{\nu)}\,,$ $(\mathcal{J}^2)^{\mu\nu} \equiv \frac{1}{2}\,\eta^{ab}\eta^{cd}\mathcal{J}_{ac}^{(\mu}\mathcal{J}_{db}{}^{\nu)}\,,$ $(\mathcal{P}^2)^{\mu\nu} \equiv \eta^{ab}\mathcal{P}_a{}^{(\mu}\mathcal{P}_b{}^{\nu)} = \eta^{\mu\nu}$

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- Traceless Killing tensors do not span anymore a subalgebra, but…

$$\mathcal{I}_{abc}{}^{\mu\nu} \equiv 2\,\mathcal{J}_{[ab}{}^{(\mu}\,\mathcal{P}_{c]}{}^{\nu)} = 0\,, \quad \mathcal{I}_{abcd}{}^{\mu\nu} \equiv 2\,\mathcal{J}_{[ab}{}^{(\mu}\,\mathcal{J}_{cd]}{}^{\nu)} = 0$$

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$$\eta^{\mu\nu} \simeq id \quad \Longrightarrow \quad \mathcal{P}^2 \sim \nu\, id,\, \mathcal{I}_{abc} \sim 0,\, \mathcal{I}_{abcd} \sim 0$$

Schouten bracket algebra as HS algebra?

- Double interpretation for the Schouten bracket algebra
  - Rigid symmetries for unconstrained Fronsdal transformations
  - Inönü-Wigner contraction of the rigid symmetries of partially-massless fields
- Any examples in flat space?

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Francia; Joung, Mkrtchyan (2012)

  - Higher-derivative theories:

$$\mathcal{R}^{[\frac{s}{2}]}{}_{\mu(s)} = 0 \qquad \text{for } s \text{ even},$$

$$\partial \cdot \mathcal{R}^{[\frac{s-1}{2}]}{}_{\mu(s)} = 0 \qquad \text{for } s \text{ odd},$$

  - Partially-massless-like eom:

AC, Francia, Heissenberg (2020)

$$\Box\, \varphi_{\mu(s)} - \frac{s(D+2s-4)}{(t+1)(D+2s-t-4)} \left(\partial_\mu \partial \cdot \varphi_{\mu(s-1)} - \frac{s-1}{D+2(s-2)}\, g_{\mu\mu} \partial \cdot \partial \cdot \varphi_{\mu(s-2)} \right) = 0$$

Summary & overview

- One can build non-Abelian HS algebras including subalgebras $h = iso(1,D-1)$ or $h = gca_{D-1}$ (with the same spectrum as in AdS)
- "Good" Lorentz commutators guaranteed in UEA constructions
- Atypical commutators with translations (counterpart of the absence of "naive" minimal coupling?)

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What's next?

- Asymptotic symmetries?
- Modules associated to our algebras?
- Linearised curvatures?
- Recovering the algebras in interacting theories?