Geometric Partial Differential Equations (Qing Liu)

FY2023 Annual Report

Geometric Partial Differential Equations Unit
Associate Professor Qing Liu

Abstract

Our research focuses on geometric properties of various nonlinear partial differential equations and applications in surface evolutions, optimal control, and image processing. We study convexity and asymptotic behavior of solutions to nonlinear partial differential equations within the framework of viscosity solution theory. We also develop novel analysis techniques for partial differential equations defined on spaces with less smooth structure and extend existing fundamental results to sub-Riemannian manifolds and general metric spaces.

1. Staff

- Prof. Qing Liu, Group Leader
- Dr. Erbol Zhanpeisov, Postdoctoral Scholar
- Mr. Chenming Zhen, Rotation Student
- Mr. Kyle Grant, Rotation Student
- Ms. Yukiko Nakagawa, Administrative Assistant

2. Collaborations

2.1 Nonlocal Hamilton-Jacobi equations and applications

- Description: This project is concerned with geometric motion of a closed surface whose velocity depends on a nonlocal quantity of the enclosed region. Using the level set formulation, we study a class of nonlocal Hamilton-Jacobi equations and establish a control-based representation formula for solutions.
- Type of collaboration: Joint research
- Researchers:
    - Takashi Kagaya, Muroran Institute of Technology
    - Qing Liu, OIST
    - Hiroyoshi Mitake, University of Tokyo

2.2 Discontinuous eikonal equations in metric measure spaces

- Description: We study the eikonal equation in metric measure spaces, where the inhomogeneous term is allowed to be discontinuous, unbounded and merely integrable in the domain. Generalizing the notion of Monge solutions in our setting, we establish uniqueness and existence results for the associated Dirichlet boundary problem.
- Type of collaboration: Joint research
- Researchers:
    - Qing Liu, OIST
    - Nageswari Shanmugalingam, University of Cincinnati
    - Xiaodan Zhou, OIST

2.3 Horizontal quasiconvexity in the Heisenberg group

- Description: We establish a PDE approach to understand a certain weak type of convexity for sets in the Heisenberg group based on a nonlinear second order elliptic operator.
- Type of collaboration: Joint research
- Researchers:
    - Antoni Kijowski, OIST
    - Qing Liu, OIST
    - Ye Zhang, OIST
    - Xiaodan Zhou, OIST

2.4 Liouville-type theorems for fully nonlinear PDEs with boundary degeneracy

- Description: For a general class of fully nonlinear boundary-degenerate elliptic or parabolic equations, we provide a sufficient condition to guarantee nonexistence of solutions other than the trivial one.
- Type of collaboration: Joint research
- Researchers:
    - Qing Liu, OIST
    - Erbol Zhanpeisov, OIST

3. Activities and Findings

3.1 Nonlocal Hamilton-Jacobi equations and applications

We are interested in geometric motion of a closed surface whose velocity depends on a nonlocal quantity of the enclosed region. Using the level set formulation, we study a class of nonlocal Hamilton-Jacobi equations. This type of equations arise in models of crystal growth and also have important applications in image processing. While the well-posedness of such nonlocal evolution equations is well studied in the literature, the behavior of solutions is less well studied. In this work we establish a representation formula for the solutions from the viewpoint of optimal control theory. It can be viewed as a nonlocal counterpart for the strong connection between Hamilton-Jacobi equations and control theory. We also apply the formula to discuss the fattening phenomenon and large-time asymptotics of the solutions.

3.2 Discontinuous eikonal equations in metric measure spaces

This work is concerned with the eikonal equation, which is a fully nonlinear equation well known for its applications in geometric optics, optimal control, image processing, etc. Motivated by new applications to control theory, we study the eikonal equation in metric measure spaces, where the inhomogeneous term is allowed to be discontinuous, unbounded and merely integrable in the domain. For continuous eikonal equations, it is known that the notion of Monge solutions is equivalent to the standard definition of viscosity solutions. Generalizing the notion of Monge solutions in our setting, we establish uniqueness and existence results for the associated Dirichlet boundary problem. The key in our approach is to adopt a new metric, based on the optimal control interpretation, which integrates the discontinuous term and converts the eikonal equation to a standard continuous form. We also discuss the Hölder continuity of the unique solution with respect to the original metric under regularity assumptions on the space and the inhomogeneous term.

3.3 Horizontal quasiconvexity in the Heisenberg group

Continuing our project in the previous year, we provide a new PDE approach to understand horizontally quasiconvex (h-quasiconvex) functions in the Heisenberg group based on a nonlinear second order elliptic operator. We discuss sufficient conditions and necessary conditions for upper semicontinuous, h-quasiconvex functions in terms of the viscosity subsolution to the associated elliptic equation. Since the notion of h-quasiconvexity is equivalent to the horizontal convexity (h-convexity) of the function's sublevel sets, we further adopt these conditions to study the h-convexity preserving property for horizontal curvature flow in the Heisenberg group. Under the comparison principle, we show that the curvature flow starting from a star-shaped h-convex set preserves the h-convexity during the evolution.

3.4 Liouville-type theorems for fully nonlinear PDEs with boundary degeneracy

We study a general class of fully nonlinear boundary-degenerate elliptic or parabolic equations that admit a trivial solution. Although no boundary conditions are posed together with the equations, we show that the operator degeneracy actually generates an implicit boundary condition. Under appropriate assumptions on the degeneracy rate and regularity of the operator, we then prove that there exist no bounded solutions other than the trivial one. Our method is based on the arguments for uniqueness of viscosity solutions to state constraint problems for Hamilton-Jacobi equations. We also have several concrete examples that confirm our results.

4. Publications

4.1 Journals

1. Kijowski, A., Liu, Q., Zhou, X., Horizontally quasiconvex envelope in the Heisenberg group, ***Rev. Mat. Iberoam. 40 (2024), 57--92.***
2. Kagaya, T., Liu, Q., Mitake, H., A representation formula for viscosity solutions of nonlocal Hamilton-Jacobi equations and applications, to appear in ***SIAM J. Math. Anal.***
3. Kagaya, T., Liu, Q., Mitake, H., Quasiconvexity preserving property for first order nonlocal evolution equations, ***RIMS Kokyuroku,*** 2277 (2024), 57--69.
4. Zhanpeisov, E., Existence of solutions to fractional semilinear parabolic equations in Besov-Morrey spaces, ***Discrete Contin. Dyn. Syst.***, 43(2023), 3969--3986.

4.2 Books and other one-time publications

Nothing to report

4.3 Oral and Poster Presentations

1. Liu, Q., ***Hamilton-Jacobi equations on metric spaces***, Meeting on Multidisciplinary Study in Math, Tokyo Metropolitan University, March 25, 2024.
2. Liu, Q., ***Monge solutions of eikonal equations in metric spaces***, invited talk at 2024 Spring Meeting of the Mathematical Society of Japan, Osaka Metropolitan University, March 18, 2024.
3. Liu, Q., ***A PDE-based approach to Borell-Brascamp-Lieb inequality***, Fukae nonlinear PDE conference, Kobe University, March 8, 2024.
4. Liu, Q., ***A representation formula for viscosity solutions of nonlocal Hamilton-Jacobi equations and applications***, Conference on probabilistic and game theoretical interpretation of PDEs, Autonomous University of Madrid, November 21, 2023.
5. Zhanpeisov, E., ***Existence of solutions for fractional semilinear parabolic equations in Besov-Morrey spaces***, Workshop on Elliptic & Parabolic PDEs 2023, Ryukoku University, Japan, November 17, 2023.

6. Liu, Q., *First order fully nonlinear nonlocal evolution equations*, Minisymposium Nonlinear PDEs and Related Diffusion Phenomena, 10th International Congress on Industrial and Applied Mathematics, Waseda University, August 25, 2023.
7. Liu, Q., *Principal eigenvalue problem for infinity Laplacian in metric spaces*, Variational Methods for Nonlinear PDEs, Cardiff University, Cardiff, UK, July 7, 2023.
8. Liu, Q., *Discontinuous eikonal equations in metric measure spaces*, OIST Workshop Potential theory and Random Walks in Metric Spaces, OIST, June 2, 2023.

5. Intellectual Property Rights and Other Specific Achievements

Nothing to report

6. Meetings and Events

6.1 Geometric PDE and Applied Analysis Seminar

1. Sep. 7, 2023
    - Speaker: Takasi Senba (Fukuoka University)
    - Title: Behavior of solutions to systems related to Keller-Segel system
2. Sep. 7, 2023
    - Speaker: Kohei Soga (Keio University)
    - Title: Several topics on the linear transport equation
3. Sep. 14, 2023
    - Speaker: Federica Dragoni (Cardiff University)
    - Title: Horizontal mean curvature flow: a Riemannian approximation
4. Oct. 12, 2023
    - Speaker: Kensuke Yoshizawa (Kyushu University)
    - Title: Li-Yau type inequality for the p-bending energy
5. Oct. 20, 2023
    - Speaker: Rudrajit Banerjee (OIST)
    - Title: Wick rotating the heat kernel
6. Oct. 26, 2023
    - Speaker: Yoshihiro Tonegawa (Tokyo Institute of Technology)
    - Title: Existence theorem of Brakke's mean curvature flow
7. Dec. 14, 2023
    - Speaker: Nao Hamamuki (Hokkaido University)
    - Title: Waiting time effects for the wearing process of a non-convex stone
8. Dec. 14, 2023
    - Speaker: Yohei Fujishima (Shizuoka University)
    - Title: Quasi self-similarity and its application to the global in time solvability of a superlinear heat equation
9. Jan. 23, 2024
    - Speaker: Minhyun Kim (Hanyang University)
    - Title: Robust near-diagonal Green function estimates
10. Jan. 25, 2024
    - Speaker: Daniel Hauer (University of Sydney)
    - Title: Bernstein functional calculus and a generalized Helmholtz problem
11. Feb. 9, 2024
    - Speaker: Yoshiyuki Kagei (Tokyo Institute of Technology)
    - Title: Bernstein functional calculus and a generalized Helmholtz problem
12. Feb. 9, 2024
    - Speaker: Ryo Takada (University of Tokyo)

    - Title: Large time behavior of global solutions to the rotating Navier-Stokes equations
13. Feb. 9, 2024
    - Speaker: Lorenzo Cavallina (Tohoku University)
    - Title: A characterization of radial symmetry for composite media by overdetermined level sets

6.2 OIST Workshop Analysis on Metric Spaces

- Date: Jan 15-18, 2024
- Venue: OIST B250
- Co-organizers: Kazuhiro Ishige (University of Tokyo), Qing Liu (OIST), Paolo Salani (University of Florence)
- Speakers:
    - Karoly Boroczky (Alfred Renyi Institue of Mathematics)
    - Alessio Figalli (ETH Zurich)
    - Katie Gittins (Durham University)
    - Pengfei Guan (McGill University)
    - Daniel Hauer (University of Sydney)
    - Norihisa Ikoma (Keio University)
    - Hiroshi Iriyeh (Ibaraki University)
    - Xinan Ma (University of Science and Technology of China)
    - Rolando Magnanini (University of Florence)
    - Carlo Nitsch (Universisty of Napels, Federico II)
    - Shinya Okabe (Tohoku University)
    - Shigeru Sakaguchi (Tohoku University)
    - Megumi Sano (Hiroshima University)
    - Ryuichi Sato (Fukuoka University)
    - Cristina Trombetti (University of Napels Federico II)
    - Xiaodan Zhou (OIST)

7. Other

Nothing to report.